Applied Mathematics 2

1 Introduction

What is applied mathematics? Many branches of knowledge (sciences, economics...) can be expressed using mathamatical language. Mathematics and other branches of knowledge have an interdependent relationship, in that each is stimulated by the other. For example, calculus needed by Newton for gravity; Dirac delta function needed for quantum mechanics. Einstein used Riemann's theory of curved spaces (slightly modified) in his theory of gravity. Lorenz discovered chaos by modelling climate.

Steps in doing applied mathematics

1. Formulate problem in mathematical terms
2. Solve maths problem
3. Interpret solution and compare with empirical results
4. Generate relevant new mathematics: new problems?

Give summary of course, homework, etc.

2 Dimensional analysis

2.1 Units and standards for M, L and T

Discrete quantities can be directly identified with integers (3 apples OK, 10^{23} electrons not very useful if you can't count them) but continuous quantities need a standard for comparison (2.6 metres, 50 minutes). There are many units and standards; for scientific use, SI units have standards for mass (kg), length (m) and time (s) and a few others.

The standards (hence units) depend on measuring technology. For example, there are actually 3 standards of mass (solar, kg, atomic) because astronomical masses are measured using gravity (and G is not known well), ordinary masses using delicate balances, and atomic masses using microscopic effects. When the number of atoms in a macroscopic sample can be counted, the standard kg will become redundent. Length and time are defined using microscopic effects, with the speed of light defined exactly. This is because lengths are measured using lasers of specified frequency.

There are prefixes for various powers of 1000, (TGMkmunp), cf the conversions used in old units. In computers, these usually refer to powers of $1024 = 2^{10}$. Decimal time was attempted during the French revolution, but did not succeed.

2.2 Combining units

Thus, when we say a certain object has a mass of 3kg, we mean that it has a mass equivalent to three standard kg masses (assuming mass, length and time can be added and multiplied - obvious?). Anything with a value proportional to a standard mass such as the kilogram is said to have dimensions of mass (M, cf L, T). We can add two such objects:

$$M_1/kg + M_2/kg = (M_1 + M_2)/kg$$

but it doesn't make sense to add objects defined with respect to different standards:

$$M_1/kg + M_2/lb = ?$$

We can multiply or divide quantities to get a derived unit,

$$\frac{x/m}{t/s} = v/(ms^{-1})$$

and we say

$$[v] = LT^{-1}$$

We can find dimensions of any quantity for which we have an equation, eg.

$$[T] = [mv^2/2] = ML^2T^{-2}$$

$$[V] = [mgh] = M(LT^{-2})L = ML^2T^{-2}$$

The SI system gives names to the more important derived units, for example Newton ($N = kgms^{-2}$) as a unit of force, Joule ($J = kgm^2s^{-2}$) as a unit of energy, etc. The symbols are capitalised because they are named after people.

Most other operations do not make sense, for example, taking an exponential:

$$e^{M/kg} \neq e^M/kg$$

or

$$e^{M/kg} = 1 + M/kg + M^2/(2kg^2) + \ldots = ?$$

Logarithms may sometimes be OK in combination:

$$\ln(M_1/kg) - \ln(M_2/kg) = \ln(M_1/M_2)$$

and occurs when evaluating certain integrals.

2.3 Other units

Angle θ is a pure number, usually measured in radians. Temperature Θ in Kelvin (same as degrees Celsius, but with a zero at absolute zero, water freezes at 273K). Electrodynamic quantities have a number of different unit conventions, for example, the electrostatic system (esu) uses the force between two charges

$$F = q^2/r^2$$

to define charge in terms of M, L and T. This leads to the equation

$$\nabla \cdot \mathbf{E} = 4\pi\rho$$

where $\mathbf{E}$ is the electric field and ρ is the charge density.

In contrast, the SI standard is that charge is defined by a standard Coulomb, so that

$$F = qE = q^2/(4\pi\epsilon_0)$$

where ϵ_0 is a constant called the permittivity of free space, and the 4π is to simplify the equation for the electric field E, that is

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0$$

where ρ is charge density, charge per unit volume. In both cases other quantities can be derived: Current (Ampere=Coulomb per second), Potential (Volt=Joule per Coulomb), etc.

2.4 Dimensional analysis; scale models

Dimensions are very useful, even when very little is known about the mathematical relationships between various quantities. Given the dimensions of all the relevant quantities, we can

- Check the results of any calculations
- Construct the relevant dimensionless parameters
- Design scale models for experiments

Consider ship design. The relevant parameters are the size $[L] = L$, the velocity $[U] = LT^{-1}$, gravity $[g] = LT^{-2}$, the power $[P] = [E/t] = ML^2T^{-3}$, and the properties of water, in particular its density $[\rho] = [M/V] = ML^{-3}$ and viscosity $[\nu] = L^2T^{-1}$ (given). The power is an unknown function of the other variables:

$$P = f(L, U, g, \rho, \nu)$$

If we have

$$P = CL^aU^bg^c\rho^d\nu^e$$

$$ML^2T^{-3} = M^dL^{a+b+c-3d+2e}T^{-b-2c-e}$$

Thus

$$d = 1$$

$$a + b + c - 3d + 2e = 2$$

$$b + 2c + e = 3$$

Fix c and e, then $b = 3 - 2c - e$ and $a + 3 - 2c - e + c - 3 + 2e = 2$, that is, $a = 2 + c - e$. Thus

$$P = CL^{2+c-e}U^{3-2c-e}g^c\rho\nu^e$$

$$P = CL^2U^3\rho(Lg/U^2)^c(\nu/LU)^e$$

Since the powers of Lg/U^2 and ν/LU are arbitrary, these must be dimensionless. The are directly related to the Froude and Reynolds numbers respectively, and both are very important in the theory of fluids.

$$Fr = U/\sqrt{gL}$$

$$Re = UL/\nu$$

Finally, the power must be

$$P = L^2U^3\rho f(Fr, Re)$$

where f is an arbitrary function. There are many possible ways of writing this result, for example, the dimensionless variables could be $Fr \times Re$ and Fr/Re; the important fact is that there are two independent dimensionless parameters. The prefactor could also be multiplied by a dimensionless parameter, for example, multiplying by Fr^4 we get

$$P = U^7\rho/g^2F(Fr, Re)$$

where the new function is related to the old one by $f(Fr, Re) = Fr^4F(Fr, Re)$.

Now we hope to calculate this unknown function of 2 variables using a scale model with $L' = L/100$, so to keep Fr the same, need $U' = U/10$, and to keep Re the same, need $\nu' = \nu/1000$. Impossible: no such liquid. Fortunately viscous effects are often unimportant in ship design, so a slightly smaller Reynolds number can also work. It is much easier to experimentally measure this function of 2 variables than the original function of 5 variables. If we do make a theoretical estimate using more knowledge of the equations, we should check to make sure the dimensions are correct.

2.5 scaling

We can use dimensional analysis to scale differential equations in order to simplify them and extract the important parameters for approximations. Example: particle projected from the Earth,

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}$$

$$y(0) = 0$$

$$\frac{dy}{dt}(0) = V$$

where R is the radius of the Earth, using inverse square law; g is the acceleration when $y = 0$. We have three parameters, $[R] = L$, $[g] = LT^{-2}$ and $[V] = LT^{-1}$. A natural unit of time is R/V, so a dimensionless time is

$$t_1 = tV/R$$

and a dimensionless distance,

$$y_1 = y/R$$

so

$$\frac{dy}{dt} = R\frac{dy_1}{dt_1}\frac{dt_1}{dt} = V\frac{dy_1}{dt_1}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt_1}(V\frac{dy_1}{dt_1})\frac{dt_1}{dt} = \frac{V^2}{R}\frac{d^2y_1}{dt_1^2}$$

and hence

$$\frac{V^2}{R}\frac{d^2y_1}{dt_1^2} = -\frac{gR^2}{(y_1R+R)^2}$$

$$\epsilon\frac{d^2y_1}{dt_1^2} = -\frac{1}{(y_1+1)^2}$$

with $\epsilon = V^2/(gR)$ a dimensionless parameter. We also have

$$\frac{dy_1}{dt_1}(0) = V/V = 1$$

$$y_1(0) = 0$$

Notice that the solution of the original problem involves three parameters $\{g, R, V\}$ while the scaled version has only a single parameter ϵ. This means that the solution in the original variables only involves the combination $\epsilon = V^2/gR$. We are expressing the problem in terms of the natural length R and natural time R/V.

For example, suppose we want to find the time at which the projectile reaches maximum height, t^M. This occurs when

$$\frac{dy}{dt}|_{t=t^M} = 0$$

$$\frac{dy_1}{dt_1}|_{t_1=t_1^M} = 0$$

but since y_1 is a function of t_1 and ϵ, the solution can only be of the form

$$t^M = t_1^M R/V = f(V^2/gR)R/V$$

Experimentally, a plot of t_1^M vs ϵ should all lie on the same curve (draw like $\sqrt{x}$ with a vertical asymptote), using data from different planets (R and g) and projectiles (different V).

The above scaling is not unique: another unit of time is $\sqrt{R/g}$ (using R and g instead of R and V). So we could use

$$t_2 = t\sqrt{g/R}$$

$$y_2 = y/R$$

Now we get

$$\frac{d^2y_2}{dt_2^2} = -\frac{1}{(y_2+1)^2}$$

$$y_2(0) = 0$$

$$\frac{dy_2}{dt_2}(0) = \epsilon^{1/2}$$

where $\epsilon = V^2/gR$ as before. The parameter has now moved into the initial condition, but the solution is still a function of the same parameter ϵ, so we have

$$t^M = t_2^M \sqrt{R/g} = F(V^2/gR)\sqrt{R/g}$$

which is equivalent to what we had before:

$$f(V^2/gR)R/V = F(V^2/gR)\sqrt{R/g}$$

$$f(V^2/gR) = F(V^2/gR)\sqrt{V^2/gR}$$

Note that this solution is exactly what we would find if we applied dimensional analysis (as with the particle of given energy and mass, etc.) without knowing the equation or initial condition, if we assume that t^M depends only on g, R and V. That is, we look for a combination

$$[g^a R^b V^c] = L^{a+b+c}T^{-2a-c}$$

Then $c = -2a - 1$ and $b = -a - c = a + 1$: The f solution is $a = 0$, $b = 1$, $c = -1$; the F solution is $a = -1/2$, $b = 1/2$, $c = 0$.

2.6 Approximations

Using two different scalings, we have replaced

$$\frac{d^2y}{dt^2} = -\frac{gR^2}{(y+R)^2}, \quad y(0)=0, \quad \frac{dy}{dt}(0) = V$$

by

$$\epsilon\frac{d^2y_1}{dt_1^2} = -\frac{1}{(y_1+1)^2}, \quad y_1(0)=0, \quad \frac{dy_1}{dt_1}(0) = 1$$

and

$$\frac{d^2y_2}{dt_2^2} = -\frac{1}{(y_2+1)^2}, \quad y_2(0)=0, \quad \frac{dy_2}{dt_2}(0) = \epsilon^{1/2}$$

where

$$\epsilon = V^2/gR, \quad y_1 = y/R, \quad t_1 = tV/R, \quad y_2 = y/R, \quad t_2 = t\sqrt{gR}$$

Now suppose $\epsilon \ll 1$ (for example $V = 100$, $g = 10$ and $R = 6 \times 10^6$ in SI units: $\epsilon \approx 10^{-4}$). Note that only a dimensionless quantity can be compared with 1. If we ignore ϵ in each of the above equations, disaster!

$$0 = \frac{1}{(y_1+1)^2}, \quad y_1(0)=0, \quad \frac{dy_1}{dt_1}(0) = 1$$

and

$$\frac{d^2y_2}{dt_2^2} = -\frac{1}{(y_2+1)^2}, \quad y_2(0)=0, \quad \frac{dy_2}{dt_2}(0) = 0$$

The first equation is contradictory, the second gives a negative solution.

Solution: Scale so that the terms in the equation are the same order of magnitude. For small velocities, we know the solution, since the acceleration is simply $-g$:

$$\frac{dy}{dt} = V - gt$$

$$y = Vt - gt^2/2$$

The time to maximum height is $t = V/g$ and this height is $y^M = V^2/2g$. Thus the times and distances are of the order of magnitude defined by these quantities. This time and length gives a new scaling:

$$y_3 = yg/V^2$$

$$t_3 = tg/V$$

in terms of which the equations are

$$g\frac{d^2y_3}{dt_3^2} = -\frac{-gR^2}{(y_3V^2/g+R)^2} \quad y_3(0) = 0 \quad \frac{dy_3}{dt_3}(0) = 1$$

that is,

$$\frac{d^2 y_3}{dt_3^2} = -\frac{1}{(1+\epsilon y_3)^2} \quad y_3(0) = 0 \quad \frac{dy_3}{dt_3}(0) = 1$$

Now, neglecting ϵ gives sensible results, and we can attempt a perturbation theory.

2.7 Regular perturbations

Define $y_3 = z$ and $t_3 = \tau$, so we have

$$\frac{d^2 z}{d\tau^2} = -\frac{1}{(1+\epsilon z)^2} \quad z(0) = 0 \quad \frac{dz}{d\tau}(0) = 1$$

where $\epsilon = V^2/gR \ll 1$. We seek an expansion in powers of ϵ, so substitute (also can use $z^{(0)}$ notation)

$$z(\tau) = z_0(\tau) + \epsilon z_1(\tau) + \epsilon^2 z_2(\tau) + O(\epsilon^3)$$

The binomial theorem gives

$$\begin{aligned}(1+\epsilon z)^{-2} &= 1 - 2\epsilon z + 3\epsilon^2 z^2 + O(\epsilon^3) \\ &= 1 - 2\epsilon z_0 + \epsilon^2(3z_0^2 - 2z_1) + O(\epsilon^3)\end{aligned}$$

The LHS is

$$\frac{d^2 z}{d\tau^2} = \frac{d^2 z_0}{d\tau^2} + \epsilon\frac{d^2 z_1}{d\tau^2} + \epsilon^2\frac{d^2 z_2}{d\tau^2} + O(\epsilon^3)$$

We can set ϵ to zero and solve for z_0, then divide the remaining equation by ϵ to solve for z_1 etc. This is the same as equating coefficents of ϵ:

$$\frac{d^2 z_0}{d\tau^2} = -1$$

$$\frac{d^2 z_1}{d\tau^2} = 2z_0$$

$$\frac{d^2 z_2}{d\tau^2} = 2z_1 - 3z_0^2$$

Each equation only involves the previous solutions, so should be easy to solve. We need the initial conditions: At $\tau = 0$, we have

$$0 = z = z_0 + \epsilon z_1 + \epsilon^2 z_2 + \dots$$

so

$$0 = z_0 = z_1 = z_2 = \dots$$

Also

$$1 = \frac{dz}{d\tau} = \frac{dz_0}{d\tau} + \epsilon\frac{dz_1}{d\tau} + \epsilon^2\frac{dz_2}{d\tau} + \dots$$

so

$$\frac{dz_0}{d\tau} = 1, \quad 0 = \frac{dz_1}{d\tau} = \frac{dz_2}{d\tau} = \dots$$

Now we are ready to solve:

$$z_0 = -\tau^2/2 + \tau$$

Thus

$$z_1 = -\tau^4/12 + \tau^3/3$$

and

$$2z_1 - 3z_0^2 = -\frac{11}{12}\tau^4 + \frac{11}{3}\tau^3 - 3\tau^2$$

$$z_2 = -\frac{11}{360}\tau^6 + \frac{11}{60}\tau^5 - \frac{1}{4}\tau^4$$

to go further, enlist the help of a computer algebra package such as Maple.

Thus

$$z = \tau - \tau^2/2 + \epsilon(\tau^3/3 - \tau^4/12) + \dots$$

Time to max height? Need $dz/d\tau = 0$, thus

$$0 = 1 - \tau + \epsilon(\tau^2 - \tau^3/3) + \dots$$

This is a cubic, but as above we can expand the solution in ϵ. We have

$$\tau^M = \tau_0 + \epsilon\tau_1 + \dots$$

leading to the equations

$$0 = 1 - \tau_0$$

$$0 = -\tau_1 + \tau_0^2 - \tau_0^3/3$$

Thus

$$\tau_0 = 1$$

$$\tau_1 = 2/3$$

$$\tau^M = 1 + 2\epsilon/3 + \dots$$

Max height is found by substituting (again to order ϵ):

$$z^M = (1 + 2\epsilon/3) - (1 - 4\epsilon/3)/2 + \epsilon(1/3 - 1/12) + O(\epsilon^2)$$

$$z^M = 1/2 + \epsilon/4 + O(\epsilon^2)$$

The maximum height can be found exactly:

$$\frac{dv}{d\tau} = v\frac{dv}{dz} = -\frac{1}{(1+\epsilon z)^2}$$

with $v = dz/d\tau$, at $\tau = 0$: $z = 0$, $v = 1$.

$$vdv = -\frac{dz}{(1+\epsilon z)^2}$$

$$v^2/2 = \frac{1}{\epsilon(1+\epsilon z)} + C$$

Using initial condition

$$1/2 = 1/\epsilon + C$$

$$(1-v^2)/2 = \frac{1}{\epsilon}(1 - \frac{1}{1+\epsilon z}) = \frac{z}{1+\epsilon z}$$

So $v = 0$ means

$$1/2 = \frac{z^M}{1+\epsilon z^M}$$

$$z^M = \frac{1}{2-\epsilon} = \frac{1}{2}(1-\frac{\epsilon}{2})^{-1} = \frac{1}{2} + \frac{\epsilon}{4} + \ldots$$

This expansion converges for $\epsilon < 2$, which is in fact the whole range of validity of z^M. The solution in terms of time is given by a difficult (elliptic) integral, so the expansion is useful in this case.

In original variables $y = zV^2/g$, $t = \tau V/g$

$$yg/V^2 = gt/V - \frac{g^2t^2}{2V^2} - \frac{V^2}{gR}\left[(\frac{gt}{V})^3/3 - (\frac{gt}{V})^4/12\right] + \ldots$$

$$y = Vt - gt^2/2 + \frac{V^2}{gR}\left[\frac{g^2t^3}{3V} - \frac{g^3t^4}{12V^2}\right] + \ldots$$

$$t^M = \frac{V}{g}(1 + \frac{2V^2}{3gR} + \ldots)$$

$$y^M = \frac{V^2}{2g}(1 + \frac{V^2}{2gR} + \ldots)$$

3 Conservation and linear first order PDE problems

3.1 Flow from an axisymmetric container

...an important idea pervading applied mathematics... involves identifying a measurable quantity and tracking its creation and disappearance during a process. For example, consider a container of axisymmetric shape $A(h)$. Conservation of volume (uniform density)

$$dV/dt = 0 - Q = -A_0 v_0$$

since there is no addition and the volume moving out is $v_0 A_0 dt$. But

$$dV = A(h)dh$$

(small slice at the top), so

$$dh/dt = -\frac{v_0 A_0}{A(h)}$$

We need to know how v_0 depends on h: Conservation of energy. A mass $dm = \rho A(h)(-dh)$ has been removed, losing potential energy $dmgh$ and gaining kinetic energy $mv_0^2/2$ (ignore velocity at top) so

$$mv_0^2/2 = mgh$$

$$v_0 = \sqrt{2gh}$$

Actually, losses at base, so assume

$$v_0 = \alpha\sqrt{2gh}$$

for $0 < \alpha < 1$. So

$$dh/dt = -\frac{\alpha A_0 \sqrt{2gh}}{A(h)}$$

If we have a circular cylinder (except for outlet at bottom), $A(h) = A$, so

$$dh/dt = -\alpha(A_0/A)\sqrt{2gh}$$

$$\int_0^H dh/\sqrt{h} = -\alpha(A_0/A)\sqrt{2g}\int dt$$

$$-2\sqrt{H} = -\alpha(A_0/A)\sqrt{2g}t$$

$$t = \frac{\sqrt{2H}}{\alpha\sqrt{g}}(A/A_0)$$

Check dimensions.

Now, if $dh/dt = -u$ constant velocity, we have

$$A(h) = \alpha A_0\sqrt{2gh}/u$$

so the radius is

$$r(h) = \sqrt{A(h)/\pi} = \frac{\alpha^{1/2}A_0^{1/2}2^{1/4}g^{1/4}h^{1/4}}{\sqrt{u\pi}}$$

(diagram). The total time is then

$$t = H/u = \frac{\sqrt{H}}{\alpha\sqrt{2g}}(A(H)/A_0)$$

half the length of time as before.

3.2 1D conservation equation

Suppose we have a conserved quantity S in one dimension, eg. along a pipe. Let the density or concentration of S per unit length be $P(x,t)$ eg. mass per unit length, number of molecules per unit length, number of cars per unit length in traffic model. Thus $P(x,t)dx$ is the amount of S contained between x and $x+dx$ at time t.

Let the flow rate or flux of S per unit time be $Q(x,t)$. Thus $Q(x,t)dt$ is the amount of S passing through the point x in time dt.

We derive a relation between $P(x,t)$ and $Q(x,t)$ as follows: at fixed t, flow in is $Q(x,t)$ and flow out is $Q(x+dx,t)$. The amount in this interval is $P(x,t)dx$. Thus

$$\frac{dS}{dt} = \frac{\partial}{\partial t}P(x,t)dx = Q(x,t) - Q(x+dx,t)$$

But

$$Q(x+dx,t) = Q(x,t) + dx\frac{\partial}{\partial x}Q(x,t) + O(dx^2)$$

so, dividing by dx and taking the limit dx to zero, we get:

$$\frac{\partial}{\partial t}P(x,t) + \frac{\partial}{\partial x}Q(x,t) = 0$$

Alternatively, we can work in a finite volume. Amount of S in region x_1 to x_2 is $\int_{x_1}^{x_2} P(x,t)dx$. Rate of change is

$$\frac{dS}{dt} = \frac{d}{dt}\int_{x_1}^{x_2} P(x,t)dx = \int_{x_1}^{x_2} \frac{\partial}{\partial t}P(x,t)dx = Q(x_1,t) - Q(x_2,t)$$

But

$$\int_{x_1}^{x_2} \frac{\partial}{\partial x}Q(x,t)dx = [Q(x,t)]_{x_1}^{x_2} = Q(x_2,t) - Q(x_1,t)$$

so

$$\int_{x_1}^{x_2} \left(\frac{\partial P}{\partial t} + \frac{\partial Q}{\partial x}\right) dx = 0$$

Provided P and Q are sufficiently smooth, since this holds for any interval (x_1, x_2), the integrand must vanish, and

$$P_t + Q_x = 0$$

In general we can consider a source $R(x,t)$ of S, leading to

$$P_t + Q_x = R$$

(problem 3.1).

Suppose that there is a velocity field $U(x,t)$ at which S is transported. Consider the amount of S between x and $x+dx$, Pdx. This is transported in a time $dt = dx/U$, thus the flux is $Q = Pdx/(dx/U) = PU$, ie

$$P_t + (PU)_x = R$$

3.3 Flow of chemical down a pipe

A fluid pouring down a pipe of area $A(x)$ carries a chemical with concentration $c(x,t)$ with velocity $u(x,t)$. The flux is $Q = Auc$ and the density is $P = Ac$ (diagrams), note that $Q = Pu$. Thus

$$0 = P_t + Q_x = (Ac)_t + (Auc)_x$$

In a uniform pipe, A is constant, so

$$0 = c_t + (uc)_x$$

This is a first order linear pde for c once u is given.

Now consider conservation of mass for the fluid. The fluid has density ρ, so mass $dm = Pdx = \rho A dx$. The mass flux $Q = Pu = \rho Au$, so

$$(\rho A)_t + (\rho Au)_x = 0$$

If ρ is constant, and A and u do not depend on time, we have

$$(Au)_x = 0$$

so

$$Au = const = A_0 u_0$$

Substituting into our previous equation,we have

$$A(x)c_t + A_0 u_0 c_x = 0$$

so we can now solve for $c(x,t)$ given $A(x)$.

Now if A is constant, so is u, and we have

$$c_t + uc_x = 0$$

with initial conditions $c(x,0) = c_0(x)$. Consider the lines in the $x - t$ plane of constant c, say given by $X(t)$. This means

$$0 = \frac{d}{dt}c(X(t),t) = c_t + X'(t)c_x$$

Thus we must have $X'(t) = u$ the constant velocity. Thus $X(t) = ut + \xi$ where ξ is the value of x at $t = 0$. These lines are called characteristic curves, and play an important role in the solution of many pdes. Thus

$$c(x,t) = c(\xi,0) = c_0(\xi)$$

along the curve $x = ut + \xi$. Thus

$$c(x,t) = c_0(x - ut)$$

The characteristics carry the initial conditions forward in time with speed u, which is not constant in general (later). Draw picture of wave moving with speed u, unchanged since u is constant.

Note that we can specify "initial" data on any curve in the $x - t$ plane that contains only point along the characteristics - no point and the solution is undefined, more than one point and the solution is multiply defined. For example we could specify data at $u(0,t)$ and $u(x,0)$ for $t > 0$ and $x > 0$ respectively, but not for $t > 0$ and $x < 0$ (diagrams).

3.4 Traffic flow

We are going to develop a continuum model, dealing with average values rather than the properties of individual cars. We look from the side of the road (Eulerian description), rather than from one of the cars (Lagrangian description). We define traffic density $\rho(x,t)$, (mean) traffic speed $u(x,t)$ and traffic flux $q(x,t)$ as before. The highway is of uniform width, and has no entrances or exits.

At fixed time, count the number of cars dN in an interval x to $x + dx$; this is $dN = \rho(x,t)dx$, then take the limit dx to zero. Similarly, at a fixed position, count the number of cars dM passing in the interval t to $t + dt$; this is $dM = q(x,t)dt$ then take the limit dt to zero. After time dt, a typical car covers a distance $dx = udt$. During this time $dN = \rho dx$ cars pass a point, so the flux at that point is measured as $q = dN/dt = \rho u$ as before.

Conservation of cars is (from above)

$$\rho_t + q_x = 0$$

$$\rho_t + (\rho u)_x = 0$$

How does the flow rate q depend on ρ? Clearly it does, for example, we expect the speed u to decrease as ρ increases, and $q = \rho u$. Let

$$q = f(\rho)$$

$$q_x = f'(\rho)\rho_x$$

What is the relation $f(\rho)$? First consider the velocity u. It takes a maximum value u_{max} at $\rho = 0$ and decreases to zero at a maximum density ρ_{max}. We can do observations to establish these parameters.

Multiplying by u to get $f(\rho) = q = \rho(u)u$ we see that $f(\rho)$ should start at zero at $u = 0$, rise to a maximum, and return to zero at ρ_{max}. Call the density at which the flux is maximum ρ^*, that is

$$f'(\rho^*) = 0$$

There are two values of the density, a high density and a low density for each value of the flux below this maximum.

The speed of travel u is $f(\rho)/\rho$, in other words the gradient of the line joining the origin with the point $(\rho, q = f(\rho))$ on the graph. The quantity $f'(\rho)$ also has dimensions of velocity, and turns out to be the signal speed (ie gives the characteristics). The signal speed is positive for low densities and negative for high densities.

We can see this for the case $f'(\rho) = c$ a constant (only true if u is constant in general). Then

$$\rho_t + c\rho_x = 0$$

and we get back to the same equation as for the chemical flowing through the pipe. ρ is constant along lines such that $x - ct$ is constant, that is

$$\rho(x,t) = \rho_0(x - ct)$$

if $\rho(x,0) = \rho_0(x)$. The argument is the same as we used before: find the lines $X(t)$ along which ρ is constant; these lines must satisfy

$$0 = \frac{d\rho}{dt} = \rho_t + X'(t)\rho_x$$

hence $X'(t) = c$, $X(t) = ct + \xi$.

Digression: If, in general, we have $\rho(x,t)$ we can describe it in two ways, either as a surface in (ρ, x, t) space, eg. a hemisphere

$$\rho^2 + x^2 + t^2 = a^2$$

corresponding to

$$\rho = \sqrt{a^2 - x^2 - t^2}$$

or as contour lines of constant ρ in (x,t) space, here circles.

$$x^2 + t^2 = a^2 - \rho^2$$

which we can write as $X(t)$:

$$X(t) = \pm\sqrt{a^2 - \rho^2 - t^2}$$

Thus

$$0 = \frac{d}{dt}\rho(X(t),t) = \rho_t + X'(t)\rho_x$$

Check:

$$\rho_t = -t/\rho$$

$$\rho_x = -x/\rho$$

$$X'(t) = -t/x$$

4 Quasilinear first order PDE problems

4.1 Traffic again

Returning to the problem of traffic flow, we have

$$\rho_t + f'(\rho)\rho_x = 0$$

with $f(0) = f(\rho_{max}) = 0$, $f(\rho) > 0$ for $0 < \rho < \rho_{max}$ and $f'(\rho^*) = 0$. Any function $\rho(x,t)$ will remain constant along paths $x = X(t)$ provided

$$0 = \frac{d}{dt}\rho(X(t),t) = \rho_t + X'(t)\rho_x$$

It follows that ρ is constant along curves $x = X(t)$ defined by

$$\frac{dx}{dt} = X'(t) = f'(\rho)$$

but since ρ is constant, so is $f'(\rho)$, and we can integrate to obtain $\rho(x,t) = \rho(\xi,0)$ along

$$x(t) - \xi = f'(\rho(\xi,0))t = f'(\rho_0(\xi))t$$

Given the initial conditions along $t = 0$, we can draw straight lines, characteristics, from the value of $f'(\rho(x,0))$ at each point. If $f'(\rho) = 0$ the line is vertical (constant x). Thus we have shown that $f'(\rho)$ is indeed the signal speed. The car speed $u = f(\rho)/\rho > f'(\rho)$.

Note that if we can solve

$$x - \xi = f'(\rho_0(\xi))t$$

for ξ in terms of x and t, we would then have the solution

$$\rho(x,t) = \rho_0(\xi(x,t))$$

which says that since the density ρ is the same along each of the characteristic lines, we need to find which initial point ξ leads to the particular point (x,t) at a velocity $f'(\rho)$.

4.2 The parabolic model

The simplest form for $f(\rho)$ satisfying the conditions is

$$q = f(\rho) = C\rho(1 - \rho/\rho_{max})$$

where C is a constant. Then

$$u = q/\rho = C(1 - \rho/\rho_m)$$

When $\rho = 0$, $u = C$ so the constant $C = u_{max}$.

$$f(\rho) = u_{max}\rho(1 - \rho/\rho_{max})$$

$$f'(\rho) = u_{max}(1 - 2\rho/\rho_{max}) = 0$$

when

$$\rho = \rho^* = \rho_{max}/2$$

$$q_m ax = u_{max}\rho_{max}/4$$

For example, at traffic lights that turn green at $t = 0$, the density is described by

$$\rho(x,0) = \begin{cases} \rho_{max} & x \leq 0 \\ \rho_{max}(1 - x/\epsilon) & 0 \leq x \leq \epsilon \\ 0 & x \geq \epsilon \end{cases}$$

where the small quantity ϵ is introduced to ensure that the density is continuous. Using the parabolic model we find

$$u(x,0) = u_m(1 - \rho(x,0)/\rho_{max}) = \begin{cases} 0 & x \leq 0 \\ u_{max}x/\epsilon) & 0 \leq x \leq \epsilon \\ u_{max} & x \geq \epsilon \end{cases}$$

At $x < 0$ before lights, cars at rest at maximum density, after which speed increases rapidly to maximum.

The characteristic lines of constant ρ are given by (from above)

$$x - \xi = f'(\rho_0(\xi))t$$

$$x - \xi = u_{max}t(1 - 2\rho/\rho_m)$$

ξ is the value of x at $t = 0$, and the lines are straight since ρ is constant. We sketch these lines as follows: for $\xi < 0$ we have $\rho_0(\xi) = \rho_{max}$ and hence

$$x = \xi - u_{max}t$$

For $\xi > \epsilon$ we have $\rho_0(\xi) = 0$ and hence

$$x = \xi + u_{max}t$$

For $0 < \xi < \epsilon$ we have $\rho_0(\xi) = \rho_{max}(1 - \xi/epsilon)$ and hence

$$x = \xi + u_{max}t(2\xi/\epsilon - 1)$$

(Diagram). There is a characteristic of infinite slope $f'(\rho) = 0$ at $x = \epsilon/2$.

In order to write down the solution explicitly, we need to solve these equations for $\xi(x,t)$, that is,

$$\xi = \begin{cases} x + u_{max}t & x < -u_{max}t \\ \frac{x + u_{max}t}{1 + 2u_{max}t/\epsilon} & -u_{max}t < x < u_{max}t + \epsilon \\ x - u_{max}t & x > u_{max}t + \epsilon \end{cases}$$

then using the initial conditions for ρ,

$$\rho(x,t) = \rho_0(\xi(x,t)) = \begin{cases} \rho_{max} & x < -u_{max}t \\ \rho_{max}(1 - \frac{x+u_{max}t}{\epsilon+2u_{max}t} & -u_{max}t < x < u_{max}t + \epsilon \\ 0 & x > u_{max}t + \epsilon \end{cases}$$

We can check this solution at $t = 0$ to make sure it reduces to the initial conditions. We can also plot ρ and u at different times, both of which have a smaller gradient at later times. The point $x = \epsilon/2$ at which the signal speed is zero corresponds to the point at which neither ρ nor u depend on time.

Example 2: Lane narrowing.

Suppose at $x = 0$ endless roadworks causing lane narrowing and loss of speed. Assume the initial velocity is

$$u(x,0) = \begin{cases} u_m & x < 0 \\ u_m(1-(1-\alpha)x/a) & 0 < x < a \\ \alpha u_m & x > a \end{cases}$$

for some $0 < \alpha < 1$. Then since $\rho = \rho_m(1 - u/u_m)$

$$\rho/\rho_m = \begin{cases} 0 & x < 0 \\ (1-\alpha)x/a & 0 < x < a \\ 1-\alpha & x > a \end{cases}$$

Draw diagrams of u and ρ. We have

$$x = u_m t(1 - 2\rho/\rho_m) + \xi$$

as the characteristics, solutions when ρ is constant. So

$$x = \begin{cases} u_m t + \xi & \xi < 0 \\ u_m t(1 - 2(1-\alpha)\xi/a) + \xi & 0 < \xi < a \\ u_m t(2\alpha - 1) + \xi & \xi > a \end{cases}$$

If $1/2 < \alpha < 1$, the characteristics to the right move with positive, but smaller velocity, while if $0 < \alpha < 1/2$ they move with negative velocity. Either way they collide with the characteristics moving with velocity u_m from the left. Diagrams.

Consider the characteristics from the middle, $0 < \xi < a$. If the coefficient of ξ from the first term becomes exactly -1, it cancels with the second term - the position x no longer depends on ξ, and all the characteristics have collided at a single "event" (point in space and time). We need

$$u_m t 2(1-\alpha)/a = 1$$

which occurs when

$$t = t_s = \frac{a}{2u_m(1-\alpha)}$$

At this time, we have

$$x|_{t=t_s} = a\frac{1-2(1-\alpha)\xi/a}{2(1-\alpha)} + \xi = \frac{a}{2(1-\alpha)} = x_s$$

independent of ξ. Are there any other intersections? No, because all of the "middle" characteristics have collided at (x_s, t_s), and the other characteristics run parallel to the outermost of the middle characteristics. Diagram of characteristics.

The solution is given as before by solving for ξ as a function of x and t. We have

$$\xi = \begin{cases} x - u_m t & x < u_m t \\ \frac{x-u_m t}{1-2u_m t(1-\alpha)/a} & u_m t < x < a + u_m t(2\alpha - 1) \\ x - u_m t(2\alpha - 1) & x > a + u_m t(2\alpha - 1) \end{cases}$$

then

$$\rho(x,t)/\rho_m = \rho_0(\xi(x,t))/\rho_m = \begin{cases} 0 & x < u_m t \\ \frac{x-u_m t}{a/(1-\alpha)-2u_m t} & u_m t < x < a + u_m t(2\alpha - 1) \\ 1-\alpha & x > a + u_m t(2\alpha - 1) \end{cases}$$

The velocity is

$$u/u_m = 1 - \rho/\rho_m = \begin{cases} 1 & x < u_m t \\ 1 - \frac{x-u_m t}{a/(1-\alpha)-2u_m t} & u_m t < x < a + u_m t(2\alpha - 1) \\ \alpha & x > a + u_m t(2\alpha - 1) \end{cases}$$

Note that at the time $t = t_s$ above, the denominators become zero in the middle region. Also, the middle region shrinks to zero, as seen by setting both boundaries of the region equal. This can be seen on diagrams of ρ and u - the functions become discontinuous. For $\alpha > 1/2$ both boundaries move to the right, while for $\alpha < 1/2$ the right boundary moves to the left. In particular, the intersection, which occurs at

$$x_s = \frac{a}{2(1-\alpha)}$$

is to the left of a for $\alpha < 1/2$ and to the right of a for $\alpha > 1/2$.

4.3 Shock waves

How can we understand what happens after the characteristics collide? Several options:

1. Give up - apply the equation only to smooth situations. The fact remains that shocks do exist (with large but not infinite gradients) and we should try to model them.

2. Something completely different takes over. For example, in the case of the traffic, we could say that the drivers can see some distance ahead, and so vary their speed according to ρ_x as well as ρ, which will lead to a ρ_{xx} term in the equation. If this term has a very small coefficient, the solution may be similar to the first order problem, but may "smooth" the solution in the vicinity of the shock. (Problem D5).

3. The differential equation doesn't make sense - but perhaps we can use the integral form of the conservation law to continue the solution. In this case, we are finding a "weak solution" of the original equation.

4. We expand the space of functions to include the derivative of a discontinuity, the Dirac delta "function". In this expanded function space, perhaps the original equation still makes sense.

Let us look at option 3 - using the integral equation. When we derived the differential equation

$$\rho_t + q_x = 0$$

one of the derivations used the integral form

$$\frac{dS}{dt} = \int_{x_1}^{x_2} \rho_t dx = q(x_1) - q(x_2)$$

We then wrote the RHS as an integral over q_x, said the equation holds for all x_1 and x_2, and so derived the differential form of the conservation law. Since it is not clear that we can define ρ_t, we should now write

$$\frac{d}{dt} \int_{x_1}^{x_2} \rho dx = q(x_1) - q(x_2)$$

Since characteristic lines are converging towards the shock, and ρ is constant on both sides of the shock (in our example), we assume that this remains true. In particular, the solution of the equation away from the shock demands that ρ remain constant. In general, the solution will be given by the same expression away from the shock as before the shock, since it is unique. If we write $x = X_s(t)$ for the position of the shock, we have

$$\int_{x_1}^{x_2} \rho dx = (x - x_1)\rho_- + (x_2 - x)\rho_+$$

where ρ_- and ρ_+ are the densities to the left and right respectively. Thus we have

$$X_s'(\rho_- - \rho_+) = q_- - q_+$$

since in our example $q(x_1) = q_-$ as long as $x_1 < x$, ie q does not depend on x away from the shock. In more complicated problems we would make x_1 and x_2 very close to the shock, and obtain the same result. We have $q = f(\rho)$ so,

$$X_s'(\rho_- - \rho_+) = f(\rho_-) - f(\rho_+)$$

$$u_s = X_s' = \frac{f(\rho_-) - f(\rho_+)}{\rho_- - \rho_+}$$

Note that it has dimensions of velocity. In this case (ρ and q constant on either side of the shock), the velocity does not depend on time, so we can write

$$x - x_s = u_s(t - t_s)$$

where x_s and t_s are the beginning of the shock. Returning to the example of the lane narrowing, we have

$$\begin{aligned} \rho_- &= 0 \\ \rho_+ &= (1-\alpha)\rho_m \\ q_- &= f(\rho_-) = 0 \\ q_+ &= f(\rho_+) = u_m\rho_m(1-\alpha)\alpha \\ u_s &= \frac{u_m\rho_m(1-\alpha)\alpha}{(1-\alpha)\rho_m} = u_m\alpha = u_+ \end{aligned}$$

In other words, the shock moves at the slower speed of the cars, which makes sense - the density to the left is zero, so the last car will travel with this velocity. Diagrams (both $\alpha > 1/2$ and $\alpha < 1/2$, characteristics, ρ and u). In general, u_s is the gradient of the chord joining the points ρ_- and ρ_+ on a graph of $f(\rho)$, and can be either positive or negative. Diagram, showing the chord, the velocities and the signal velocities.

Now we can write down the full solution. For $t < t_s$ the solution is given above. For $t > t_s$ we have

$$\rho/\rho_m = \begin{cases} 0 & x < u_s(t-t_s) + x_s \\ 1-\alpha & x > u_s(t-t_s) + x_s \end{cases}$$

and

$$u/u_m = \begin{cases} 1 & x < u_s(t-t_s) + x_s \\ \alpha & x > u_s(t-t_s) + x_s \end{cases}$$

We remark again that, as in ODE problems, PDEs that are perfectly well defined, and have smooth initial conditions can develop discontinuities at finite time.

4.4 The Dirac delta

We have continued the solution beyond the appearance of the shock, but at the expense of reverting from the differential form of the conservation equation to the integral form. We now try to make sense of the differential equation. At the shock, the density changes suddenly from ρ_- to ρ_+. Its space derivative appears to be infinite at the shock and zero elsewhere. We have that

$$\int_{x_1}^{x_2} \rho_x dx = \rho(x_2) - \rho(x_1)$$

so ρ_x must be a function which when integrated gives a finite value if the limits of integration include the shock and zero otherwise. We can consider a shock of unit strength at the origin, and call it $\delta(x)$, the Dirac delta "function". In other words

$$\int_{x_1}^{x_2} \delta(x)dx = \begin{cases} 1 & x_1 < 0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

It can be considered the limit of a function that gets steeper and narrower so that the total integral remains constant, for example, a step function

$$\delta_\epsilon^{(1)}(x) = \begin{cases} 0 & |x| > \epsilon/2 \\ \epsilon^{-1} & |x| < \epsilon/2 \end{cases}$$

or a Gaussian,

$$\delta_\epsilon^{(2)}(x) = \frac{e^{-x^2/(2\epsilon^2)}}{\epsilon\sqrt{2\pi}}$$

in the same way that a step function can be considered as a limit of smooth functions. The difference is that only integrals of the delta function make sense, not the function on its own. It is thus called a "generalised function" or a "distribution". The integrals themselves would be defined rigorously using limits like

$$\int_{x_1}^{x_2} f(x)\delta(x)dx = \lim_{\epsilon\to 0}\int_{x_1}^{x_2} f(x)\delta_\epsilon(x)dx$$

and then it is found that a variety of approximation functions $\delta_\epsilon(x)$ give the same answer, which is $f(0)$ if $x_1 < 0 < x_2$. In any case, we will omit the proofs - it turns out that standard manipulations give the right answer.

Other examples include

$$\int f(x)\delta(x-a)dx = \int f(y+a)\delta(d) = f(a)$$

where we have substituted $y = x - a$.

$$\int f(x)\delta(2x)dx = \int f(y/2)\delta(y)dy/2 = f(0)/2$$

where $y = 2x$. We can write this directly as $\delta(2x) = \delta(x)/2$. This makes sense: we have $\delta(2x)$ as a step function with half the width, thus it gives an area of $1/2$. In general, a change of variable gives

$$\int \delta(g(x))dx = \int \delta(y)|\frac{dx}{dy}|dy$$

where $y = g(x)$. Now $dy/dx = g'(x)$, so $dx/dy = 1/g'(x)$. We get a contribution from every point where $g(x) = 0$, so the result is

$$\int_{-\infty}^{\infty} \delta(g(x)) = \sum_{x:g(x)=0} \frac{1}{|g'(x)|}$$

For $g(x) = x^2 - 1$ we have $x = \pm 1$ and $g'(x) = 2x = \pm 2$, so the integral comes to 1. For $g(x) = x^2$, $x = 0$ and $g'(x) = 0$ so we get ∞.

Using integration by parts, we can also deal with derivatives of delta functions:

$$\int f(x)\delta'(x)dx = 0 - \int f'(x)\delta(x)dx = -f'(0)$$

$$\int f(x)\delta''(x)dx = f''(0)$$

What is the dimension of a delta function? We have $\int \delta(x)dx = 1$ so $\delta(x)$ must have the same dimensions as $1/x$. This applies to any dimensional quantity x (which could be time, etc.)

Returning to the problem of shocks in traffic modelling, we conclude that

$$q_x = (q_+ - q_-)\delta(x - X_s(t))$$

at the shock. Also

$$\rho_t = (\rho_- - \rho_+)\delta(t - T_s(x))$$

assuming a positive velocity of the shock. To linear approximation (eg when the shock has constant velocity)

$$X_s(t) = x_s + (t - t_s)X_s'$$

$$T_s(x) = t_s + (x - x_s)/X_s'$$

Thus

$$\rho_t + q_x = (\rho_- - \rho_+)\delta(t - t_s - (x - x_s)/X_s') + (q_+ - q_-)\delta(x - x_s - (t - t_s)X_s')$$

But we can multiply the argument of the first delta function by the constant X_s' and divide the coefficient, giving

$$\rho_t + q_x = [\frac{(\rho_- - \rho_+)}{X_s'} + (q_+ - q_-)]\delta(x - x_s - (t - t_s)X_s')$$

This will be zero, thus satisfying the equation, if the coefficient is zero, that is,

$$X_s' = \frac{q_- - q_+}{\rho_- - \rho_+}$$

the same equation we derived using the integral approach.

Dirac delta functions have many other applications, for example, if we want to describe the mass density of a point mass in three dimensions, we would write

$$\rho(\mathbf{x}) = m\delta(\mathbf{x} - \mathbf{x}_0) = m\delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

so that the integral of the mass density is m, the mass, and all of the density is concentrated at the point (x_0, y_0, z_0).

4.5 Characteristics in general first order quasilinear equations

The traffic flow problem is a special case of the equation

$$au_x + bu_t = c$$

where a, b and c are all functions of x, t and u. (If a and b do not contain u, and c is linear in u, the equation is linear, not just quasilinear). In traffic flow $b = 1$, $c = 0$ and $a = a(u)$. Again, we consider $u = u(x, t)$ as a surface in three dimensions. We can express this as $F(x, t, u) = u(x, t) - u = 0$. For example, the hemisphere $x^2 + t^2 + u^2 = R^2$ has $u(x, t) = \sqrt{R^2 - x^2 - t^2}$ and $F(x, t, u) = \sqrt{R^2 - x^2 - t^2} - u$.

Now consider the gradient vector $\nabla F = (F_x, F_t, F_u)$: this is in the direction normal to the surface $F = 0$. Since $F(x, t, u) = u(x, t) - u$, $\nabla F = (u_x, u_t, -1)$. In the hemisphere example,

$$\nabla F = (\frac{-x}{\sqrt{R^2 - x^2 - t^2}}, \frac{-t}{\sqrt{R^2 - x^2 - t^2}}, -1)$$

explicitly from $F(x, t, u)$, and also using $u(x, t)$. This is normal to any vector tangent to the surface. For example, at $x = t = 0$ we have $u = a$, and any vector in the $x - t$ direction, ie of the form $(p, q, 0)$ is normal to ∇F. If we have $x \approx R$ but $t = 0$, the x component of ∇F gets very large, so any vector $(0, q, r)$ is (nearly) normal to it.

Now we write the PDE as $au_x + bu_t - c = (a, b, c) \cdot (u_x, u_t, -1) = 0$, so that (a, b, c) at every point (x, t, u) is normal to ∇F, and hence tangent to the surface $u = u(x, t)$ whenever $\nabla F \neq 0$. The vector (a, b, c) thus determines a characteristic direction that is defined for all (x, t, u), and that lies in the tangent plane to the surface $u = u(x, t)$.

If we have a curve parametrised by s, ie $x = x(s)$, $t = t(s)$, and $u = u(s)$, the tangent vector of the curve is $(dx/ds, dt/ds, du/ds)$. In the hemisphere example we could have $x = R \sin s$, $t = 0$ and $u = R \cos s$ with $-\pi/2 < s < \pi/2$. Then the tangent vector is $(R \cos s, 0, -R \sin s)$ which is clearly orthogonal to the outward pointing vector $(R \sin s, 0, R \cos s)$.

The characteristic curves will thus be defined by $a = dx/ds$, $b = dt/ds$, $c = du/ds$. Note that, in contrast to traffic flow, the characteristic curves no longer have constant u when c is not zero. However, in the case of traffic flow, we have $dx/dt = a/b = f'(\rho)$ along the characteristic curves, ie they reduce to the curves we discussed previously. As long as a, b and c are sufficiently smooth and not all zero, it can be shown that there is a unique characteristic curve passing through each point (x_0, t_0, u_0).

The initial value problem for the PDE consists of a given function $u(x, t)$ along a given curve C in $x - t$ space. The curve C is called the initial curve. Starting on C we pass a characteristic curve through each point on C. The surface generated is our solution $F(x, t, u) = 0$.

Theorem: Let a, b, c have continuous partial derivatives in x, t, u. Suppose C is given by $x = x(\tau)$, $t = t(\tau)$, $u = u(\tau)$ has a continuous tangent vector and

$$\Delta(\tau) = \frac{dt}{d\tau} a[x(\tau), t(\tau), u(\tau)] - \frac{dx}{d\tau} b[x(\tau), t(\tau), u(\tau)] \neq 0$$

Then there exists a unique solution $u(x,t)$ defined in some neighbourhood of C satisfying the equation and the initial data.

The proof is constructive - we will see how to solve the equations. The proof is not examinable, but you need to know the method of solution. The procedure for solving quasilinear equations is as follows:

1. Parametrize the initial curve C using a parameter τ: $x = x(\tau)$, $t = t(\tau)$, $u = u(\tau)$. For example, choose $\tau = x$.

2. Write down the characteristic equations

$$\frac{dx}{ds} = a(x, t, u)$$

$$\frac{dt}{ds} = b(x, t, u)$$

$$\frac{du}{ds} = c(x, t, u)$$

   and try and solve for $x = X(s,\tau)$, $t = T(s,\tau)$, $u = U(s,\tau)$ with the initial conditions $X(0,\tau) = x(\tau)$, $T(0,\tau) = t(\tau)$, $U(0,\tau) = u(\tau)$. From the theory of ODEs, the solution is unique under the stated conditions (ie that a, b and c have continuous derivatives).

3. Invert $x = X(s,\tau)$, $t = T(s,\tau)$ to get s and τ as functions of x and t. This is only possible if the Jacobian is nonzero, ie if

$$\Delta(s,\tau) = \begin{vmatrix} \frac{\partial X}{\partial s} & \frac{\partial X}{\partial \tau} \\ \frac{\partial T}{\partial s} & \frac{\partial T}{\partial \tau} \end{vmatrix}$$

   is nonzero at $s = 0$, then this inversion can be accomplished in a neighbourhood of $s = 0$. At $s = 0$, $\Delta(s,\tau) = a dt/d\tau - b dx/d\tau$, thus it is nonzero by the stated condition. Of course, this makes no guarantee about how far the solution can be extended.

4. Substitute these values into $u = U(s,\tau) = U(s(x,t), \tau(x,t)) = u(x,t)$, the solution

Example 1: $u_x - u_t = 1$ with $u = x^2$ on $x = t$. Parametrise initial curve C, eg $x = \tau$, $t = \tau$, $u = \tau^2$. Char. eqs are $dx/ds = 1$, $dt/ds = -1$, $du/ds = 1$. Thus we have $x = X(s,\tau) = \tau + s$, $t = T(s,\tau) = \tau - s$, $u = U(s,\tau) = \tau^2 + s$. We also check $\Delta = a dt/d\tau - b dx/d\tau = (1)(1) - (-1)(1) = 2 \neq 0$. Solve for s, τ in terms of x,t: $\tau = (x+t)/2$, $s = (x-t)/2$. Diagram of characteristics. So $u(x,t) = U(s(x,t),\tau(x,t)) = (x+t)^2/4 + (x-t)/2$. We check: $u_x = (x+t)/2 + 1/2$, $u_t = (x+t)/2 - 1/2$, $u_x - u_t = 1$ as required, also $u(x,x) = x^2$ satisfies the initial data.

Example 2: $xu_x + tu_t = u^2$ with $u = x^2$ on $x+t=1$. Initial curve C: $x=\tau$, $t = 1-\tau$ $u = \tau^2$. Char. eqs: $dx/ds = x$, $dt/ds = t$, $du/ds = u^2$. Thus we have $x = \tau e^s$, $t = (1-\tau)e^s$,

$$ds = du/u^2$$

$$s = -1/u + C$$

$$u = -1/(s-C)$$

$$u = \tau^2/(1-s\tau^2)$$

Characteristic lines radiate out from the origin (diagram). To invert we check $\Delta = adt/d\tau - bdx/d\tau = -x - t = -1 \neq 0$, then write

$$x + t = e^s$$

$$s = \ln(x+t)$$

$$\tau = xe^{-s} = \frac{x}{x+t}$$

So finally $u = \frac{1}{\tau^{-2}-s} = \frac{1}{\frac{(x+t)^2}{x^2} - \ln(x+t)} = \frac{x^2}{(x+t)^2 - x^2\ln(x+t)}$. Check: on $x+t=1$, $u = x^2$. Write $D = (x+t)^2 - x^2\ln(x+t)$ as the denominator. Then

$$u_x = \frac{2x(x+t)^2 - 2x^3\ln(x+t) - 2x^2(x+t) + 2x^3\ln(x+t) + x^4/(x+t)}{D^2}$$

$$u_x = \frac{2(x+t)(x^2+tx-x^2) + x^4/(x+t)}{D^2}$$

$$u_x = \frac{2xt(x+t) + x^4/(x+t)}{D^2}$$

$$u_t = \frac{-2x^2(x+t) + x^4/(x+t)}{D^2}$$

$$xu_x + tu_t = \frac{x^4}{D^2} = u^2$$

Example 3: $u_t + cu_x = 0$, $u(x,0) = f(x)$. Initial curve is $x = \tau$, $t = 0$, $u = f(\tau)$. $\Delta = adt/d\tau - bdx/d\tau = c(0) - 1(1) = -1 \neq 0$. Char. eqns: $dx/ds = c$, $dt/ds = 1$, $du/ds = 0$. Thus $x = cs + \tau$, $t = s$, $u = f(\tau)$. We have $\tau = x - cs = x - ct$, so $u = f(x-ct)$ as before. Diagram of characteristics.

Example 4: $u_t + uu_x = 0$, $u(x,0) = f(x)$. Initial curve as in 3, above. $\Delta = adt/d\tau - bdx/d\tau = 0 - 1(1) = -1 \neq 0$. Char. eqns: $dx/ds = u$, $dt/ds = 1$, $du/ds = 0$, thus $u = f(\tau)$ (constant), $x = f(\tau)s + \tau$, $t = s$. Thus $\tau = x - sf(\tau) =$

$x - tf(\tau)$, $u = f(x - tu)$ must be solved implicitly. This means that the speed of the characteristics (ie the wave) is given by the height of the wave, thus the wave will eventually break if $du/dx < 0$ at any point. Diagram of a stretched pulse. Actually we can calculate the time to breaking: $u_x = f'(x - tu)(1 - tu_x)$ thus

$$u_x = \frac{f'(x - tu)}{1 + tf'(x - tu)}$$

which becomes infinite when $t = -1/f'(x - tu)$, so breaking occurs for the smallest such value.

5 Waves

5.1 definitions

We have seen that the equation

$$u_t + cu_x = 0$$

with c constant has the solution

$$u = f(x - ct)$$

for any function f, which gives a wave moving with speed c in the positive x direction. Diagram of propagating wave. Similarly,

$$u_t - cu_x = 0$$

has solution

$$u = f(x + ct)$$

which is a wave moving in the negative x direction.

One such solution is a sinusoidal wave $u = \sin k(x-ct)$ satisfying $u_t + cu_x = 0$ (check). Diagram of $u(x, 0)$. We can also write this as $u = \sin(kx - \omega t)$ where $\omega = kc$. We know that sin is a periodic function, so $\sin(\phi + 2\pi) = \sin\phi$. Thus the wave solution is periodic in space at fixed time t, ie $u(x + 2\pi/k, t) = \sin(k(x + 2\pi/k) + \omega t) = \sin(kx + \omega t + 2\pi) = \sin(kx + \omega t) = u(x, t)$. We write $\lambda = 2\pi/k$ and call it the *wavelength* (mark on diagram). We call $k = 2\pi/\lambda$ the *wave number*, because $k/2\pi$ is the number of waves per unit length.

Now at fixed position x we also have $u(x, t+2\pi/\omega) = \sin(kx+\omega(t+2\pi/\omega)) = \sin(kx + \omega t + 2\pi) = \sin(kx + \omega t) = u(x, t)$. We write $T = 2\pi/\omega$ and call it the *period* of the wave, the time for one complete wave to pass the fixed point x. We define $f = 1/T = \omega/2\pi$ as the *frequency*, the number of waves passing per unit time, with unit Hertz (Hz) $= s^{-1}$. We call $\omega = 2\pi f = 2\pi/T$ the *radian frequency* or *angular frequency*. We call c the phase velocity, equal to $c = \omega/k = f\lambda$.

If we have $u = A\sin(kx - \omega t)$, A is the *amplitude* and $2A$ is the *height*, from the crest to the trough.

Notice that the equations $u_t \pm cu_x = 0$ are linear and homogeneous. Thus the sum of two solutions is a solution. We could have

$$u = A\sin(kx - \omega t) + B\cos(kx - \omega t)$$

as a solution of $u_t + cu_x = 0$.

Phase: Let

$$u_1(x,t) = A\cos(kx - \omega t)$$

$$u_2(x,t) = A\cos(kx - \omega t + \epsilon) = A\cos(k(x + \epsilon/k) - \omega t)$$

In both cases the amplitude is A and the speed is $c = \omega/k$, but u_2 is shifted by a distance $-\epsilon/k$ in the x-direction relative to u_1. We call ϵ the *phase* of the wave. For example if $\epsilon = 2\pi, 4\pi, \ldots$ the waves are exactly in phase, while if $\epsilon = \pi, 3\pi, \ldots$ they are exactly out of phase and the sum cancels to zero.

5.2 Complex notation

Consider $u(x,t) = a\cos(kx - \omega t + \epsilon)$, a wave travelling in the positive x direction with phase ϵ and amplitude a. We can write this as

$$u(x,t) = \mathrm{Re}(ae^{i\epsilon}e^{i(kx-\omega t)})$$

using the relation $e^{a+b} = e^a e^b$ and $e^{i\theta} = \cos\theta + i\sin\theta$. The wave can thus be written

$$u(x,t) = Ae^{i(kx-\omega t)}$$

$$A = ae^{i\epsilon}$$

is complex; the real part is understood to be taken at the end of the calculation. Note that $|A| = |a|$ is still the amplitude and $\arg A = \epsilon$ is the phase. In the negative x direction we have

$$u(x,t) = b\cos(kx + \omega t + \delta) = \mathrm{Re}(be^{i\delta})e^{i(kx+\omega t)}$$

$$u(x,t) = Be^{i(kx+\omega t)}$$

with the real part understood. Note that we now have $e^{+i\omega t}$ rather than $e^{-i\omega t}$. We can keep both as $e^{-i\omega t}$ by writing

$$u = B'e^{-i(kx+\omega t)}$$

$$B' = be^{-i\delta}$$

and taking the real part. In general we have written a sinusoidal wave

$$u(x,t) = a\cos(kx \mp \omega t + \epsilon)$$

in complex form as

$$u = Ae^{i(\pm kx - \omega t)}$$

with

$$A = ae^{\pm i\epsilon}$$

and $|A| = a$, $\arg A = \pm\epsilon$, and a real part implied at the end of the calculation.

For example,

$$u(x,t) = a\cos(kx - \omega t) + b\sin(kx - \omega t)$$
$$u(x,t) = \mathrm{Re}[ae^{i(kx-\omega t)} + b(-i)e^{i(kx-\omega t)}]$$
$$u(x,t) = \mathrm{Re}[(a - bi)e^{i(kx-\omega t)}]$$

This is a single sinusoidal wave with amplitude $\rho = |a - bi| = \sqrt{a^2 + b^2}$ and phase $\delta = \arg(a - bi) = \tan^{-1}(b/a)$, ie

$$u(x,t) = \rho\cos(kx - \omega t + \delta)$$

we could also obtain this by assuming the above form, expanding out the cosine of a sum, and solving for ρ and δ in terms of a and b. The wave has the same period and wavelength, but a different amplitude and phase to its constituents. The complex form has the advantage that it is easier to do this addition calculation to find the amplitude and phase of the sum of two or more constituents. We can also take partial derivatives:

$$u(x,t) = Ae^{i(kx-\omega t)}$$
$$\frac{\partial}{\partial x}u(x,t) = ikAe^{i(kx-\omega t)}$$
$$\frac{\partial}{\partial t}u(x,t) = -i\omega Ae^{i(kx-\omega t)}$$

This makes sense: the derivative of a cosine is a sine and vice versa; this is taken care of by the factor of i, which singles out the sine or cosine when the real part is taken. For example

$$u(x,t) = a\cos(kx - \omega t) = \mathrm{Re}\, ae^{i(kx-\omega t)}$$
$$u_x(x,t) = -ak\sin(kx - \omega t) = \mathrm{Re}\, aike^{i(kx-\omega t)}$$

since $aike^{i(kx-\omega t)} = aik(\cos(kx - \omega t) + i\sin(kx - \omega t)) = aik\cos(kx - \omega t) - ak\sin(kx - \omega t)$. These calculations work because

$$\mathrm{Re}(Z_1 + Z_2) = \mathrm{Re}(Z_1) + \mathrm{Re}(Z_2)$$
$$\mathrm{Re}(Z') = \mathrm{Re}(Z)'$$

however we cannot multiply two such waves:

$$\mathrm{Re}(Z_1 Z_2) \neq \mathrm{Re}(Z_1)\mathrm{Re}(Z_2)$$

in general (for example: $-1 = \mathrm{Re}(ii) \neq \mathrm{Re}(i)\mathrm{Re}(i) = 0$).

5.3 Waves on strings

We want to model the wave motion of a stretched string (guitar string, skipping rope, cables supporting Clifton suspension bridge, ...). Assumptions:

1. $y = 0$ in equilibrium
2. Transverse motion only (ie in y direction)
3. Large tension T, so the string is straight in equilibrium.
4. Uniform properties eg, mass per unit length ρ, constant small thickness.
5. No forces except tension (ignore gravity, air resistance, ...)
6. Position of string defined by $y = u(x,t)$
7. Slope of string $\partial u/\partial x$ small

Aim: to find simplest theory which is close to observed behaviour; we can add the other effects later as perturbations.

Consider a small piece of string (see diagram) δs of mass $\rho\delta s$. The force in the y direction is $T_0 \sin(\psi + \delta\psi) - T_0 \sin\psi = T_0 \frac{\partial}{\partial x} \sin\psi \delta x + O(\delta x^2)$ from a Taylor expansion of ψ as a power series in x. The slope of the string at x is $u_x = \tan\psi$ thus $\sin\psi = u_x/\sqrt{1+u_x^2} \approx u_x$ since $u_x \ll 1$ by assumption. So the net force is $T_0 u_{xx}\delta x$ which is mass times acceleration, $\rho\delta s u_{tt}$. But $\delta s = \sqrt{\delta x^2 + \delta u^2} = \sqrt{1+u_x^2}\delta x \approx \delta x$ (diagram). So we have $u_{xx}\delta x = \rho u_t t\delta x$, or

$$u_{tt} = c^2 u_{xx}$$

with $c^2 = T_0/\rho$. This second order equation is called the 1D wave equation. Like the first order equations we considered, it is also linear and homogeneous, so the sum of two solutions is a solution.

5.4 Solutions of the wave equation

We can factorise the differential operator as follows:

$$(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2})u = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})u$$

This works because the partial derivatives commute, ie we can take them in either order. In this form, it is clear that any solution of $u_t - cu_x = 0$ will be a solution of the wave equation, that is, any function $u(x,t) = g(x+ct)$ corresponding to a wave moving to the left.

On the other hand, we can write the factors in the opposite order,

$$(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2})u = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})u$$

showing that any solution of $u_t + cu_x = 0$ is also a solution of the wave equation, that is $u(x,t) = f(x-ct)$ corresponding to a wave moving to the right.

From linearity we can write a solution as

$$u(x,t) = f(x-ct) + g(x+ct)$$

This is in fact the general solution because it requires two arbitrary functions and the equation is linear and second order. This is called D'Alembert's solution. It is a superposition (sum) of two waves moving in opposite directions with speed $c = \sqrt{T_0/\rho}$.

An alternative method of finding this solution and showing its generality: transform to new variables $\xi = x - ct$ and $\eta = x + ct$. We write $u(x,t) = U(\xi, \eta)$ Then $u_x = U_\xi \xi_x + U_\eta \eta_x = U_\xi + U_\eta$ and $u_t = U_\xi \xi_t + U_\eta \eta_t = -cU_\xi + cU_\eta$. We can now write down the solutions of the first order equations: $0 = u_t + cu_x = 2cU_\eta$ so U does not depend on η, but may be any function of $\xi = x - ct$. Similarly $0 = u_t - cu_x = -2cU_\xi$ so U does not depend on ξ, but may be any function of $\eta = x + ct$.

Taking a second derivative, we get

$$u_{xx} = (\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})(U_\xi + U_\eta) = U_{\xi\xi} + 2U_{\xi\eta} + U_{\eta\eta}$$

similarly

$$u_{tt} = c^2(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta})(-U_\xi + U_\eta) = c^2(U_{\xi\xi} - 2U_{\xi\eta} + U_{\eta\eta}$$

thus

$$u_{tt} - c^2 u_{xx} = -4c^2 U_{\xi\eta}$$

so

$$U_{\xi\eta} = 0$$

This means that U_ξ does not contain η, call it an arbitrary function $f'(\xi)$. Integrating, we find that $U = f(\xi) + g(\eta)$ where g is another arbitrary function. It is clear using this method that no other solutions are possible.

5.5 Waves on an infinite stretched string

Case 1: Suppose that a string is initially at rest $u_t(x,0) = 0$, with transverse displacement $u(x,0) = \phi(x)$ for some function ϕ (Gaussian type graph). What happens subsequently? D'Alembert's solution is

$$u(x,t) = f(x-ct) + g(x+ct)$$

so substituting these conditions we find

$$u_t = -cf'(x-ct) + cg'(x-ct) = 0$$

$$u_t|_{t=0} = -cf'(x) + cg'(x) = 0$$
$$u|_{t=0} = f(x) + g(x) = \phi(x)$$

Integrating we find
$$-cf(x) + cg(x) = A$$
where A is a constant. Thus
$$f(x) - g(x) = -A/c$$
$$f(x) + g(x) = \phi(x)$$
and we find
$$f(x) = (\phi(x) - A/c)/2$$
$$g(x) = (\phi(x) + A/c)/2$$
thus
$$u(x,t) = f(x-ct)+g(x+ct) = (\phi(x-ct)-A/c)/2+(\phi(x+ct)+A/c)/2 = (\phi(x-ct)+\phi(x+ct))/2$$
so the constant A is irrelevant. We could also have removed it by adding it to f and subtracting it from g, noting that there is ambiguity of an additive constant in the definition of these functions. Thus the initial hump splits into two pieces, one travelling in each direction (diagram).

For example, suppose
$$\phi(x) = \begin{cases} 0 & |x| > \pi/2 \\ A\cos^2 x & |x| < \pi/2 \end{cases}$$
Find displacement at times $t = \pi/4c, \pi/2c, \pi/c$. We have
$$u(x,t) = (\phi(x-ct) + \phi(x+ct))/2$$
At time $t = \pi/4c$ the two waves are a distance $\pi/2$ apart, so they add to a constant $A/2$ for $|x| < \pi/4$ with tails on either side (diagram). At time $t = \pi/2c$ the two waves just separated, and at time $t = \pi/c$ there is a gap of π between them (diagrams).

In general, a wave of initial size L will completely separate when the right edge of the left-moving wave touches the left edge of the right moving wave, ie after a time $t = L/2c$ (here $L = \pi$).

Case 2: Waves with zero initial displacement and given initial velocity. That is $u(x,0) = 0$ and $u_t(x,0) = \psi(x)$. We use the D'Alambert solution,
$$u = f(x-ct) + g(x+ct)$$
we find
$$u|_{t=0} = f(x) + g(x) = 0$$

so

$$g(x) = -f(x)$$

we also have

$$u_t|_{t=0} = -cf'(x) + cg'(x) = \psi(x)$$

Integrate:

$$f(x) - g(x) = 2f(x) = -\frac{1}{c}\int_a^x \psi(s)ds$$

for some constant a. Thus

$$f(x) = -g(x) = -\frac{1}{2c}\int_a^x \psi(s)ds$$

substituting back we have

$$u(x,t) = -\frac{1}{2c}\int_a^{x-ct} \psi(s)ds + \frac{1}{2c}\int_a^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

where again the unknown constant a drops out. We check the solution:

$$u(x,0) = \int_x^x \psi(s)ds = 0$$

$$u_t(x,0) = \frac{1}{2c}(c\psi(x+ct) + c\psi(x-ct)) = \psi(x)$$

We have used the derivative of an integral with respect to its bound:

$$f(t) = \int_a^{g(t)} \psi(s)ds$$

$$f'(t) = \frac{df}{dg}\frac{dg}{dt} = \psi(g(t))g'(t)$$

In this case it is the velocity that separates into two halves which propagate in each direction. We can now solve the more general case by superposition:

$$u(x,0) = \phi(x)$$

$$u_t(x,0) = \psi(x)$$

has solution

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

by superposition (linear and homogeneous equation).

Example: Suppose we have $\phi(x) = 0$ and $\psi(x) = f'(x)$. Then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} f'(s)ds = \frac{1}{2c}[f(x+ct) - f(x-ct)]$$

This corresponds to two waves which are equal and opposite, moving in each direction. If

$$f(x) = \begin{cases} 0 & |x| > \pi/2 \\ A\cos^2 x & |x| < \pi/2 \end{cases}$$

we can find the displacement for times $\pi/4c, \pi/2c, \pi/c$ as before:

$$u(x, \pi/4c) = \begin{cases} 0 & |x| > 3\pi/4 \\ A\cos^2(x+\pi/4)/2c & -3\pi/4 < x < -\pi/4 \\ A\cos(2(x+\pi/4))/2c & -\pi/4 < x < \pi/4 \\ -A\cos^2(x-\pi/4)/2c & \pi/4 < x < 3\pi/4 \end{cases}$$

$$u(x, \pi/2c) = \begin{cases} 0 & |x| > \pi \\ A\sin^2 x/2c & -\pi < x < 0 \\ -A\sin^2 x/2c & 0 < x < \pi \end{cases}$$

Graphs, also for $t = \pi/c$.

Example: What initial conditions produce a positive wave only? In the solution, the parts that give $x + ct$ must be constant, ie

$$\phi(\eta)/2 + \frac{1}{2c}\int^{\eta} \psi(s)ds = const.$$

where $\eta = x + ct$. Differentiating, we find

$$\phi'(\eta)/2 + \frac{1}{2c}\psi(\eta) = 0$$

thus

$$\psi(\eta) = -c\phi'(\eta)$$

This is clearly necessary. Also sufficient since subsituting back into the equation we find

$$u(x,t) = \frac{1}{2}[\phi(x-ct) + \phi(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} (-c\phi'(s))ds = \phi(x-ct)$$

5.6 Semi-infinite problems

Case 1: reflection at a fixed end. Suppose a string in $x < 0$ is fixed at $x = 0$, ie $u(0,t) = 0$ for all t. We know the general solution is

$$u(x,t) = f(x-ct) + g(x+ct)$$

thus

$$0 = u(0,t) = f(-ct) + g(ct)$$

If we write $\xi = ct$ we have $g(\xi) = -f(-\xi)$, thus

$$u(x,t) = f(x-ct) - f(ct-x)$$

Now the initial conditions are

$$u(x,0) = f(x) - f(-x)$$

$$u_t(x,0) = -c[f'(x) + f'(-x)$$

so for example take

$$f(x) = \begin{cases} A\cos^2 x & |x-\pi| < \pi/2 \\ 0 & x > 0 \end{cases}$$

Diagrams of an initial hump moving to the right, and being inverted (ie the negative part reaches the left half plane).

Another example: $f(x) = A\cos kx$. We have

$$u(x,t) = f(x-ct) - f(-(x+ct)) = A\cos(kx-\omega t) - A\cos(kx+\omega t)$$

where $\omega = kc$ as before and we have used the even property of the cosine function. We can expand the cosines to give

$$u(x,t) = A[\cos kx\cos\omega t + \sin kx\sin\omega t] - A[\cos kx\cos\omega t - \sin kx\sin\omega t]] = 2A\sin kx\sin\omega t$$

This is a standing wave (diagram); a similar result occurs if $f(x) = A\sin kx$.

We can also do this calculation using the complex representation:

$$f(x) = Ae^{ikx}$$

$$u(x,t) = Ae^{i(kx-\omega t)} - Ae^{-i(kx+\omega t)} = A(e^{ikx} - e^{-ikx})e^{-i\omega t} = 2iA\sin kxe^{-i\omega t}$$

from which the real part is the same as before.

Reflection at a boundary:

Suppose $u(x,t)$ satisfies $u_{tt} = c^2 u_{xx}$ for $t > 0$, $x < 0$ with $u(x,0) = a(x)$ and $u_t(x,0) = b(x)$ with $x < 0$. Extend a and b to the whole real axis and let $\nu(x,t)$ be the solution, ie

$$a(x) = b(x) = 0 \qquad x > 0$$

then

$$\nu(x,t) = f(x-ct) + g(x+ct)$$

with

$$f(\xi) = a(\xi)/2 + \frac{1}{2c}\int_\xi^0 b(s)ds$$

and

$$g(\xi) = a(\xi)/2 - \frac{1}{2c}\int_{\xi}^{0} b(s)dx$$

for $\xi < 0$, otherwise both these functions are zero.

Now we apply boundary conditions at $x = 0$. Let our solution be

$$u(x,t) = \nu(x,t) + F(x-ct) + G(x+ct)$$

Then the initial conditions give

$$a(x) = u(x,0) = a(x) + F(x) + G(x)$$

$$b(x) = u_t(x,0) = b(x) - cF'(x) + cG'(x)$$

These give

$$F(x) + G(x) = 0$$

$$F'(x) - G'(x) = 0$$

for $x < 0$. Thus

$$F'(x) = G'(x) = 0$$

for $x < 0$, and F and $G = -F$ are constants for $x < 0$. This constant is arbitrary since the initial definition of F and G allows an arbitrary constant; thus set $F = 0$ for its argument less than zero. We have

$$F(x-ct) = 0$$

for $x - ct < 0$, or $x < ct$. But this includes the whole domain $x < 0$. Thus we have

$$u(x,t) = f(x-ct) + g(x+ct) + G(x+ct)$$

where $G(x + ct) = 0$ when $x < -ct$. This result can be interpreted as an incoming and outgoing initial waves f and g together with a reflected wave G (show x, t plane with validity).

Now at $x = 0$ we have

$$u(0,t) = f(-ct) + G(ct)$$

since $g(ct) = 0$ since $ct > 0$ (see above for definition of g). In general we can ignore g; the interest is in the relationship between f and G. Note that f is defined for negative values of the argument, and G is defined for positive values of the argument.

For example, we considered a fixed end, and have already found that

$$G(\xi) = -f(-\xi)$$

leading to inversion upon reflection of the wave (diagram). A free end has $0 = u_t(0,t) = -cf'(-ct) + cG'(ct)$ so we find

$$G(\xi) = f(-\xi)$$

A damped end has a combination of space and time derivatives zero:

$$Ru_t(0,t) + Tu_x(0,t) = 0$$

for some constants R and T (think reflection and transmission) leading to

$$G(\xi) = fracT - cRT + cRf(\xi)$$

Case 2: Suppose at $x = 0$ we have a ring of mass M which can slide up and down the y-axis without friction. The condition at $x = 0$ is the equation of motion of M:

$$Mu_{tt}(0,t) = -T_0 \sin\psi \approx -T_0 u_x(0,t)$$

$$Mu_t t + T_0 u_x = 0$$

at $x = 0$ for all t. Suppose there is a wave incident from $x = -\infty$

$$Ae^{i(kx-\omega t)}$$

(with $\omega = kc$ and real part assumed). We expect a reflected wave of the form

$$Be^{-i(kx+\omega t)}$$

that is, with the same time dependence (otherwise we will not satisfy the boundary condition) moving in the opposite direction. Recall that the complex numbers A and B contain information about both amplitude and phase. Because the string is no longer fixed, we do not expect simply $B = -A$ as we had before. So we have

$$u(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)} = (Ae^{ikx} + Be^{-ikx})e^{-i\omega t}$$

and hence

$$Mu_t t + T_0 u_x = (-M\omega^2)(Ae^{ikx} + Be^{-ikx})e^{-i\omega t} + ikT_0(Ae^{ikx} - Be^{-ikx})e^{-i\omega t}$$

at $x = 0$ this is

$$0 = (-M\omega^2(A+B) + ikT_0(A-B))e^{-i\omega t}$$

or

$$B(M\omega^2 + ikT_0) = A(-M\omega^2 + ikT_0)$$

$$\frac{B}{A} = \frac{ikT_0 - M\omega^2}{ikT_0 + M\omega^2} = -\frac{1-i\alpha}{1+i\alpha}$$

where $\alpha = kT_0/(M\omega^2) = T_0/(Mkc^2) = \rho/Mk = \rho\lambda/(2\pi M)$ is the dimensionless ratio relating the mass M and the mass of one wavelength $\rho\lambda$.

We have $|A| = |B|$ since

$$\left|\frac{1-i\alpha}{1+i\alpha}\right| + 1$$

so there is no change in amplitude. The change in phase is

$$B/A = e^{i\delta}$$

where

$$\delta = \arg\frac{1+i\alpha^{-1}}{1-i\alpha^{-1}} = \arg(1+i\alpha^{-1}) - \arg(1-i\alpha^{-1}) = 2\tan^{-1}\alpha^{-1}$$

So for example if A is real, we have

$$u(x,t) = ReA(e^{i(kx-\omega t)} + e^{-i(kx+\omega t)+i\delta})$$

$$u(x,t) = A\cos(kx-\omega t) + A\cos(kx+\omega t-\delta)$$

We check the limits: $M \to \infty$ leads to $\alpha = 0$, and $\delta = \pi$ which is what we expect - for a fixed end, there is a change of sign, equivalent to a change of phase of π. $m \to 0$ leads to $\alpha = \infty$ and $\delta = 0$ corresponding to the free end.

Case 3: String on the other side of the ring. Now the boundary condition is

$$Mu_{tt}(0,t) = T_0 u_x(0^+,t) - T_0 u_x(0^-,t)$$

Why is the slope discontinuous at $x = 0$? Otherwise there would be no net force to produce the acceleration of the mass. Note that we can combine this condition and the wave equation valid elsewhere as

$$u_{xx} = \frac{\rho + M\delta(x)}{T_0} u_{tt}$$

that is, the point mass has a density of $M\delta(x)$ in that it is concentrated at the origin and has total mass M.

The displacement itself is continuous,

$$u(0^+,t) = u(0^-,t)$$

In the region to the right, we expect only a wave travelling away from the origin:

$$u(x,t) = \begin{cases} Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)} & x < 0 \\ Ce^{i(kx-\omega t)} & x > 0 \end{cases}$$

Real parts are assumed, with the same time dependence.

Continuity of u gives

$$A + B = C$$

while the derivative condition gives

$$-M\omega^2 C = ikT_0(C - A + B)$$

$$A - B = C(1 - i\alpha^{-1})$$

(recall $\alpha = T_0/kc^2M$). Adding we find

$$2A = C(2 - i\alpha^{-1})$$

$$C/A = \frac{2}{2 - i\alpha^{-1}}$$

The modulus squared $|C/A|$ is the *transmission coefficient*. Subtracting we find

$$2B = Ci\alpha^{-1}$$

so

$$B/A = B/CC/A = \frac{i\alpha^{-1}}{2 - i\alpha^{-1}}$$

The modulus squared $|B/A|^2$ is the *reflection coefficient*.

Taking the limits: $M = \infty$ and $\alpha^{-1} = \infty$ we find $B = -A$ and $C = 0$, as expected. In the limit $M = 0$ and $\alpha^{-1} = 0$ we find $B = 0$ and $C = A$ as expected. We also have

$$|B^2/A^2| = \frac{\alpha^{-2}}{4 + \alpha^{-2}}$$

$$|C^2/A^2| = \frac{4}{4 + \alpha^{-2}}$$

These add to one, perhaps indicating some kind of conservation law...

5.7 Energy in waves

The kinetic energy of an element δs of the string is

$$\frac{1}{2}\rho\delta s u_t^2 \approx \frac{1}{2}\rho u_t^2 \delta x$$

so that the total kinetic energy is

$$\int \frac{1}{2}\rho u_t^2 dx$$

and the kinetic energy per unit length is

$$K = \frac{1}{2}\rho u_t^2$$

The potential energy is the work done in extending the element from δx to δs. This is force T_0 multiplied by distance $\delta s - \delta x = (\sqrt{u_x^2 + 1} - 1)\delta x \approx \frac{1}{2}u_x^2 \delta x$. Thus we have total potential energy

$$\int \frac{1}{2} T_0 u_x^2 dx$$

and potential energy density

$$V = \frac{1}{2} T_0 u_x^2$$

What is the total energy of a travelling wave?

$$u(x,t) = f(x - ct)$$

$$K + V = \frac{1}{2}[\rho c^2 f'(x-ct)^2 + T_0 f'(x-ct)^2] = T_0 f'(x-ct)^2$$

since $c^2 = T/\rho$.

What about the general case

$$u(x,t) = f(x - ct) + g(x + ct)$$

$$K + V = \frac{1}{2}[\rho c^2 (f'^2 - 2f'g' + g'^2) + T_0(f'^2 + 2f'g' + g'^2)] = T_0(f'^2 + g'^2)$$

and there is no cross term. This is natural because the waves are not actually interacting with each other.

Energy flow: The wave carries energy from one part of the string to another, because each part doing work on another part. At $x = 0$ the string to the left exerts a force T_0 at angle ψ on the string to the right (diagram). The rate of work done (power) is force times velocity, ie

$$-T_0 \sin\psi u_t(0,t) \approx -T_0 u_x u_t$$

Of course the same argument holds at any point, not just zero. This is the amount of flow of energy (flux) to the right. For example, a wave moving to the right

$$u(x,t) = f(x - ct)$$

$$-T_0 u_x u_t = (-T_0) f'(x-ct)(-c) f'(x-ct) = cT_0 f'^2 > 0$$

positive as expected. This is also just velocity c times energy density the same as the flux of other conserved quantities. The flux divided by the density is the speed at which energy is transported, called the *group velocity*. In this case it is just the speed of the waves c, but this is not true in other wave problems. In the general case we have

$$u(x,t) = f(x - ct) + g(x + ct)$$

$$-T_0 u_x u_t = (-T_0)(f' + g')(-cf' + cg') = T_0({f'}^2 - {g'}^2)$$

again there is no interaction between the two.

Sinusoidal waves: We have

$$u(x,t) = Ae^{i(kx-\omega t)}$$

real part assumed. Thus

$$u(x,t) = a\cos(kx - \omega t + \delta)$$

where $a = |A|$. Thus the kinetic and potential energies are

$$K = \frac{1}{2}\rho u_t^2 = \frac{a^2\omega^2\rho}{2}\sin^2(kx - \omega t + \delta)$$

$$V = \frac{1}{2}T_0 u_x^2 = \frac{a^2 k^2 T_0}{2}\sin^2(kx - \omega t + \delta)$$

that is, they are equal. It is usual to consider the average energy, obtained by integrating over time or space of one period;

$$\frac{1}{\pi}\int_b^{b+\pi}\sin^2\theta d\theta = \frac{1}{2\pi}\int_b^{b+\pi}(1-\cos 2\theta)d\theta = \frac{1}{2\pi}[\theta - \frac{1}{2}\sin 2\theta]_b^{b+\pi} = \frac{1}{2}$$

so average energy is

$$\bar{E} = \frac{1}{2}a^2\omega^2\rho = \frac{|A|^2\omega^2\rho}{2}$$

Similarly, the energy flux is just the energy times c, so it averages to

$$c\bar{E}$$

Both of them are proportional to the square of the amplitude a.

Recall the discussion about the mass at $x = 0$: we observed that the incoming wave A led to a reflected wave B and transmitted wave C such that

$$|B/A|^2 + |C/A|^2 = 1$$

We could write this as

$$|B|^2 + |C|^2 = |A|^2$$

and since the radian frequency ω and the density is the same, it is clear that this is just conservation of mean energy, that is, the flux is balanced.

Example: Suppose that the string has two different densities, ρ_- for $x < 0$ and ρ_+ for $x > 0$. What are the reflection and transmission coefficients for a sinusoidal wave entering from the left?

The different densities will give rise to different speeds $c_\pm^2 = T_0/\rho_\pm$, however the time dependence $e^{-i\omega t}$ must be the same, as it is the time dependence of the point $x = 0$. This means that the values of $k_\pm = \omega/c_\pm$ and hence the wavelength $\lambda_\pm = 2\pi/k_\pm$ will be different on each side of $x = 0$. The displacement u and its derivative u_x must match up, otherwise there would be a finite force, leading to infinite acceleration of this point. Hence we have

$$u(x,t) = \begin{cases} Ae^{i(k_-x-\omega t)} + Be^{-i(k_-x+\omega t)} & x < 0 \\ Ce^{i(k_+x-\omega t)} & \end{cases}$$

with both u and its x derivative matching at the point $x = 0$. These conditions give

$$A + B = C$$

$$ik_-(A - B) = ik_+C$$

or

$$A - B = \frac{k_+}{k_-}C$$

Adding we find

$$2A = (1 + \frac{k_+}{k_-})C$$

$$C/A = \frac{2k_-}{k_- + k_+}$$

Subtracting we find

$$2B = (1 - \frac{k_+}{k_-})C$$

$$B/A = (B/C)(C/A) = \frac{k_- - k_+}{k_- + k_+}$$

In the original notation we have (if A is real)

$$u(x,t) = \begin{cases} A(\cos(k_-x - \omega t) + \frac{k_- - k_+}{k_- + k_+}\cos(k_-x + \omega t) & x < 0 \\ A\frac{2k_-}{k_- + k_+}\cos(k_+x - \omega t) & x > 0 \end{cases}$$

The reflection coefficient is $|B/A|^2 = (k_- - k_+)^2/(k_- + k_+)^2$ and the transmission coefficient is $|C/A|^2 = 4k_-^2/(k_- + k_+)^2$. We note that in the limit that the two densities are equal, $B = 0$ and $C = A$; in the limit that the right density is infinite, and hence $k_+ = \infty$, $B = -A$ (inversion at a fixed end) while $C = 0$; in the limit that the right density is zero, $k_+ = 0$, $B = A$ and $C = 2A$. Note that the transmitted amplitude can be larger than the incident amplitude.

We check that energy is conserved: the average energy flux is

$$\bar{Q}_{\text{in}} = \frac{1}{2}|A|^2\omega^2\rho_-c_-$$

for the incoming wave, and

$$\bar{Q}_{\text{out}} = \frac{1}{2}\omega^2(|B|^2\rho_-c_- + |C|^2\rho_+c_+) = \frac{1}{2}\omega^2A^2((\frac{k_- - k_+}{k_- + k_+})^2\rho_-c_- + (\frac{2k_-}{k_- + k_+})^2\rho_+c_+) = \bar{Q}_{\text{in}}$$

since $\rho_+ = \rho_- k_+^2/k_-^2$ and $c_+ = c_- k_-/k_+$.

Example: energy loss.

Consider a mass M at $x = 0$ subject to a resistance force $-\mu\dot{y}$ which changes the energy at a rate of force times velocity, $-\mu\dot{y}^2$. If we have $x < 0$ only, we have at $x = 0$,

$$Mu_{tt} = -T_0u_x - \mu u_t$$

We have

$$u(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

thus

$$-M\omega^2(A+B) = -T_0ik(A-B) - \mu(-i\omega)(A+B)$$

$$B(M\omega^2 + ikT_0 + i\omega\mu) = A(-M\omega^2 + ikT_0 - i\omega\mu)$$

$$B/A = -\frac{1 - i(\alpha - \beta)}{1 + i(\alpha + \beta)}$$

where $\alpha = T_0/kc^2M = \rho/Mk$ gives the ratio of the single mass to the mass in a wavelength as before, and $\beta = \mu/M\omega$ compares the time scales of $1/\omega$ and M/μ. The reflection coefficient is now

$$|B/A|^2 = \frac{1 + (\alpha - \beta)^2}{1 + (\alpha + \beta)^2}$$

less than one unless $\beta = 0$. The rest,

$$1 - |B/A|^2 = \frac{4\alpha\beta}{1 + (\alpha + \beta)^2}$$

gives the proportion of the energy that is lost at $x = 0$.

In particular, the average energy flux incident (from A) is

$$A^2\omega^2\rho c/2$$

assuming A real. The rate of energy loss at $x = 0$ is

$$\mu u_t^2 = \mu(Re[-Ai\omega e^{i(kx-\omega t)} - Bi\omega e^{-i(kx+\omega t)}])^2 = \mu\omega^2A^2(Re\frac{2\alpha}{1 + i(\alpha + \beta)}e^{-i\omega t})^2$$

This is

$$\mu\omega^2A^2 4\frac{\alpha^2}{(1 + (\alpha + \beta)^2)^2}(\cos(\omega t) - (\alpha + \beta)\sin(\omega t)^2$$

Averaging over a period we find

$$\mu\omega^2 \frac{4A^2\alpha^2}{(1+(\alpha+\beta)^2)^2}(1+(\alpha+\beta)^2)/2 = \mu\omega^2 \frac{2A^2\alpha^2}{1+(\alpha+\beta)^2}$$

Dividing this by the total incident energy, we have

$$\frac{\mu}{\rho c}\frac{4\alpha^2}{1+(\alpha+\beta)^2}$$

Now $\mu = \beta M\omega = \beta\omega\rho/\alpha k = \beta c\rho/\alpha$ so we have finally

$$\frac{4\alpha\beta}{1+(\alpha+\beta)^2}$$

as required.

5.8 Waves on a finite stretched string

This has been omitted from the course this year as it overlaps with DE2. The basic ideas are as follows. Using the technique of separation of variables, $u(x,t) = X(x)T(t)$ we arrive at the standing wave solution obtained by reflecting a sinusoidal wave from a boundary. We can then match this solution to fixed boundary conditions, for example ensuring that the sine or cosine function is automatically zero at the boundaries when the boundaries are fixed, and so on. We get an infinite number of such solutions, given by various numbers of wavelengths in the finite region. These solutions can be combined using superposition to obtain a general solution. The solution given the initial state of the string is obtained by writing the function as a Fourier series, and hence as a sum of the solutions found above. It is also instructive to study energy conservation in this system.

6 Revision of 3D vector calculus

6.1 Scalar and vector fields

A *scalar field* is a function $\phi(x,y,z)$ defined at each point of space. For example, temperature at each point in a room (may also depend on time t, but still a scalar). We construct the surfaces along which ϕ is constant, the *level surfaces*. Different values of ϕ give different level surfaces. In 2D the level surfaces look like coutour lines on a map.

A *vector field* is a vector $\mathbf{F}(x,y,z)$ defined at each point of space. For example velocity in a liquid varies in both speed and direction from point to point.

6.2 Curves in $\mathcal{R}^3$

We have a parametric representation

$$\mathbf{r} = \mathbf{X}(t)$$

for $a < t < b$; this is actually three equations

$$x = X_1(t)$$

$$y = X_2(t)$$

$$z = X_3(t)$$

and t is a parameter. For example

$$\mathbf{r} = a\cos t\mathbf{i} + a\sin t\mathbf{j} + bt\mathbf{k}$$

is a helix (diagrams).

The tangent at point P is the vector PQ as $\delta t \to 0$ (diagram). But this is just the derivative:

$$\lim_{\delta t\to 0} \frac{\mathbf{X}(\mathbf{t}+\delta\mathbf{t}) - \mathbf{X}(t)}{\delta t} = \mathbf{X}'(t) = \frac{d\mathbf{r}}{dt}$$

so the tangent vector to the curve $\mathbf{X}(\mathbf{t})$ is $\mathbf{X}'(t)$ and the unit tangent vector is

$$\mathbf{X}'(t)/|\mathbf{X}'(t)|$$

Note that we could have the same curve with a different parameter u which is a function of t. In this case we get

$$\frac{d\mathbf{X}}{dt} = \frac{du}{dt}\frac{d\mathbf{X}}{du}$$

so the tangent vector has the same direction but a different magnitude. The unit tangent vector would then be the same.

The *length* of a curve between a and b is

$$l(a,b) = \int_a^b |\mathbf{X}'(t)|dt = \int_a^b |\frac{d\mathbf{r}}{dt}|dt$$

for example the helix,

$$\frac{d\mathbf{r}}{dt} = -a\sin t\mathbf{i} + a\cos t\mathbf{j} + b\mathbf{k}$$

$$l(\alpha,\beta) = \int_\alpha^\beta \sqrt{a^2+b^2}dt = (\beta-\alpha)\sqrt{a^2+b^2}$$

The arc length is

$$s(t) = \int_{t_0}^{t} |\mathbf{X}'(\tau)| d\tau$$

$$s'(t) = |\mathbf{X}'(t)|$$

$$\frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}/dt}{ds/dt} = \frac{\mathbf{X}'(t)}{|\mathbf{X}'(t)|} = \mathbf{T}(t)$$

the unit tangent vector to the curve. For example, the helix has

$$s'(t) = |\mathbf{X}'(t)| = \sqrt{a^2 + b^2}$$

$$s(t) = (t - t_0)\sqrt{a^2 + b^2}$$

so the arc length is proportional to t but not equal to it. But

$$\frac{d\mathbf{r}}{ds} = \frac{-a \sin t}{\sqrt{a^2 + b^2}}\mathbf{i} + \frac{a \cos t}{\sqrt{a^2 + b^2}}\mathbf{j} + \frac{b}{\sqrt{a^2 + b^2}}\mathbf{k}$$

is of unit magnitude since $\mathbf{r}$ and s are both distances.

6.3 Line integrals

Let $\mathbf{F}(\mathbf{r})$ be a vector field defined in some region of $\mathcal{R}^3$ containing a simple smooth curve C: $a < t < b$ defined by $\mathbf{r} = \mathbf{X}(t)$. The line integral of $\mathbf{F}$ along C is defined by

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{X}(t)) \cdot \frac{d\mathbf{r}}{dt} dt$$

For example if $\mathbf{F}$ is the force in moving a particle from A to B, W is just the work done in going from A to B.

6.4 Field lines

A picture of a vector field is obtained from field lines. Given a vector field $\mathbf{F}(\mathbf{r})$ a *field line* is a curve whose tangent at any point P is in the direction of $\mathbf{F}$ at P. Let

$$\mathbf{r} = X(t)\mathbf{i} + Y(t)\mathbf{j} + Z(t)\mathbf{k} = \mathbf{X}(t)$$

be the field line at P. Its tangent vector at P is

$$\frac{d\mathbf{r}}{dt} = \frac{dX}{dt}\mathbf{i} + \frac{dY}{dt}\mathbf{j} + \frac{dZ}{dt}\mathbf{k} = \frac{d\mathbf{X}}{dt}$$

so we must have for some constant λ,

$$\frac{dX}{dt} = \lambda F_1$$

$$\frac{dY}{dt} = \lambda F_2$$
$$\frac{dZ}{dt} = \lambda F_3$$

at P, where

$$\mathbf{F}(\mathbf{r}) = F_1(\mathbf{r})\mathbf{i} + F_2(\mathbf{r})\mathbf{j} + F_3(\mathbf{r})\mathbf{k}$$

It follows that

$$\frac{dX}{F_1(x,y,z)} = \frac{dY}{F_2(x,y,z)} = \frac{dZ}{F_3(x,y,z)}$$

determine the field lines. If $\mathbf{F}$ is the velocity vector field of a fluid, the field lines are called stream lines. For example

$$\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$$

lines are

$$\frac{dx}{-y} = \frac{dy}{x} = \frac{dz}{0}$$

so z is constant, $xdx = -ydy$, $x^2 + y^2 = const$, lines are concentric circles with anticlockwise arrows.

6.5 The vector gradient and directional derivative

Consider two neighbouring level surfaces S and S': f, $f + \delta f$ of a scalar field $f(x, y, z)$. We have a point P on S and a point P' on S' so that OP=$\mathbf{r} = (x, y, z)$, PP'=$\delta\mathbf{r} = \delta s\mathbf{a}$ where $\delta s = |\delta\mathbf{r}|$. We also have a point M on S' such that PM is perpedicular to S', and the angle MPP' is θ. We have PM=$\delta n\mathbf{n}$ where $\mathbf{n}$ is a unit vector. As S' approaches S, PM becomes normal to S, so $\mathbf{n}$ becomes a unit normal to S at P. Going from P to P' we have

$$\frac{\delta f}{\delta s} = \frac{\delta f}{\delta n}\frac{\delta n}{\delta s}$$

but $\delta n = \delta s \cos\theta$ so we take the limit S' to S and P' to P,

$$\frac{\partial f}{\partial s} = \cos\theta \frac{\partial f}{\partial n}$$

so the maximum rate of change takes place when $\cos\theta = 1$, ie along direction $\mathbf{n}$, perpendicular to S at P. We define

$$\nabla f = \frac{\partial f}{\partial n}\mathbf{n}$$

the vector gradient of f at P. We have

$$|\nabla f| = |\frac{\partial f}{\partial n}|$$

is the maximum rate of change of f.

Now $\mathbf{n} \cdot \mathbf{a} = \cos\theta$ so

$$\frac{\partial f}{\partial s} = \cos\theta \frac{\partial f}{\partial n} = \mathbf{a} \cdot \mathbf{n} \frac{\partial f}{\partial n} = \mathbf{a} \cdot \nabla f$$

which is called the directional derivative of f in the direction $\mathbf{a}$.

Example: if $\mathbf{a} = \mathbf{i}$, s=x,

$$\frac{\partial f}{\partial x} = \mathbf{i} \cdot \nabla f$$

and similarly for y,z directions. This means that we can write the gradient explicitly as

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Example: find the directional derivative of $f = 2x^2 - 3xy + 9z - 2$ at $(1, 0, 0)$ in the direction specified by $(2, -2, 1)$.

$$\nabla f = (4x - 3y)\mathbf{i} - 3x\mathbf{j} + 9\mathbf{k}$$

at the point $(1, 0, 0)$ this is

$$4\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$$

The unit vector

$$\mathbf{a} = \frac{(\mathbf{2}, -\mathbf{2}, \mathbf{1})}{\mathbf{3}}$$

so

$$\frac{\partial f}{\partial s} = \mathbf{a} \cdot \nabla f = (4, -3, 9) \cdot \frac{(2, -2, 1)}{3} = 23/3$$

Example, find $\nabla(1/r)$ where $\mathbf{r} = (x, y, z)$ Method 1: write $\phi = 1/r$ so the level surfaces are spheres r constant.

$$\nabla\phi = \frac{\partial\phi}{\partial n}\mathbf{n} = \frac{\partial\phi}{\partial r}\frac{\mathbf{r}}{r} = -\frac{\mathbf{r}}{r^3}$$

Method 2: write

$$\phi = \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}}$$

$$\phi_x = \frac{-x}{r^3}$$

and similarly for y and z, so

$$\nabla\phi = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^3} = \frac{-\mathbf{r}}{r^3}$$

6.6 Surfaces and normals

We can represent a surface in two ways, as an equation

$$F(x, y, z) = 0$$

for example

$$F(x, y, z) = x^2 + y^2 + z^2 - a^2$$

gives a sphere of radius a. A normal to the surface is given by ∇F and unit normals by $\pm\nabla F/|\nabla F|$. For example, the sphere:

$$\pm\frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{2\sqrt{x^2 + y^2 + z^2}} = \pm\frac{\mathbf{r}}{a} = \pm\hat{\mathbf{r}}$$

where the plus sign corresponds to an outward normal.

Alternatively, a surface can be represented as a vector function $\mathbf{r}(u, v)$ of two parameters u and v. As u and v vary a point P traces a surface in $\mathcal{R}^3$. For example spherical polar coordinates for the sphere:

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$x = r \sin u \cos v$$

$$y = r \sin u \sin v$$

$$z = r \cos u$$

Diagram showing $0 \leq r < \infty$, $0 \leq u \leq \pi$ and $0 \leq v < 2\pi$. Usually, $u = \theta$, $v = \phi, \psi$ (the reverse is possible - beware!). Fixing $r = a$ traces out a sphere. The inverse equations are (taking care with the signs):

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\cos u = z/\sqrt{x^2 + y^2 + z^2}$$

$$\tan v = y/x$$

Now we know that $d\mathbf{r}/du$ is a tangent vector to a line formed by $\mathbf{r}(u, v)$ with v fixed, and $d\mathbf{r}/dv$ is a tangent vector to a line formed by $\mathbf{r}(u, v)$ with u fixed, thus both of these are tangent to the surface, and a normal to the surface is

$$\frac{d\mathbf{r}}{du} \times \frac{d\mathbf{r}}{dv}$$

We can check the normal to the sphere:

$$\frac{d\mathbf{r}}{du} = a \cos u \cos v\mathbf{i} + a \cos u \sin v\mathbf{j} - a \sin u\mathbf{k}$$

$$\frac{d\mathbf{r}}{dv} = -a\sin u \sin v\mathbf{i} + a\sin u\cos v\mathbf{j}$$

$$\frac{d\mathbf{r}}{du}\times\frac{d\mathbf{r}}{dv} = a^2\sin^2 u\cos v\mathbf{i}+a^2\sin u\sin v\mathbf{j}+a^2\sin u\cos u\cos^2 v\mathbf{k}+a^2\cos u\sin u\sin^2 v\mathbf{k}$$

which simplifies to

$$a\sin u(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

Surface integrals:

Consider a surface S and its projection on the x,y plane, so we have

$$0 = F(x,y,z) = z - f(x,y)$$

If the surface has more than one part projecting on the same part of the x, y plane, we consider each separately. A small element of surface dS (unit normal $\mathbf{n}$) is projected on an element dA (unit normal $\mathbf{k}$). We have

$$dS|\cos\gamma| = dA$$

where γ is the angle between $\mathbf{n}$ and $\mathbf{k}$. So we have (taking limits as the elements get small):

$$\int_S \phi(x,y,z)dS = \int_A \phi(x,y,f(x,y))|\sec\gamma|dA$$

integrating over the projected area in the x, y plane.

But

$$\nabla F\cdot\mathbf{k} = |\nabla F|\cos\gamma$$

so we can write

$$|\sec\gamma| = \frac{|\nabla F|}{|\nabla F\cdot\mathbf{k}|} = \frac{\sqrt{F_x^2+F_y^2+F_z^2}}{|F_z|} = \sqrt{1+f_x^2+f_y^2}$$

so

$$\int_S \phi(x,y,z)dS = \int_A \phi(x,y,f(x,y))\sqrt{1+f_x^2+f_y^2}dxdy$$

For example, if we want to calculate the area of a surface, we use $\phi(x,y,z) = 1$. In the case of the sphere we have

$$z = \sqrt{a^2-x^2-y^2}$$

$$f_x = -x/\sqrt{a^2-r^2}$$

$$f_y = -y/\sqrt{a^2-r^2}$$

where $r = \sqrt{x^2+y^2}$. Thus

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{1+\frac{x^2}{a^2-r^2}+\frac{y^2}{a^2-r^2}} = \frac{a}{\sqrt{a^2-r^2}}$$

Thus

$$\int_S dS = \int_A \sqrt{1 + f_x^2 + f_y^2} dxdy = \int_0^{2\pi} \int_0^a \frac{a}{\sqrt{a^2 - r^2}} r dr d\theta$$

This is

$$2\pi a \int_0^a \frac{r}{\sqrt{a^2 - r^2}} dr = 2\pi a [-\sqrt{a^2 - r^2}]_0^a = 2\pi a^2$$

which is indeed the surface area of the hemisphere.

Normal flux of a vector over a surface:

Let $\mathbf{F}(\mathbf{r})$ be a vector field defined in a region, and a volume V bounded by a surface S=∂V. Consider a small element of area dS at the point P on S having unit normal $\mathbf{n}$ *out* of V. Then

$$d\mathbf{S} = \mathbf{n} dS$$

is the vector area element of the surface.

The normal flux of $\mathbf{F}$ across element dS is

$$F \cos\theta dS = \mathbf{F} \cdot \mathbf{n} dS = \mathbf{F} \cdot d\mathbf{S}$$

Over the whole of S the normal flux of $\mathbf{F}$ is

$$\int_S F \cos\theta dS = \int_S \mathbf{F} \cdot \mathbf{n} dS = \int_S \mathbf{F} \cdot d\mathbf{S}$$

This leads us to the idea of the *divergence* of a vector field $\mathbf{F}$.

6.7 Divergence of a vector field

Let ΔV denote an element of volume of space containing a point P, and enclosed by a closed surface ΔS. Let $\mathbf{n}$ be the unit normal at any surface elemene $ds \ll \Delta S$ drawn outwards from ΔV. Then the total normal flux of $\mathbf{F}$ over ΔS is

$$\int_{\Delta S} \mathbf{n} \cdot \mathbf{F} dS = \int_{\Delta S} \mathbf{F} \cdot d\mathbf{S}$$

and the outward normal flux of $\mathbf{F}$ per unit volume is

$$\frac{1}{\Delta V} \int_{\Delta S} \mathbf{F} \cdot \mathbf{n} dS$$

Keeping P fixed, we shrink the volume, so that $\Delta V \to 0$ and ΔS shrinks to the point P. Then we define the *divergence* of $\mathbf{F}$ at P as

$$\text{div}\mathbf{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\Delta S} \mathbf{F} \cdot \mathbf{n} dS$$

if the limit exists. In words, the divergence of a vector field is the outward flux of the vector field per unit volume at each point.

In cartesian coordinates we calculate as follows: The components of $\mathbf{F}$ are

$$\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$$

Let P be at the centre of a cuboid with sides Δx, Δy, Δz. That is

$$\Delta V = \Delta x \Delta y \Delta z$$

Towards the positive x direction we have a side where

$$\Delta\mathbf{S} = \Delta y \Delta z \mathbf{i}$$

and so the flux through this side is

$$\mathbf{F}\cdot\Delta\mathbf{S} = F_1 \Delta y \Delta z$$

This side is at $x + \Delta x/2$, so we expand to linear order to get

$$[\mathbf{F}\cdot\Delta\mathbf{S}]_{x+} = (F_1(x,y,z) + \Delta x\frac{\partial F_1}{\partial x}(x,y,z)/2)\Delta y\Delta z$$

The opposite side is at $x - \Delta x/2$ and has

$$\Delta\mathbf{S} = -\Delta y \Delta z \mathbf{i}$$

so it contributes

$$[\mathbf{F}\cdot\Delta\mathbf{S}]_{x-} = -(F_1(x,y,z) - \Delta x\frac{\partial F_1}{\partial x}(x,y,z)/2)\Delta y\Delta z$$

The sum of these two contributions is

$$[\mathbf{F}\cdot\Delta\mathbf{S}]_{x} = \frac{\partial F_1}{\partial x}(x,y,z)\Delta x\Delta y\Delta z$$

Adding the contributions from y and z we find

$$\int_{\Delta S}\mathbf{F}\cdot d\mathbf{S} = (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z})\Delta x\Delta y\Delta z$$

so we have

$$\mathrm{div}\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = \nabla\cdot\mathbf{F}$$

if we recall that

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Examples: 1. $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then

$$\nabla\cdot\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

2. Suppose there exists a scalar function ϕ so that $\mathbf{F} = \nabla\phi$. Then

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial z}\frac{\partial \phi}{\partial z} = \phi_{xx} + \phi_{yy} + \phi_{zz} = \nabla^2\phi$$

where $\nabla^2 = \nabla \cdot \nabla$ is called the Laplacian operator.

If $\nabla^2\phi = 0$ then ϕ is said to be a *harmonic* function at to satisfy Laplace's equation.

For example, prove that $\phi = 1/r$ is harmonic (except at the origin).

$$\nabla\phi = -\mathbf{r}/r^3$$

$$\nabla^2\phi = -\nabla \cdot \mathbf{r}/r^3 = -\frac{1}{r^3}\nabla \cdot \mathbf{r} + \left(\nabla\frac{}{r^3}\right) \cdot \mathbf{r}$$

we have used the product rule

$$\nabla \cdot (\phi\mathbf{F}) = \phi\nabla \cdot \mathbf{F} + \nabla\phi \cdot \mathbf{F}$$

Now

$$\nabla(1/r^3) = \frac{\partial}{\partial r}(1/r^3)(\mathbf{r}/r) = -3\mathbf{r}/r^5$$

so

$$\nabla^2\phi = -\frac{3}{r^3} + \frac{3\mathbf{r} \cdot \mathbf{r}}{r^5} = 0$$

We could also do this calculation in Cartesian coordinates:

$$\phi = \frac{1}{(x^2 + y^2 + z^2)^{1/2}}$$

$$\phi_x = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\phi_{xx} = \frac{-1(x^2 + y^2 + z^2) + 3x^2}{(x^2 + y^2 + z^2)^{5/2}}$$

so that

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

6.8 The divergence theorem

Let S denote a closed surface containing a volume V. Divide V into many elemental volumes δV, using cartesian coordinate planes. Then each element δV will either be a cuboid $\delta x\delta y\delta z$ or a fraction of one, cut by the surface element δS.

If $\mathbf{F}$ is a differentiable vector field defined in V and on S, then

$$\text{div}\mathbf{F}\delta V = \int_{\Delta S} \mathbf{F} \cdot \mathbf{n} dS$$

where ΔS is the total surface of the volume element. Summing over all such δV, to get

$$\int_V \mathrm{div}\mathbf{F}\delta V = \lim_{\delta V\to 0}\sum_{\delta V}\int_{\Delta S}\mathbf{F}\cdot\mathbf{n}dS$$

In the interior, contributions from neighbouring cells cancel (diagram), so this is

$$\int_V \mathrm{div}\mathbf{F}\delta V = \lim_{\delta V\to 0}\sum_{\delta S}\int_{\delta S}\mathbf{F}\cdot\mathbf{n}dS = \int_S \mathbf{F}\cdot\mathbf{n}dS$$

which is the divergence theorem.

6.9 Change of coordinates

An important technique in applied mathematics is choosing coordinate systems to suit the problem, for example using spherical coordinates when there is a sphere, cylindrical coordinates when there is a cylinder. We need to be able to express the gradient and divergence in terms of these and other coordinate systems. Then we have $\nabla^2 = \nabla\cdot\nabla$ which appears in many important equations.

We introduce curvilinear coordinates (u, v, w) which are related to Cartesian coordinates (x, y, z) by

$$x = f(u, v, w)$$

$$y = g(u, v, w)$$

$$z = h(u, v, w)$$

for example, in the case of cylindrical polar coordinates these are

$$x = u\cos v$$

$$y = u\sin v$$

$$z = w$$

where $0 \le u < \infty$, $-\pi < v \le \pi$, $-\infty < w < \infty$ and more usual notation is $u = \rho$, $v = \theta$, $w = z$.

Holding two of the (u, v, w) constant and varying the other, we have a *coordinate curve.* There are three families of these curves obtained by varying u, v and w respectively. If

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = f(u, v, w)\mathbf{i} + g(u, v, w)\mathbf{j} + h(u, v, w)\mathbf{k}$$

then the vectors $\partial\mathbf{r}/\partial u$ etc. are tangent vectors to the coordinate curves. From these we define *unit* tangent vectors, $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, that is,

$$\frac{\partial\mathbf{r}}{\partial u} = h_1\mathbf{e}_1$$

etc. where

$$h_1 = |\frac{\partial \mathbf{r}}{\partial u}|$$

etc. and the chain rule says that

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u}du + \frac{\partial \mathbf{r}}{\partial v}dv + \frac{\partial \mathbf{r}}{\partial w}dw = h_1 du \mathbf{e}_1 + h_2 dv \mathbf{e}_2 + h_3 dw \mathbf{e}_3$$

Suppose that the unit vectors are *orthogonal*. That is,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

This is the *Kronecker delta*, and is a discrete version of the Dirac delta, ie compare

$$\sum_i a_i \delta_{ij} = a_j$$

$$\int f(x)\delta(x-y)dx = f(y)$$

Then the length ds of an arc between points (u, v, w) and $(u+du, v+dv, w+dw)$ is

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dv^2$$

using the chain rule equation. In particular the arc length along coordinate curves is given by one of

$$ds_1 = h_1 du$$

$$ds_2 = h_2 dv$$

$$ds_3 = h_3 dw$$

Example: cylindrical polar coordinates.

$$\mathbf{r} = u \cos v \mathbf{i} + u \sin v \mathbf{j} + w \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial u} = \cos v \mathbf{i} + \sin v \mathbf{j} = h_1 \mathbf{e}_1$$

$$h_1 = |\frac{\partial \mathbf{r}}{\partial u}| = 1$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u \sin v \mathbf{i} + u \cos v \mathbf{j} = h_2 \mathbf{e}_2$$

$$h_2 = |\frac{\partial \mathbf{r}}{\partial v}| = u$$

$$\frac{\partial \mathbf{r}}{\partial w} = \mathbf{k} = h_3 \mathbf{e}_3$$

$$h_3 = |\frac{\partial \mathbf{r}}{\partial w}| = 1$$

so we have $h_1 = h_3 = 1$ and $h_2 = u$.

$$\mathbf{e}_1 = \cos v\mathbf{i} + \sin v\mathbf{j}$$

$$\mathbf{e}_2 = -\sin v\mathbf{i} + \cos v\mathbf{j}$$

$$\mathbf{e}_3 = \mathbf{k}$$

Note that $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ if $i \neq j$, so this coordinate system is orthogonal. The arc length is

$$ds^2 = (h_1 du)^2 + (h_2 dv)^2 + (h_3 dw)^2 = du^2 + u^2 dv^2 + dw^2$$

that is,

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2$$

The gradient in orthogonal curvilinear coordinates:

We have that

$$\nabla f \cdot \mathbf{a} = \frac{\partial f}{\partial s}$$

where s is arc length along a curve, $\mathbf{a}$ is a unit tangent vector, and f is a scalar field. If we choose $\mathbf{a}$ as the three unit vectors in turn,

$$\nabla f = \frac{\partial f}{\partial s_1}\mathbf{e}_1 + \frac{\partial f}{\partial s_2}\mathbf{e}_2 + \frac{\partial f}{\partial s_3}\mathbf{e}_3$$

But

$$\frac{\partial f}{\partial s_1} = \lim_{\delta u \to 0} \frac{f(u + \delta u, v, w) - f(u, v, w)}{h_1 \delta u} = \frac{1}{h_1}\frac{\partial f}{\partial u}$$

and similarly with v, w. So we have

$$\nabla f = \frac{1}{h_1}\frac{\partial f}{\partial u}\mathbf{e}_1 + \frac{1}{h_2}\frac{\partial f}{\partial v}\mathbf{e}_2 + \frac{1}{h_3}\frac{\partial f}{\partial w}\mathbf{e}_3$$

For example in cylindrical polars, $h_1 = h_3 = 1$, $h_2 = u$, so

$$\nabla f = \frac{\partial f}{\partial u}\mathbf{e}_1 + \frac{1}{u}\frac{\partial f}{\partial v}\mathbf{e}_2 + \frac{\partial f}{\partial w}\mathbf{e}_3$$

usually written

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{\partial f}{\partial z}\mathbf{e}_z$$

The divergence in orthogonal curvilinear coordinates

We evaluate $\lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\Delta S} \mathbf{F} \cdot d\mathbf{S}$ on the (almost) rectangular box bounded by the surfaces u, $u + \delta u$, etc. with $\mathbf{F} = F_1(u, v, w)\mathbf{e}_1 + \ldots$. Diagram, showing box with side lengths $h_1 \delta u$ etc.

The area of the surface in the u direction is approximately $h_2\delta v h_3 \delta w$, so the flux on this surface is approx $-F_1 h_2 \delta v h_3 \delta w$ (minus sign because flux is inward). The flux on the $u + \delta u$ surface is approx

$$F_1 h_2 \delta v h_3 \delta w|_{u+\delta u} = F_1 h_2 \delta v h_3 \delta w + \frac{\partial}{\partial u}(F_1 h_2 h_3)\delta u \delta v \delta w$$

so the net flux from both of these is

$$\frac{\partial}{\partial u}(F_1 h_2 h_3)\delta u \delta v \delta w$$

and similarly for the other two pairs of faces. The volume is

$$\Delta V = h_1 h_2 h_3 \delta u \delta v \delta w$$

Thus

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V} \int_{\Delta S} \mathbf{F} \cdot d\mathbf{S} = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u}(F_1 h_2 h_3) + \frac{\partial}{\partial v}(F_2 h_3 h_1) + \frac{\partial}{\partial w}(F_3 h_1 h_2)]$$

This is the divergence $\nabla \cdot \mathbf{F}$ in the new coordinate system.

We can also write down the Laplacian,

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u}(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial u}) + \frac{\partial}{\partial v}(\frac{h_3 h_1}{h_2}\frac{\partial \phi}{\partial v}) + \frac{\partial}{\partial w}(\frac{h_1 h_2}{h_3}\frac{\partial \phi}{\partial w})]$$

Example: cylindrical polars:

$$\nabla^2 \phi = \frac{1}{r}[\frac{\partial}{\partial r}(r\frac{\partial \phi}{\partial r}) + \frac{\partial}{\partial \theta}(\frac{1}{r}\frac{\partial \phi}{\partial \theta}) + \frac{\partial}{\partial z}(r\frac{\partial \phi}{\partial z}) = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Example: spherical polars: $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$. This leads to $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$. Hence

$$\nabla^2 f = \frac{1}{r^2\sin\theta}[\frac{\partial}{\partial r}(r^2\sin\theta\frac{\partial f}{\partial r}) + \frac{\partial}{\partial \theta}(\sin\theta\frac{\partial f}{\partial \theta}) + \frac{\partial}{\partial \phi}(\frac{1}{\sin\theta}\frac{\partial f}{\partial \phi})]$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial^2 \phi}$$

6.10 Conservation in 3D

In 1D we had $P(x,t)$ density, per unit length and $Q(x,t)$ flux, per unit time. In 3D length becomes volume, so $P(x,y,z,t)$ is density of the conserved quantity per unit volume, and $\mathbf{Q}(x,y,z,t)$ is flow of the quantity per unit time through a surface. It is a vector in the direction of the flow. We can also have $R(x,y,z,t)$ as the rate of generation per unit volume.

Consider a volume V bounded by a surface S. The rate of change of substance in V is equal to the amount flowing in through S [plus any sources or sinks]. At time t, the amount of the substance in V is

$$\int_V P(\mathbf{r},t)dV = \int\int\int P(x,y,z,t)dxdydz = \int\int\int P(r,\theta,z,t)rdrd\theta dz$$

in cartesian or cylindrical coordinates respectively. For a fixed volume V,

$$\frac{d}{dt}\int_V PdV = \int_V \frac{\partial P}{\partial t}dV$$

The total flux out of V through S is

$$\int_S \mathbf{Q}\cdot d\mathbf{S} = \int_S \mathbf{Q}\cdot\mathbf{n}dS$$

so

$$\int_V \frac{\partial P}{\partial t}dV = -\int_S \mathbf{Q}\cdot\mathbf{n}dS = -\int_V \nabla\cdot\mathbf{Q}dV$$

using the divergence theorem. Hence we have

$$\int_V (\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{Q})dV = 0$$

(or $\int_V RdV$ if there is a source) for any volume, hence,

$$\frac{\partial P}{\partial t} + \nabla\cdot\mathbf{Q} = 0$$

(or R). This is the 3D statement of conservation.

For example a fluid of density ρ and velocity $\mathbf{u}$ has flux $\mathbf{Q} = \rho\mathbf{u}$. We can see this as follows: Align the motion with the x-axis. The amount passing through a surface $\delta y\delta z$ in time δt has thickness $\delta x = u\delta t$, so the amount is $\rho\delta x\delta y\delta z = \rho u\delta y\delta z\delta t$.

Conservation of mass is

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

also called the equation of continuity. If the fluid is incompressible, ρ is constant, and we have simply

$$\nabla\cdot\mathbf{u} = 0$$

Our main application is the flow of heat...

7 Heat Conduction

7.1 The heat equation

Diffusion: A process by which a (conserved) substance is transported from one place to another by *random molecular motion.* For example a container with iodine (brown) on the bottom and water on the top, will lead after time to a blurred boundary, and then completely mix (three diagrams).

Heat conduction is a process in which heat flows in a solid from regions of higher to lower temperature by diffusion. *Specific heat capacity* c is the amount of heat required to raise a unit mass of a substance one degree in temperature. (Warning: heat capacity can be measured in terms of number of particles or moles of a substance instead of mass). We use symbols c_p, c_v to denote heating at constant pressure and volume respectively for a gas (solids and liquids are typically at constant pressure). Since heat is a form of energy,

$$[c] = \frac{\text{energy}}{\text{mass} \times \text{degrees}}$$

It follows that the total heat per unit volume is

$$P = \rho c u(\mathbf{r}, t)$$

assuming that c doesn't depend on temperature - should be good for small temperature ranges at least. ρ is mass per unit volume. Here, u is temperature, measured with respect to some reference temperature, say $u = 0$ at large distances. Since we are talking about transfer of heat energy, rather than the total amount of heat energy, we can add a constant to u (and hence to P) without changing anything.

Heat flow in most isotropic, homogeneous media obeys Fourier's law: heat flow proportional to the temperature difference. In 3D the heat flow per unit time per unit area is

$$\mathbf{Q}(\mathbf{r}, t) = -k \nabla u(\mathbf{r}, t)$$

The minus sign ensures that heat flows from higher to lower temperature regions, and k is the thermal conductivity of the medium.

Note: we ignore heat flow due to convection which would give a contribution $\rho c u \mathbf{v}$ where $\mathbf{v}$ is the velocity. Heat can also travel via radiation, in a medium or in vacuum.

Conservation of energy (in the form of heat) is thus

$$\frac{\partial P}{\partial t} + \nabla \cdot \mathbf{Q} = R$$

R gives the sources of heat, amount of heat generated per unit volume per unit time. We usually take ρ and c constant, and $R = 0$. Thus

$$\frac{\partial u}{\partial t} = \kappa \nabla^2 u$$

where $\kappa = k/(\rho c)$ is the *thermal diffusivity*. This is called the *heat equation*, or more generally the *diffusion equation* and is relevant to many other processes involving diffusion.

Some typical values:

Medium	ρ	c	k	κ
Water	998	3900	0.591	0.015×10^{-5}
Air	1.29	993	0.024	1.87×10^{-5}
Glass	2600	670	1.0	0.057×10^{-5}
Iron	7700	450	100	2.9×10^{-5}

Boundary conditions: At a boundary separating two regions of different thermal conductivity k, we usually have 1. Temperature continuous, ie $u_1 = u_2$, and 2. heat flux $\mathbf{q} \cdot \mathbf{n}$ continuous, ie $(-k\nabla u) \cdot \mathbf{n}$ is continuous, hence $k_1 du/dn = k_2 du/dn$.

A solid may be bounded by (a) an insulating material, so no heat flow across boundary, $du/dn = 0$; (b) a fluid such as air or water which transports heat via convection, this can be strong, leading to a fixed temperature $u = u_0$, or moderate, described by Newton's law of cooling,

$$\mathbf{q} \cdot \mathbf{n} = -k\nabla u \cdot \mathbf{n} = -k du/dn = \lambda(u - u_0)$$

ie the rate of heat loss is proportional to the difference in temperature. The constant λ is called the surface conductivity: the strong convection corresponds to the limit $\lambda \to \infty$, while the insulating material (although not convecting) corresponds to the limit $\lambda \to 0$.

Initial conditions: The diffusion equation has only one time derivative, so we usually take

$$u(\mathbf{r}, 0) = f(\mathbf{r})$$

a given function. Are these conditions sufficient for a unique solution?

Theorem: If u satisfies $u_t = \kappa \nabla^2 u$ in a domain V, with initial conditions $u(\mathbf{r}, 0) = f(\mathbf{r})$ given in V and either (i) $u = g(\mathbf{r})$ given on S, or (ii) du/dn=0 on S, or (iii) $du/dn = -\lambda u$ on S. Then the solution is unique.

Proof (sketch): Given two solutions, the difference $\phi = u_1 - u_2$ is a solution with zero initial and boundary conditions (the condition (iii) is unchanged), so we need to show that such a solution is always zero. We do this by considering

$$\int_V \phi^2 dV \geq 0$$

But we also have

$$\begin{aligned}
\frac{d}{dt} \int_V \phi^2 dV &= \int_V 2\phi \frac{\partial \phi}{\partial t} dV \\
&= \int_V 2\phi \kappa \nabla^2 \phi dV \\
&= \int_V 2\kappa [\nabla \cdot (\phi \nabla \phi) - (\nabla \phi)^2] dV
\end{aligned}$$

$$\leq \int_V 2\kappa\nabla\cdot(\phi\nabla\phi)dV$$

$$= \int_S 2\kappa\phi\nabla\phi\cdot d\mathbf{S}$$

$$= 0$$

where we used the diffusion equation and the divergence theorem. With zero initial conditions, ϕ can only remain zero.

Solutions of $u_t = \kappa\nabla^2 u$. An explicit solution is not known in general, but there are some very important solutions known for some simple geometries.

7.2 Time independent solutions

A solution is said to be a steady state (or time independent) solution if u depends only on $\mathbf{r}$, not on t. This can occur in the limit that t becomes large, the amount of heat flowing into a region is balanced by the amount flowing out, and the solution depends only on the boundary conditions, the initial conditions are irrelevant. The full solution can then be expressed as the sum of the steady state solution, and a decaying solution, the 'transient' solution.

Since u does not depend on t, $u_t = 0$, so the equation becomes

$$\nabla^2 u = 0$$

We also know $\mathbf{q} = -k\nabla u$, so we can say

$$\nabla\cdot\mathbf{q} = 0$$

In 1D this is simply

$$\frac{d^2u}{dx^2} = 0$$

$$u = Bx + C$$

Consider a rod of length l. The surface is insulated, so no heat flows into or out of y or z direction. If $u(0,t) = u_1$ and $u(l,t) = u_2$ the solution is clearly

$$u(x,t) = u_1 + (u_2 - u_1)x/l$$

and

$$\mathbf{q} = -k\nabla u = -k\frac{du}{dx}\mathbf{i} = q\mathbf{i}$$

$$q = -k\frac{du}{dx} = -\frac{k}{l}(u_2 - u_1)$$

If $u_1 > u_2$ this is a positive quantity - heat is moving to the right. The same equations hold for an infinite plate or slab held at u_1 and u_2 at the ends.

The total heat in the bar is

$$\int_0^l \rho c u dx = \rho c \int_0^l [u_1 + (u_2 - u_1)x/l] dx = \rho c[u_1 l + (u_2 - u_1)l/2] = \rho c(u_1 + u_2)l/2 = \rho c l u_m$$

where u_m is the mean temperature.

Hot water pipe (steady state): Radial flow only. We need the Laplacian in cylindrical polar coordinates (r, θ, z). It is (from before):

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

Diagram of concentric circles, radius $a < b$. The boundary conditions will be $u = u_1$ at $r = a$ (strong convection of water), and $du/dr = -\lambda(u - u_0)$ at $r = b$ (Newton's law of cooling) - but choose $u_0 = 0$ without loss of generality (clearly it is possible to add a constant to u and still satisfy the equation).

The solution will be independent of θ and z (since the bc's are and it is unique), so we have

$$\frac{\partial u}{\partial t} = \kappa(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}) = \frac{\kappa}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})$$

In the steady state it is also independent of t so we need to solve

$$\frac{1}{r}\frac{d}{dr}(r\frac{du}{dr}) = 0$$

That is

$$r\frac{du}{dr} = A$$

$$\frac{du}{dr} = \frac{A}{r}$$

$$u = A \ln r + B$$

$$u = A \ln(r/a) + B_1$$

where $B_1 = B + A \ln a$ for convenience. On $r = a$ we have

$$u = A \ln(1) + B_1 = u_1$$

so

$$B_1 = u_1$$

On $r = b$ we have

$$\frac{du}{dr} = \frac{A}{r} = -\lambda u = -\lambda(A \ln(r/a) + u_1)$$

$$A(\frac{1}{b} + \lambda \ln(b/a)) = -\lambda u_1$$

$$A = \frac{-\lambda u_1 b}{1 + b\lambda \ln(b/a)}$$

so

$$u = u_1[1 - \frac{\lambda b}{1 + b\lambda \ln(b/a)} \ln(r/a)]$$

Special case $\lambda = 0$ (insulator) leads to $u = u_1$ as expected. The heat flow is $\mathbf{q} = -k\nabla u = -k\partial u/\partial r\hat{\mathbf{r}}$. Across a cylinder of radius r this is

$$\mathbf{q} \cdot \hat{\mathbf{r}} = -k\frac{\partial u}{\partial r} = \frac{-kA}{r}$$

So the total rate of heat loss across a cylinder per unit length is

$$Q = \int_0^{2\pi} \mathbf{q} \cdot d\mathbf{S} = -kA \int_0^{2\pi} \frac{r d\theta}{r} = -2\pi kA$$

This is independent of r, because energy is conserved. In fact this argument could be reversed to deduce that $q = C/r$ for some constant C. The time independent heat equation is

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{\partial}{\partial r}(rq_r) = -\frac{1}{r}\frac{\partial A}{\partial r} = 0$$

We have

$$Q = -2\pi kA = \frac{2\pi k\lambda b u_1}{1 + \lambda b \ln(b/a)}$$

For what value of b is the flux a maximum (eg to warm a room)? Write $b/a = x > 1$ and $\lambda a = \alpha$ (these are both dimensionless).

$$Q = (2\pi k u_1 \alpha)\frac{x}{1 + \alpha x \ln x}$$

$$Q'(x) = (2\pi k u_1 \alpha)\frac{(1 + \alpha x \ln x) - x(\alpha \ln x + \alpha)}{(1 + \alpha x \ln x)^2} = (2\pi k u_1 \alpha)\frac{1 - \alpha x}{(1 + \alpha x \ln x)^2}$$

which is zero when

$$x = \alpha^{-1}$$

Graph showing Q going from the constant $2\pi k u_1 \alpha$ either directly towards zero ($\alpha > 1$) or via a maximum ($\alpha < 1$).

Example: spherical heat source. Uranium pellets in a reactor: Total rate of heat generation Q in radius $r < a$, passive coating conductivity k in $a < r < b$, fast moving coolant, $u = 0$ at $r = b$.

Due to symmetry we have $u(r)$ only. Flux through any radius $a < r < b$ is Q by conservation of energy. Flux is

$$Q = \int \mathbf{q} \cdot d\mathbf{S} = 4\pi r^2 q = -4\pi r^2 k\frac{\partial u}{\partial r}$$

$$\frac{du}{dr} = -\frac{Q}{4\pi r^2 k}$$

$$u = \frac{Q}{4\pi r k} + const$$

and since $u = 0$ on $r = b$,

$$u = \frac{Q}{4\pi k}\left(\frac{1}{r} - \frac{1}{b}\right)$$

If $b \to \infty$ we have

$$u = u = \frac{Q}{4\pi r k}$$

If $a \to 0$ this solution is still valid (does not depend on a), and so we have the solution corresponding to a point source, ie

$$R = Q\delta(\mathbf{r}) = Q\delta(x)\delta(y)\delta(z)$$

If the point source is at a point $\mathbf{r}_1$ the solution is

$$u = \frac{Q}{4\pi k|\mathbf{r} - \mathbf{r}_1|}$$

We could also have used the differential equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = 0$$

$$r^2\frac{\partial u}{\partial r} = C$$

$$\frac{\partial u}{\partial r} = \frac{C}{r^2}$$

$$u = -\frac{C}{r} + D$$

with C found using the flux Q and D found from the boundary condition at $r = b$.

7.3 Unsteady problems - spherical symmetry

Omitted this year (involves Fourier series) - brief summary:

Suppose that the sphere $0 \leq r < a$ at initial temperature $f(r)$ and surface temperature U, constant. That is,

$$\frac{\partial u}{\partial t} = \kappa\nabla^2 u$$

$$u(r, 0) = f(r)$$

$$u(a, t) = U$$

Since initial and boundary conditions are independent of θ, ϕ, we have

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})$$

This equation can be reduced to the 1D heat equation by the following trick: Write

$$u = v/r$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial r} - \frac{1}{r^2}v$$

$$r^2\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial r} - v$$

$$\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = r\frac{\partial^2 v}{\partial^2 r} + \frac{\partial v}{\partial r} - \frac{\partial v}{\partial r} = r\frac{\partial^2 v}{\partial^2 r}$$

$$\frac{\partial v}{\partial t} = r\frac{\partial u}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = \frac{\partial^2 v}{\partial^2 r}$$

with $v(r,0) = rf(r)$, $v(0,t) = 0$, $v(a,t) = aU$.

We then solve this like the finite string - separation of variables, $v(r,t) = R(r)T(t)$, giving sinusoidal functions. The initial condition is expanded in a Fourier series, leading to a superposition of these sinusoidal solutions.

7.4 Similarity solutions

Consider an infinite slab, or an insulated rod, and look for a solution that varies in x only (ie a 1D solution).

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2}$$

We have a concentrated source of heat at the $x = 0$, ie

$$u(x,0) = Q\delta(x)$$

$$\lim_{x\to\infty} u(x,t) = 0$$

There is no natural length or time scale in the solution, so we are well placed to use dimensional analysis. $u = u(x,t,\kappa,Q)$ where $[x] = L$, $[t] = T$, $[\kappa] = L^2T^{-1}$ (from the heat equation), $[Q] = UL$ where U is dimensions of temperature (recall $[\delta(x)] = L^{-1}$). Dimensional analysis gives:

$$u = x^\alpha t^\beta \kappa^\gamma Q^\delta$$

$$U = L^{\alpha+2\gamma+\delta}T^{\beta-\gamma}U^\delta$$

thus $\delta = 1$, $\beta = \gamma$, $\alpha = -1 - 2\gamma$.

$$u = \frac{Q}{x}\eta^{-2\gamma}$$

where $\eta = x/\sqrt{\kappa t}$ and γ arbitrary. We will write this solution as

$$u = \frac{Q}{\sqrt{\kappa t}} f(\eta)$$

The dimensionless variable η is called a *similarity* variable. Now we need to find the function $f(\eta)$ (from which we can find the solution $u(x,t)$). We substitute the similarity solution into the equation

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial u}{\partial x} = \frac{Q}{\kappa t} f'(\eta)$$

$$\frac{\partial^2}{\partial x} = \frac{Q}{(\kappa t)^{3/2}} f''(\eta)$$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\frac{Q}{\sqrt{\kappa t}}) f(\eta) + \frac{Q}{\sqrt{\kappa t}} f'(\eta) \frac{\partial \eta}{\partial t} = \frac{Q}{\sqrt{\kappa}}(\frac{f(\eta)}{-2t^{3/2}} + \frac{f'(\eta)}{\sqrt{t}} \frac{-x}{2\sqrt{\kappa t^3}}) = \frac{-Q}{2\sqrt{\kappa t^3}}(f(\eta) + \eta f(\eta))$$

so we have

$$\frac{-Q}{2\sqrt{\kappa t^3}}(f(\eta) + \eta f(\eta)) = \frac{\kappa Q}{(\kappa t)^{3/2}} f''(\eta)$$

$$f''(\eta) + \frac{\eta f'(\eta) + f(\eta)}{2} = 0$$

Initial and boundary conditions:
The solution decays at infinity for fixed t so we require

$$\lim_{\eta \to \infty} f(\eta) = 0$$

The initial condition is $u(x,0) = Q\delta(x)$ which is zero if $x \neq 0$, so

$$0 = \lim_{t \to 0} u(x,t) = \lim_{t \to 0} \frac{Q\eta f(\eta)}{x}$$

so

$$\lim_{\eta \to \infty} (\eta f(\eta)) = 0$$

Finally,

$$\int_{-\infty}^{\infty} u(x,t) dx = \int_{-\infty}^{\infty} \frac{Q}{\sqrt{\kappa t}} f(\eta)(\sqrt{\kappa t} d\eta) = Q \int_{-\infty}^{\infty} f(\eta) d\eta$$

This is Q if

$$\int_{-\infty}^{\infty} f(\eta) d\eta = 1$$

Now the equation is

$$\frac{d^2f}{d\eta^2}+\frac{1}{2}\frac{d}{d\eta}(\eta f)=0$$

$$\frac{df}{d\eta}+\frac{1}{2}\eta f=C$$

The original problem is symmetric about $x=0$, ie

$$u(-x,t)=u(x,t)$$

$$f(-\eta)=f(\eta)$$

$$-f'(-\eta)=f'(\eta)$$

$$f'(0)=0$$

thus $C=0$.

$$\frac{df}{d\eta}+\frac{1}{2}\eta f=0$$

$$\frac{df}{f}=-\frac{\eta d\eta}{2}$$

$$\ln|f|=\frac{-\eta^2}{4}+const$$

$$f=Ae^{-\eta^2/4}$$

Clearly f and ηf approach zero as $\eta\to\infty$ as required.

$$1=A\int_{-\infty}^{\infty}e^{-\eta^2/4}d\eta=A\int_{-\infty}^{\infty}e^{-t^2}2dt=2A\sqrt{\pi}$$

where $\eta=2t$. Thus $A=1/(2\sqrt{\pi})$. Proof:

$$\int_{-\infty}^{\infty}e^{-x^2}dx\int_{-\infty}^{\infty}e^{-y^2}dy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-(x^2+y^2)}dxdy=\int_0^{2\pi}\int_0^{\infty}e^{-r^2}rdrd\theta=2\pi\int_0^{\infty}e^{s}(ds/2)=\pi$$

where $s=r^2$.

We have $f(\eta)=\frac{e^{-\eta^2/4}}{2\sqrt{\pi}}$ and

$$u(x,t)=\frac{Q}{2\sqrt{\pi\kappa t}}e^{-x^2/(4\kappa t)}$$

This is the fundamental source solution in 1D. Diagram showing the Gaussian spreading over time.

7.5 Superposition

This fundamental solution can be used to solve more general problems. For example suppose that the infinite slab has initial temperature profile $u(x, 0) = u_0(x)$. Now a solution is

$$u(x,t) = \frac{Q}{2\sqrt{\pi\kappa t}} e^{-(x-x')^2/(4\kappa t)}$$

for any fixed x' - this is just an initial point source at $x = x'$ rather than $x = 0$. From linearity, so is a superposition (sum, or in this case, an integral) of such solutions at different x':

$$u(x,t) = \frac{1}{2\sqrt{\pi\kappa t}} \int_{-\infty}^{\infty} u_0(x') e^{-(x-x')^2/(4\kappa t)} dx'$$

Write

$$s = \frac{x' - x}{2\sqrt{\kappa t}}$$

$$dx' = 2\sqrt{\kappa t} ds$$

$$u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u_0(x + 2s\sqrt{\kappa t}) e^{-s^2} ds$$

which as $t \to 0$ gives (not rigorously)

$$u(x,0) = \frac{u_0(x)}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-s^2} ds = u_0(x)$$

that is, this solution satisfies the initial conditions. We could also have taken the limit on the shifted solution to get $\delta(x - x')$, with the same result.

For example, suppose

$$u_0(x) = \begin{cases} 0 & x < 0 \\ u_0 & x > 0 \end{cases}$$

diagram. From above, the solution is

$$u(x,t) = \frac{u_0}{2\sqrt{\pi\kappa t}} \int_0^{\infty} e^{-(x-x')^2/(4\kappa t)} dx'$$

Put

$$s = \frac{x' - x}{2\sqrt{\kappa t}}$$

$$dx' = 2\sqrt{\kappa t} ds$$

then

$$u(x,t) = \frac{u_0}{\sqrt{\pi}} \int_{-\eta/2}^{\infty} e^{-s^2} ds$$

where $\eta = x/\sqrt{\kappa t}$ as before. Note that the solution is a function of only η, as there are still no length or time scales in the problem.

Definition: the error function erf$y$ is

$$\text{erf}y = \frac{2}{\sqrt{\pi}} \int_0^y e^{-s^2} ds$$

We have $\text{erf}(0) = 0, \text{erf}(\infty) = 1, \text{erf}(-y) = -\text{erf}(y)$ (graph). So

$$1 - \text{erf}y = \frac{2}{\sqrt{\pi}} \int_0^\infty - \frac{2}{\sqrt{\pi}} \int_0^y = \int_y^\infty e^{-s^2} ds$$

Hence

$$1 - \text{erf}(-y) = 1 + \text{erf}y = \frac{2}{\sqrt{\pi}} \int_{-y}^\infty e^{-s^2} ds$$

$$u(x,t) = \frac{u_0}{2}[1 + \text{erf}(\eta/2)]$$

This approches zero as $\eta \to -\infty$ and u_0 as $\eta \to \infty$. Diagram.

Example: suppose

$$u(x,0) = \begin{cases} 0 & |x| > a \\ u_0 & |x| < a \end{cases}$$

(diagram) then the solution is

$$u(x,t) = \frac{u_0}{2\sqrt{\pi\kappa t}} \int_{-a}^a e^{-(x-x')^2/(4\kappa t)} dx'$$

which we can express in terms of the error function, as follows:

The solution to the first example is

$$v(x,t) = \frac{u_0}{2}[1 + \text{erf}(\eta/2)]$$

and satisfies the heat equation with initial conditions

$$v(x,0) = \begin{cases} 0 & x < 0 \\ u_0 & x > 0 \end{cases}$$

Now

$$u(x,t) = v(x+a,t) - v(x-a,t)$$

satisfies the equation by linearity, and has initial conditions

$$u(x,0) = v(x+a,0) - v(x-a,0) = \begin{cases} 0 & x < -a \\ u_0 & -a < x < a \\ 0 & x > a \end{cases}$$

(diagram), ie is the desired solution.

$$\begin{aligned} u(x,t) &= \frac{u_0}{2}[1 + \mathrm{erf}((x+a)/(2\sqrt{\kappa t})) - 1 - \mathrm{erf}((x-a)/(2\sqrt{\kappa t}))] \\ &= \frac{u_0}{2}[\mathrm{erf}((x+a)/(2\sqrt{\kappa t})) - \mathrm{erf}((x-a)/(2\sqrt{\kappa t}))] \end{aligned}$$

Diagram.

In summary, we have used a number of methods developed earlier in the course to treat the heat equation: dimensional analysis to obtain the similarity solution, conservation (or change of variable) to find the time indepedent solutions, and superposition (as in the wave equation) to solve the general infinite case.