Chapter 5

The Wave Equation in One Dimension

We concentrate on the **wave equation**:

$$u_{tt} = c^2 u_{xx}$$

2nd order in t so vibrations, not decay. Also need two initial conditions, $u(x,0)$, $u_t(x,0)$.

Dimensions of c ? Take dimensions of the PDE:

$$\frac{[u]}{[t^2]} = [c^2]\frac{[u]}{[x^2]}$$

so that $[c^2] = [c]^2 = \text{length}^2/\text{time}^2$ or $[c] = LT^{-1}$. I.e. c has dimensions of speed (e.g. metres per second). It is the wave speed.

We've already looked at the wave equation on bounded domains – sep. of vars. Could use F.T.'s for infinite domains (see Problems sheets)... better way:

5.1 d'Alembert's Solution

We've already seen (in Problem sheets) that

$$u(x,t) = f(x-ct) + g(x+ct)$$

is the **general solution** of $u_{tt} = c^2 u_{xx}$. Why ? Differentiate twice w.r.t. x to get $u_{xx} = f''(x-ct) + g''(x-ct)$ and twice w.r.t. t to get $u_{tt} = c^2 f''(x-ct) + c^2 g''(x-ct)$.

Now suppose we are given

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x)$$

Then from gen. soln. we have

$$f(x) + g(x) = \phi(x), \qquad -cf'(x) + cg'(x) = \psi(x)$$

Take the last equation and integrate up w.r.t. x to get

$$f(x) - g(x) = -\frac{1}{c}\int_a^x \psi(\xi)d\xi$$

where a plays the role of the integration constant. We can now eliminate $f(x)$ and $g(x)$ in turn to get

$$f(x) = \frac{1}{2}\left[\phi(x) - \frac{1}{c}\int_a^x \psi(\xi)d\xi\right]$$

and

$$g(x) = \frac{1}{2}\left[\phi(x) + \frac{1}{c}\int_a^x \psi(\xi)d\xi\right]$$

Then our solution is $u(x,t) = f(x-ct) + g(x+ct)$ which gives

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\phi(x-ct)+\phi(x+ct)\right] + \frac{1}{2c}\left[\int_a^{x+ct}\psi(\xi)d\xi + \int_{x-ct}^{a}\psi(\xi)d\xi\right] \\ &= \frac{1}{2}\left[\phi(x-ct)+\phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(\xi)d\xi \end{aligned}$$

This is **d'Alembert's solution**, the **general solution for any I.C.'s**. Doesn't include B.C's in x - applies as it stands to an infinite domain.

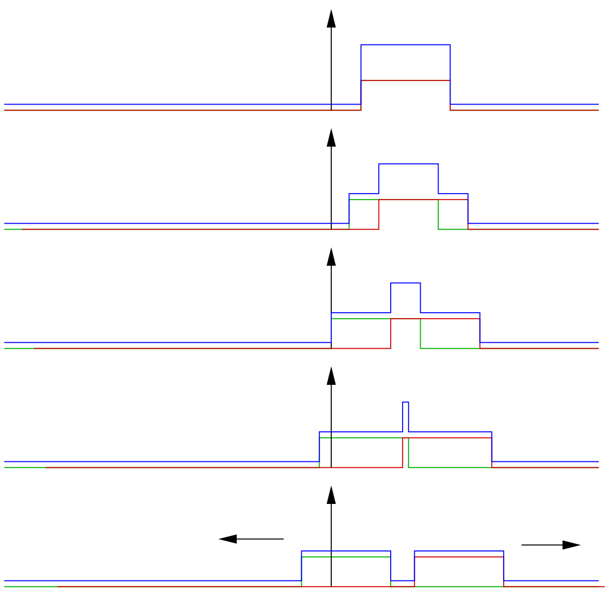
Example 1. $u(x,0) = \phi(x) \equiv \begin{cases} 1 & \text{for } \alpha < x < \beta \\ 0 & \text{for otherwise} \end{cases}$,

and $u_t(x,0) = 0$.

Since $\psi \equiv 0$, the integral vanishes and so

$$u(x,t) = \tfrac{1}{2}\phi(x-ct) + \tfrac{1}{2}\phi(x+ct)$$

This equation tells you that the initial fn. ϕ splits into two halves, each of height $\frac{1}{2}$ which move apart in opposite directions with speed c.



Rule: Blue = Green + Red

Example 2. $u(x,0) \equiv 0$, $u_t(x,0) = \psi(x)$.

In this case $\phi \equiv 0$, and so

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi$$

Now consider the function: $\psi(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$.

This corresponds to an initial impulse or 'hammer blow' to the string across the range $|x| < a$.

The best way to approach this is as follows. Let

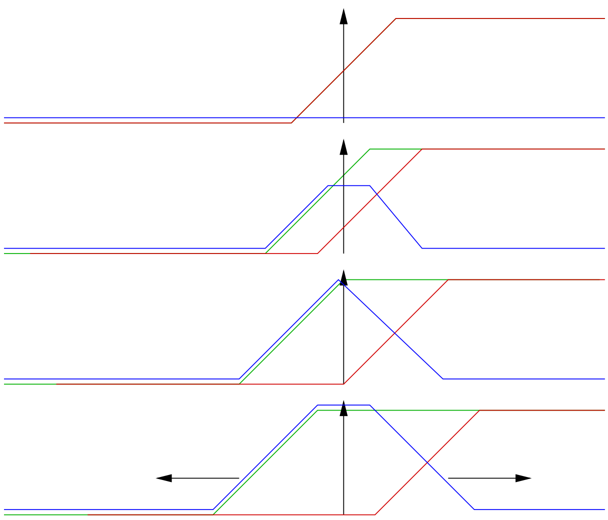
$$\Psi(x) = \frac{1}{2c}\int_{-\infty}^{x} \psi(\xi)\, d\xi = \begin{cases} 0 & \text{for } x < -a \\ (a+x)/2c & \text{for } -a < x < a \\ (2a)/2c & \text{for } x > a \end{cases}$$

which we can establish by careful consideration of the integral.

Then it is clear that

$$u(x,t) = \Psi(x+ct) - \Psi(x-ct)$$

Need pictures to see how this wave evolves. Two linear ramps, the green one moves to the left with increasing time at a speed c and represents the first term above and the red one goes right with speed c. The solution (blue) is the value of the red line subtracted from the value of the green line.



Rule: Blue = Green−Red

5.2 Reflection of Waves

Consider $u_{tt} = c^2 u_{xx}$ for $x > 0$ with the B.C. at $x = 0$ given by $u(0,t) = 0\ \forall t$.

I.C's are $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, for $x > 0$.

Gen. Soln. is $u(x,t) = f(x-ct) + g(x+ct)$.

Try the trick: let $\phi_o(x), \psi_o(x)$ be the *odd* extensions of $\phi(x)$ and $\psi(x)$ into $x < 0$. Now we have a problem for all x using ϕ_o and ψ_o and we can use D'Alembert's solution:

$$\boxed{u(x,t) = \tfrac{1}{2}[\phi_o(x-ct) + \phi_o(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_o(\xi)\,d\xi}$$

It satisfies the PDE and the IC. What about the B.C? Plug in and see

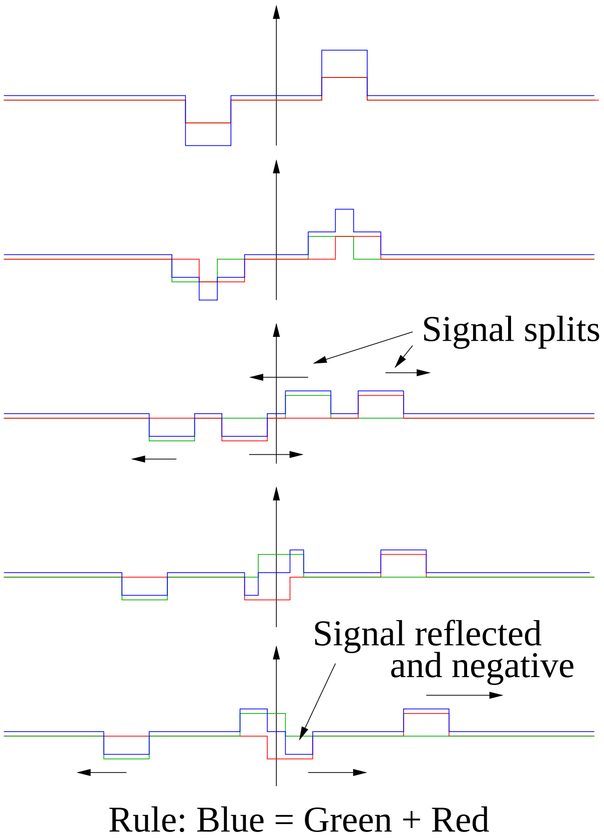
$u(0,t) = \frac{1}{2}[\phi_o(-ct) + \phi_o(ct)] + \frac{1}{2c}\int_{-ct}^{ct} \psi_0(\xi)\,d\xi = 0$

after using the oddness of the functions. This is as required.

So the boxed eqn. above is the soln. of our problem.

Example: $\phi(x) = \begin{cases} 1 & \text{for } 1 < x < 2 \\ 0 & \text{for } 0 < x < 1 \text{ or } x > 2 \end{cases}$, $\psi(x) \equiv 0$. Take the odd extension of ϕ, the first graph on the left below. Then $u = \frac{1}{2}[\phi_o(x-ct) + \phi_o(x+ct)]$

The series of pictures are increasing in time. The pictures show the two separate parts of u (red and green) and the blue line is the sum of those two parts. It can be seen that the reflection at the wall flips the wave over and reflects it back towards the right.



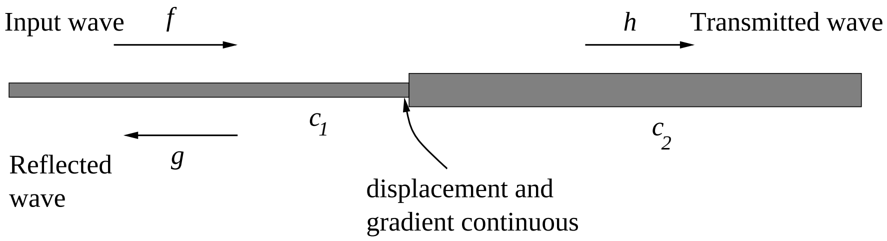
5.3 Wave reflection and transmission at interfaces

More interesting effects can happen when a boundary is not perfectly reflecting, as above

E.g.: Two semi-infinite strings of different densities are joined at $x = 0$. So

$$u_{tt} = c_1^2 u_{xx}, \qquad x < 0, \qquad u_{tt} = c_2^2 u_{xx}, \qquad x > 0,$$

with 'B.C.s' that u, u_x are both continuous at $x = 0$. A wave of known form $u = f(x - c_1 t)$ (so f is given) is incoming on $x = 0$ from $x = -\infty$.



The general solution satisfying the wave equation in $x < 0$ is

$$u(x,t) = \underbrace{f(x - c_1 t)}_{input\ wave} + \underbrace{g(x + c_1 t)}_{reflected\ wave},$$

and in $x > 0$

$$u(x,t) = \underbrace{h(x - c_2 t)}_{transmitted\ wave}$$

Now we apply $u(0^-, t) = u(0^+, t)$:

$$f(-c_1 t) + g(c_1 t) = h(-c_2 t)$$

and $u_x(0^-, t) = u_x(0^+, t)$:

$$f'(-c_1 t) + g'(c_1 t) = h'(-c_2 t)$$

Integrate up the last equation w.r.t. t to get

$$-\frac{1}{c_1} f(-c_1 t) + \frac{1}{c_1} g(c_1 t) = -\frac{1}{c_2} h(-c_2 t) + A$$

where A is a constant, which we can set to zero (by assuming $f(s), g(s), h(s) \to 0$ as $s \to \pm\infty$.)

Eliminating g first, we get

$$2f(-c_1 t) = h(-c_2 t)(1 + c_1/c_2)$$

and letting $s = -c_2 t$ and $\mu = c_1/c_2$ gives

$$h(s) = \frac{2f(\mu s)}{(1 + \mu)}$$

Similarly, eliminate h (check) to get

$$g(c_1 t) = \left(\frac{1-\mu}{1+\mu}\right) f(-c_1 t), \qquad \text{or, letting } s = c_1 t \text{ now,} \qquad g(s) = \left(\frac{1-\mu}{1+\mu}\right) f(-s),$$

Special cases: $\mu = 1$ implies total transmission, $\mu \to \infty$ implies total reflection.

5.4 Unidirectional Waves

$u_{tt} - c^2 u_{xx} = 0$ has general solution $u = f(x - ct) + g(x + ct)$. Can factorise the wave eqn. as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = 0$$

It follows that *either*

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x = 0$$

or

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = u_t - cu_x = 0$$

In the first case, the solution is $u = f(x - ct)$ (wave travelling to the right) and in the second case $u = g(x + ct)$ (to the left).

