Chapter 7

PDEs in Three Dimensions

7.1 Equilibrium Solutions: Laplace's Equation.

7.1.1 Harmonic Functions

The three key eqns introduced in Chapter 2 were:

(i) $u_{tt} = c^2\nabla^2 u$, the wave equation,

(ii) $u_t = D\nabla^2 u$, the diffusion equation,

(iii) $\nabla^2 u = 0$, Laplace's equation.

equilibrium solutions are independent of time (i.e. $u_t = u_{tt} = 0$). So (i), (ii) reduce to (iii)

Defn Solutions of $\nabla^2 u = 0$ are called **harmonic functions**, which are different in 1D (trivial), 2D and 3D (highly non-trivial). 2D harmonic functions are very important in complex analysis as they correspond to real and imaginary parts of all analytic functions.

7.1.2 Properties of harmonic functions

In 1D, $u_{xx} = 0 \implies u(x) = px + q$ for constants p, q. A trivial calculation shows that $u(x) = \frac{1}{2}[u(x+a) + u(x-a)] = \frac{1}{2}(p(x-a) + q + p(x-a) + q) = u(x)$ for any a. I.e., $u(x) =$ average value of two points a distance a from x.

Corollary: On the interval $c \leq x \leq d$, $u(x)$ satisfying $u_{xx} = 0$ must take its max/min values at c or d, not in $c < x < d$.

In 2D and 3D, essentially the same thing...

The Mean Value Property: In 2D/3D, the value of a harmonic function $u(\mathbf{x})$ is the average of the values on any circle/sphere centred on $\mathbf{x}$.

(Proof by complex variables/vector calculus)

Maximum Principle: A harmonic function in a domain $\mathscr{D}$ cannot have a strict local min/max within $\mathscr{D}$.

Proof: follows from the MVP above, by contradition.

Corollary: min/max values must occur on the boundaries of a domain $\mathscr{D}$.

For harmonic functions, u, the values of u are determined by the values on the enclosing curves/surfaces in 2D/3D.

The Zero Solution Property: Suppose $u(\mathbf{x}) = 0$ on a closed curve/surface S, and u is harmonic (i.e. $\nabla^2 u = 0$) inside S (i.e. in $\mathscr{D}$) then $u \equiv 0$ in $\mathscr{D}$.

Proof: Suppose $u(\mathbf{x}) \gtrless 0$ for some $\mathbf{x} \in \mathscr{D}$. Then it has a max/min somewhere in $\mathscr{D}$ with a value $\gtrless 0$. Violates the max/min principle. Hence contradiction.

Uniqueness Theorem: If $u(\mathbf{x})$ is a function satisfying $\nabla^2 u = 0$ inside $\mathscr{D}$ with $u(\mathbf{x}) = f$ on S, a closed curve/surface surrounding $\mathscr{D}$ then it is unique.

Proof. Let $u_1(\mathbf{x}) \not\equiv u_2(\mathbf{x})$ both satisfy $\nabla^2 u_1 = \nabla^2 u_2 = 0$ in $\mathscr{D}$ with $u_1(\mathbf{r}) = u_2(\mathbf{r}) = f$ on S. Then let $u(\mathbf{x}) = u_1(\mathbf{x}) - u_2(\mathbf{x}) \not\equiv 0$ by assumption. Clearly, $\nabla^2 u = \nabla^2 u_1 - \nabla^2 u_2 = 0$ whilst $u = u_1 - u_2 = 0$ on S. By zero property solution, $u \equiv 0$. Hence contradiction.

The Dirichlet Problem: is one in which $u(\mathbf{x})$ is given for $\mathbf{x}$ on S, the boundary of $\mathscr{D}$.

The Uniqueness Theorem says that the Dirichlet problem has at most one solution. Existence is beyond the scope of this course in general; typically shown by finding a solution.

Application: Electrostatics For time-independent problems the electric potential in free space satisfies Laplace's equation. This means it is not possible to construct a time-independent trap for charged particles.

7.2 The Laplacian in non-Cartesian Coordinates

7.2.1 2D Polars (plane polars)

We transform $\nabla^2 u = u_{xx} + u_{yy}$ to (r, θ) coordinates, where

$$\left.\begin{array}{l} x = r\cos\theta, \\ y = r\sin\theta \end{array}\right\},$$

Application of the chain rule (see prob sheet 2, Q8) eventually gives:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{7.1}$$

7.2.2 3D: Cylindrical Polar Coordinates

Cylindrical polar coordinates are (r, θ, z) with $x = r\cos\theta$, $y = r\sin\theta$ as before, Then

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} \tag{7.2}$$

7.3 Separation solutions

7.3.1 Cartesian Coordinates (2D)

Consider $\nabla^2 u = 0$ inside a rectangular domain, $0 < x < a$, $0 < y < b$, say. Then

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let $u(x, y) = X(x)Y(y)$. Then $X''(x)Y(y) + X(x)Y''(y) = 0$ and so

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = k$$

where k is the separation constant. Need B.C's to determine k.

Example: if $u(0, y) = 0$, $u(a, y) = 0$ then $k = -\mu^2$ and $X(x) = \sin(n\pi x/a)$, where $\mu = n\pi/a$.

Then solving for $Y(y)$, ($Y''(y) = \mu^2 Y(y)$) gives

$$Y(y) = A_n \sinh(n\pi y/a) + B_n \cosh(n\pi y/b)$$

or

$$Y(y) = C_n \mathrm{e}^{(n\pi y/a)} + D_n \mathrm{e}^{-(n\pi y/a)}$$

(typical to use the former representation if the y-domain is finite, latter if infinite).

E.g. 1 Let $u(x, 0) = 0$ and $u(x, b) = f(x)$. Then

$$u(x, y) = \sum_{n=1}^{\infty} \left(A_n \sinh(n\pi y/a) + B_n \cosh(n\pi y/a)\right)\sin(n\pi x/a)$$

So $u(x, 0) = 0$ implies $B_n = 0$ for all n and $u(x, b) = f(x)$ implies

$$f(x) = \sum_{n=1}^{\infty} \left(A_n \sinh(n\pi b/a)\right)\sin(n\pi x/a)$$

and then, using expansion formula,

$$A_n \sinh(n\pi b/a) = \frac{\langle f, \sin(n\pi x/a)\rangle}{||\sin(n\pi x/a)||^2}$$

determines A_n and hence u.

E.g. 2 if $u(x,0) = f(x)$ and $u(x,y) \to 0$ as $y \to \infty$. For $0 < x < a$ then $C_n = 0$ in above (for bounded solutions) and

$$u(x,y) = \sum_{n=1}^{\infty} D_n \mathrm{e}^{(-n\pi y/a)} \sin(n\pi x/a)$$

is general solution. Find D_n by putting $y = 0$ with

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} D_n \sin(n\pi x/a)$$

and continue as in E.g. 1.

Of course, D_n (and previously $A_n \sinh(n\pi b/a)$) are the coefficients of the Fourier Sine Series for $f(x)$ (see section 3).

7.3.2 Plane Polars

If a 2D problem has boundaries which fit naturally to a circular geometry then separation in polars is natural.

For example, solving $\nabla^2 u = 0$ inside a circle, $r < a$, with $u(r,\theta) = f(\theta)$ on $r = a$.

I.e. Solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Separation ? Look for solutions of the form

$$u(r,\theta) = R(r)\Theta(\theta)$$

Plug in

$$\Theta R'' + \frac{\Theta R'}{r} + \frac{R\Theta''}{r^2} = 0$$

and divide by $R\Theta/r^2$

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = k$$

where k is a separation constant.

So we have

$$r^2R'' + rR' - kR = 0, \qquad \text{and} \qquad \Theta'' + k\Theta = 0. \tag{7.3}$$

7.3.3 The Θ Equation

To find the separation constant, we want an inhomog. equation with inhomog BC's. The $R(r)$-eqn won't do it, but the $\Theta(\theta)$-eqn will...

General solutions are

$$\Theta = A\cos(\sqrt{k}\,\theta) + B\sin(\sqrt{k}\,\theta) \tag{7.4}$$

Note that if $k < 0$ then cos and sin become cosh and sinh.

On our original problem, we assume u and its derivatives are continuous for all r, θ. So we must insist that $u(r,\theta) = u(r,\theta + 2\pi)$ and $u_\theta(r,\theta) = u_\theta(r,\theta + 2\pi)$.

Looking at (7.4) we can do this if $\sqrt{k} = m$ (or $k = m^2$) where m is an integer. Then

$$\Theta = A_m \cos(m\theta) + B_m \sin(m\theta), \qquad m \in \mathbb{Z}$$

Notes:

- Only need $m \geq 0$ since $m < 0$ gives the same functions with B_m replaced with $-B_m$.
- with $k < 0$ cosh and sinh functions won't work.

7.3.4 The R Equation

The R equation in (7.3) with $k = m^2$ gives

$$r^2 R'' + rR' - m^2 R = 0 \tag{7.5}$$

Solution ? Note non-constant coefficients. Try $R(r) = r^\alpha$ where α is a constant. Then (7.5) is

$$\alpha(\alpha - 1)r^\alpha + \alpha r^\alpha - m^2 r^\alpha = 0$$
$$\implies \qquad (\alpha^2 - m^2)r^\alpha = 0,$$

so $\alpha^2 = m^2$, and $\alpha = \pm m$.

General solution is

$$R(r) = C_m r^m + D_m / r^m \tag{7.6}$$

However, when $m = 0$, r^m and r^{-m} are the same functions – 1, so there must be another...

The equation for $m = 0$ is $r^2 R'' + rR' = 0$. Easy to solve.

For $r \neq 0$ we have $r\dfrac{dR'}{dr} = -R'$. Separate variables and integrate to get $\log R = -\log r + \log D_0$ so that $R'(r) = D_0/r$. Then integrate again to get R, giving

$$R(r) = C_0 + D_0 \log r \tag{7.7}$$

7.3.5 The full solution

Putting all different solutions together using superposition gives the general solution

$$u(r,\theta) = (C_0 + D_0 \log r)A_0 + \sum_{m=1}^{\infty} \left(C_m r^m + \frac{D_m}{r^m} \right) (A_m \cos m\theta + B_m \sin m\theta) \tag{7.8}$$

[Note: For $m = 0$ we have $A_0 \cos 0\theta + B_0 \sin 0\theta = A_0$, giving $u(r,\theta) = A_0(C_0 + D_0 \log r)$.]

Example 1. $\nabla^2 u = 0$ for $r < a$ with BC $u(a, \theta) = f(\theta)$ where f is a given function for $0 < \theta < 2\pi$.

The domain includes the point $r = 0$. Must avoid singularities (infinities) in the solution and so $D_m = 0$ for all m and $D_0 = 0$.

Hence

$$u(r, \theta) = a_0 + \sum_{m=1}^{\infty} r^m (a_m \cos m\theta + b_m \sin m\theta)$$

where $a_m = A_m C_m$ and $b_m = B_m C_m$ in the notation of (7.8).

The B.C. at $r = a$ gives

$$a_0 + \sum_1^{\infty} a^m (a_m \cos m\theta + b_m \sin m\theta) = f(\theta) \ \text{ for } \ 0 < \theta < 2\pi$$

where f is a given function. Thus a_m, b_m are (apart from factors of a^m) the Fourier Series coefficients. Find using expansion formula (section 3).

Example 2. $\nabla^2 u = 0$ in $1 < r < 2$. This region is called an **annulus**.

B.C.'s needed on $r = 1, 2$:

$$u(1, \theta) = f(\theta), \quad u(2, \theta) = g(\theta) \quad \text{for } \ 0 < x < 2\pi$$

where f and g are given functions.

The solution is given by (7.8), but can include all the D_m's and D_0 as $r = 0$ is not part of the annular region. Follow as before but apply conditions on both $r = 1$ and $r = 2$ and get coupled equations for C_n and D_n.

7.4 The Wave Equation: Normal Modes

7.4.1 Normal modes for the 2D wave equation

Consider

$$u_{tt} = c^2 \nabla^2 u \equiv c^2 (u_{xx} + u_{yy}) \tag{7.9}$$

inside a domain $\mathscr{D}$.

Solution determined by:

- Initial values of u and u_t at all points of $\mathscr{D}$,
- Values of u on S, boundary of $\mathscr{D}$ for all t.

We shall only consider the case where $u = 0$. This corresponds to vibrations on a drum skin with fixed edges. Easy to generalise to setting the normal derivative of u equal to zero on S.

The simplest vibration is sinusoidal in time. I.e. motion is proportional to $\sin \omega t$ or $\cos \omega t$ where **period** of oscillations is $2\pi/\omega$.

A **normal-mode solution** of (7.9) to be a solution of the form

$$u(x, y, t) = \phi(x, y) \cos(\omega t + \delta) \tag{7.10}$$

where δ is constant phase-shift.

Plugging (7.10) into (7.9) gives

$$-\omega^2 \phi(x, y) \cos(\omega t + \delta) = c^2 (\nabla^2 \phi) \cos(\omega t + \delta)$$

$$\implies \qquad -\nabla^2 \phi = (\omega^2/c^2)\phi$$

The function ϕ is an eigenfunction of $-\nabla^2$ with eigenvalue $\lambda = \omega^2/c^2$. So the angular frequency is

$$\omega = c\sqrt{\lambda}$$

in terms of the eigenvalue.

7.4.2 An Example

We find the eigenvalues and eigenfunctions of $-\nabla^2$ on rectangle $0 < x < a$, $0 < y < b$ with $u = 0$ on the boundary.

I.e. solve

$$-(\phi_{xx} + \phi_{yy}) = \lambda\phi \tag{7.11}$$

with $\phi(0, y) = \phi(a, y) = \phi(x, 0) = \phi(x, b) = 0$ Solve (7.11) by separating variables. I.e. let $\phi(x, y) = X(x)Y(y)$, with $X(0) = X(a) = 0$ and $Y(0) = Y(b) = 0$. Then, substitute into (7.11), and the usual argument gives

$$\frac{X''}{X} = -\frac{Y''}{Y} - \lambda = k$$

where k is a separation constant.

Solve for $X(x)$, so that $k = -n^2\pi^2/a^2$ and $X(x) = \sin(n\pi x/a)$.

The $Y(y)$ equation is then

$$Y''(y) + (k + \lambda)Y(y) = 0$$

Since λ is unknown, we let $k + \lambda = \mu$ so that the Y-eqn is $Y'' + \mu Y = 0$ with with $Y(0) = Y(b) = 0$. Just like the eqn for X, we have $\mu = m^2\pi^2/b^2$ with $Y(y) = \sin(m\pi y/b)$ for $m = 1, 2, \ldots$ and so $k + \lambda = m^2\pi^2/b^2$ implying:

$$\lambda = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}, \qquad n, m = 1, 2, \ldots$$

and $\phi(x, y) = \sin(n\pi x/a)\sin(m\pi y/b)$. The frequencies of the normal modes are $\omega = c\sqrt{\lambda}$ so that

$$\omega = \omega_{nm} = c\pi\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2}}$$

There exist an infinite, discrete set of frequencies.

The shape of the normal mode is constructed from the separate components so that

$$u(x,y,t) = \phi_{nm}(x,y)\cos(\omega_{nm}t+\delta) = \sin(n\pi x/a)\sin(m\pi y/b)\cos(\omega_{nm}t+\delta)$$

Defn The **fundamental frequency** means the lowest value of ω_{nm} which is when $n = m = 1$ and

$$\omega_{11} = c\pi\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}$$

and the corresponding **fundamental mode** is $\phi_{11}(x,y) = \sin(\pi x/a)\sin(\pi y/b)$

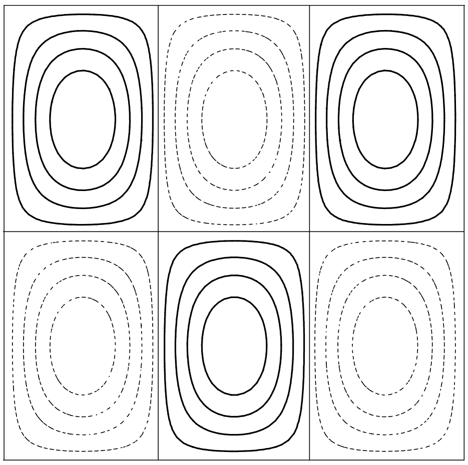
7.4.3 Square domain

In the simplest case where the domain is a square, with $a = b$, the frequencies ω_{nr} are given by the infinite matrix

$$\omega_{nr} = \frac{c\pi}{a}\begin{pmatrix} \sqrt{2} & \sqrt{5} & \sqrt{10} & \dots \\ \sqrt{5} & \sqrt{8} & \sqrt{13} & \dots \\ \sqrt{10} & \sqrt{13} & \sqrt{18} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{7.12}$$

The $3,2$ mode $\phi_{32}(x,y) = \sin\left(\frac{3\pi x}{a}\right)\sin\left(\frac{2\pi y}{a}\right)$ is illustrated below by a contour diagram, showing the curves in the x,y plane along which $\phi_{32}(x,y)$ is constant.

The solid contours are where $\phi_{32}(x,y) > 0$ and the dotted contours are where $\phi_{32}(x,y) < 0$. There are three maxima and three minima. The straight lines are where $\phi(x,y) = 0$. They divide the rectangle into six regions, called *cells*; each cell consists of a single peak or valley.



As time increases the peaks and valleys each oscillate up and down with angular frequency $\omega_{32} = (c\pi/a)\sqrt{13}$. When ϕ is increasing in one cell, it is decreasing in the adjacent cells; the peaks become valleys and the valleys become peaks after a time π/ω_{32}.

The other normal modes are similar, but with different numbers of cells in the x and y directions. The fundamental mode has just one cell.

The solution of an initial value problem can be found as a superposition of normal modes. So when you bang a drum, the sound produced is a combination of the normal modes. The principle is similar to Fourier series solutions, but the details are lengthy and beyond the scope of this course.

7.4.4 Why the guitar is tuneful and drums are noisy

Guitars

A guitar string satisfies the 1-d wave equation with boundary conditions that $u = 0$ at the endpoints, $x = 0$ and a say. It is easy to see that it has normal modes

$$\sin\left(\frac{n\pi x}{a}\right)\cos\left(\frac{n\pi ct}{a}+\delta\right),\ n = 1, 2, \ldots$$

The angular frequencies are $c\pi/a,\ 2c\pi/a,\ 3c\pi/a,\ \ldots$; they are integer multiples of the fundamental frequency $c\pi/a$. So the sound wave that travels to your ears is a combination of frequencies which are integer multiples of the fundamental (angular) frequency $c\pi/a$. It is therefore a periodic function of time with period $2a/c$; the higher frequencies correspond to higher terms in the Fourier series solution of the wave equation.

A periodic sound wave like this is heard by the ear as a musical note. The pitch of the note[1] is determined by the period of the wave; high frequencies give high notes. The Fourier coefficients a_n, b_n determine the character of the sound. If a_1 or b_1 is much larger than all the $n > 1$ coefficients, then the note sounds flute-like and smooth. But if a_n or b_n does not decrease rapidly with n (for example, if the n-th coefficient behaves like $1/n$) then the note sounds quite sharp in character and perhaps even harsh. Thus you can *hear* something about the Fourier coefficients in a musical sound.

Drums

The vertical vibration of a drumskin satisfies the wave equation in 2d. The boundary condition is zero displacement at the edge of the drum. If the drum is rectangular, its vibration is a combination of the normal modes derived above. For a square drum, where $a = b$, the normal modes have frequencies ω given by (7.12); the first few are

$$\omega = \sqrt{2}\pi c/a,\ \sqrt{5}\pi c/a,\ \sqrt{8}\pi c/a,\ \sqrt{10}\pi c/a, \ldots$$

They are *not* integer multiples of the fundamental frequency. Therefore the sound produced by a drum is *not* heard as a musical note, it is heard as a noise.

Of course most drums are not square but round. We will work out the normal modes for a circular drum, and the answer shows that their frequencies are not integer multiples of the fundamental frequency. That is why drums bang while strings play tunes.

[1] *pitch* describes whether it is a high or a low note

7.5 The Wave Equation in Plane Polar Coordinates

7.5.1 Separation of Variables

Consider the wave equation in a circular domain (vibrations of a circular drumskin, oscillations on the surface of a cup of tea):

$$u_{tt} = c^2\nabla^2 u \equiv c^2\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right], \qquad 0 < r < a \tag{7.13}$$

with $u = 0$ on $r = a$.

Let $u(x, y, t) = \phi(r, \theta)\cos(\omega t + \delta)$ as before.

Then

$$-\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right) = \left(\frac{\omega}{c}\right)^2 \phi = \lambda\phi$$

and λ is the eigenvalue, to be found.

Separate variables: $\phi(r, \theta) = R(r)\Theta(\theta)$ and then above is

$$-\left(R''\Theta + \frac{R'\Theta}{r} + \frac{R\Theta''}{r^2}\right) = \lambda R\Theta \tag{7.14}$$

Divide by $R(r)\Theta(\theta)/r^2$ to get

$$\frac{r^2 R''}{R} + \frac{rR'}{R} + \lambda r^2 = -\frac{\Theta''}{\Theta} = k$$

where k is sep. const.

The Θ Equation

We have $\quad \Theta'' + k\Theta = 0.$

Since we are solving inside a circle, need $\Theta(0) = \Theta(2\pi)$ and $\Theta'(0) = \Theta'(2\pi)$ and so

$$k = m^2 \qquad \text{and} \qquad \Theta = A_m \cos m\theta + B_m \sin m\theta, \qquad \text{for } m = 0, 1, 2, \ldots$$

where A_m, B_m are constants.

The R Equation

With $k = m^2$, so the R equation in (7.14) is

$$R'' + \frac{R'}{r} - \frac{m^2}{r^2}R + \lambda R = 0$$

where λ is unknown eigenvalue (once λ is known then so is ω) and m is an integer.

This equation cannot be solved in terms of elementary functions. But it can be analysed by Sturm-Liouville theory. Instead, put into SL form as

$$(rR')' - \frac{m^2}{r}R + \lambda r R = 0, \qquad 0 < r < a \tag{7.15}$$

This is a SL equation with $p(r) = \sigma(r) = r$, $q(r) = -m^2/r$. Must have boundedness of R and R' at $r = 0$ whilst $R(a) = 0$ because u vanishes on the circle $r = a$.

Simplifying the Equation

Rescale the independent variable: $x = r\sqrt{\lambda}$ and let $y(x) = R(x/\sqrt{\lambda})$ or $R(r) = y(r\sqrt{\lambda})$. Then $d/dr = \sqrt{\lambda}d/dx$ so that

$$\sqrt{\lambda}(xy')' - \frac{m^2\sqrt{\lambda}}{x}y(x) + \frac{\lambda x}{\sqrt{\lambda}}y(x) = 0$$

$$\implies \qquad x^2y''(x) + xy'(x) + (x^2 - m^2)y(x) = 0 \tag{7.16}$$

This is called **Bessel's equation**.

7.5.2 Solutions of Bessel's Equation

Bessel's equation (7.16) does not have solutions in terms of elementary functions. Their solutions are called Bessel functions. The are well-studied and have many useful properties.

There are two linearly independent solutions:

$$\begin{cases} J_m(x), & \text{(bounded at the origin} \sim x^m) \\ Y_m(x), & \text{(singular at the origin} \sim x^{-m}\log(x)) \end{cases}$$

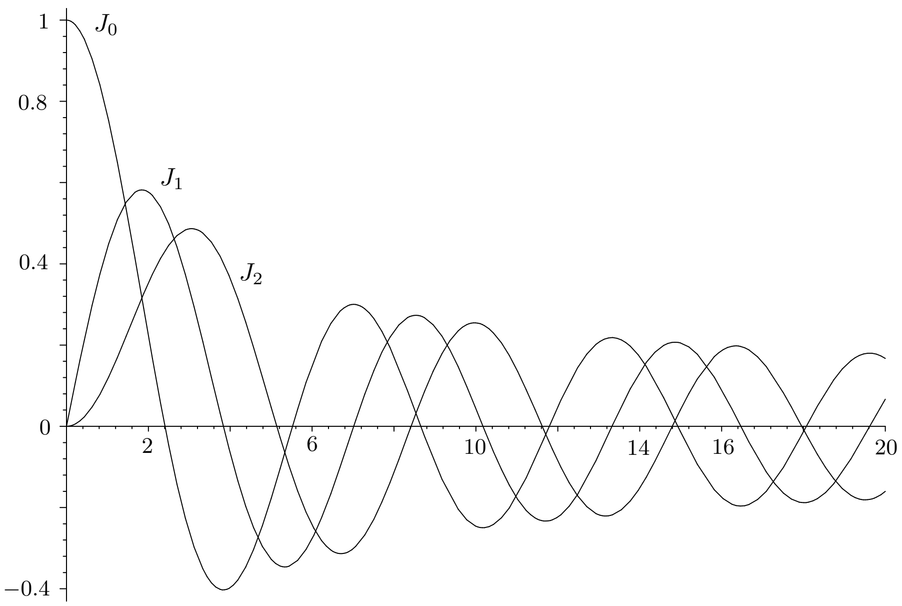
[We won't include Y_m as we don't want to include singularities at $x = 0$ ($r = 0$) in our solution, although for problems which exclude the origin you must include them (not in this course)]

The functions $J_m(x)$ have the following features:

1. Power series representation (cf. cos or sin)

$$J_m(x) = \frac{x^m}{2^m m!}\left[1 - \frac{x^2}{2^2 1!(m+1)} + \frac{x^4}{2^4 2!(m+1)(m+2)} - \dots\right] \tag{7.17}$$

2. Sketch:



The first three Bessel functions.

3. $J_m(x)$ are roughly like $\cos(x+\epsilon)/x^{1/2}$ for large x.
4. $J_0(0) = 1$ and $J_m(0) = 0$ for $m \geq 1$
5. (Important) $J_m(x) = 0$ has infinitely many solutions. Label these roots, $x = z_{m,i}$, $i = 1, 2, 3, \dots$.

7.5.3 Normal Modes of a Circular Membrane

Go back to the problem: circular membrane, radius $r = a$.

Since $y(x) = J_m(x)$ and $R(r) \equiv y(x) = y(r\sqrt{\lambda})$, the general solutions, bounded at $r = 0$ of the $R(r)$-eqn are given by

$$R(r) = C_m J_m(r\sqrt{\lambda})$$

where C_m an arbitrary constant.

The boundary condition at the edge of the drum gives $R(a) = 0$. So

$$J_m(a\sqrt{\lambda}) = 0. \tag{7.18}$$

Therefore we must have

$$a\sqrt{\lambda} = z_{m,i}, \qquad i = 1, 2, \ldots$$

where $z_{m,i}$ are the zeros of $J_m(x)$. Hence

$$R(r) = C_{m,i} J_m\left(\frac{z_{m,i}\, r}{a}\right) \text{ for } r \le a, \quad i = 1, 2, \ldots \tag{7.19}$$

with $C_{m,i}$ constants, after modifying the notation.

Hence, the **frequencies** of oscillations are given by

$$\omega/c = \sqrt{\lambda} = z_{m,i}/a, \qquad \text{or} \qquad \omega_{m,i} = \frac{z_{m,i}c}{a}$$

The modal shape of the membrane comes from reconstructing the solution from its separable parts

$$u(r,\theta,t) = \phi_{m,i}(r,\theta)\cos(\omega_{m,i}t + \delta) = C_{m,i} J_m\left(\frac{z_{m,i}\, r}{a}\right)[A_m \cos m\theta + B_m \sin m\theta]\cos(\omega_{m,i}t + \delta)$$

where $\omega_{m,i} = z_{m,i}c/a$.

The first few zeros of the Bessel functions (approx)

$$z_{0,1} = 2.4\ldots,\ z_{1,1} = 3.8\ldots,\ z_{2,1} = 5.1\ldots,\ z_{0,2} = 5.5\ldots$$

So the fundamental (lowest-frequency) mode has frequency $\approx 2.4c/a$ where a is the radius of the drum and c is the speed of waves on the drumskin. The larger the radius, the lower the frequency. This is why a bass drum must be big.

7.5.4 The Initial-value problem

In both the rectangular and circular membrane problem, an initial value problem in which u and u_t are specified at $t = 0$, a general solution is formed by the superposition of all possible normal modes. The unknown coefficients can, in principle, be found by applying initial conditions on u and u_t at $t = 0$, but this is too complicated for this course.

7.6 Diffusion in a Cylinder

Consider diffusion in a long cylinder (e.g. heat flow in a hot water pipe). Choose cylindrical polars, z along cylinder axis.

Assume u is independent of z and θ. So $u = u(r,t)$ and satisfies

$$u_t = D\left(u_{rr} + \frac{1}{r}\,u_r\right)\left(\equiv D\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)\right) \tag{7.20}$$

where $D > 0$ is the diffusion coefficient.

We need an initial condition:

$$u(r,0) = f(r) \;\text{ for }\; 0 < r < a \tag{7.21}$$

We also need a B.C. on $r = a$, so consider

$$u(a,t) = 0 \;\text{ for }\; t > 0 \tag{7.22}$$

Separation of Variables

Let $u(r,t) = R(r)T(t)$, substitute into (7.20):

$$\frac{T'}{DT} = \frac{1}{R}\left(R'' + \frac{R'}{r}\right) = -k.$$

(We chose $-k$, because from what we know about diffusion we expect exponential decay in time, thus implying that $k > 0$ in the above assignment)

The T eqn: Easy $T' = -kDT$ has solutions Ce^{-kDt}. Still need to know what values k takes.

The R eqn: is

$$(rR')' + krR = 0. \tag{7.23}$$

This is the same as (7.15) with $m = 0$ and $k = \lambda$. So solutions bounded at $r = 0$ given by Bessel functions $J_0(r\sqrt{k})$ and

$$R(r) = BJ_0(r\sqrt{k}) \tag{7.24}$$

for constant B.

Values of k determined by B.C. $R(a) = 0$. I.e. $J_0(a\sqrt{k}) = 0$ so $\sqrt{k} = z_{0,i}/a$, $i = 1, 2, \ldots$ and

$$R(r) = B_i J_0(z_{0,i}r/a)$$

are the radial solutions.

General solution

Superposition of all sep. solutions gives a general solution

$$u(r,t) = \sum_{i=1}^{\infty} a_i\, e^{\left(\frac{-z_{0,i}^2\, D\, t}{a^2}\right)} J_0\left(\frac{z_{0,i}r}{a}\right) \tag{7.25}$$

for unknown coefficients a_i, which are contracted from C and B_i.

To find a_i, apply the I.C. $u(r,0) = f(r)$, $r < a$ so

$$\sum_{i=1}^{\infty} a_i J_0(z_{0,i}r/a) = f(r) \text{ for } 0 < r < a \tag{7.26}$$

From the expansion theorem (this is all S-L),

$$a_i = \frac{\langle f(r), J_0(z_{0,i}r/a)\rangle}{\langle J_0(z_{0,i}r/a), J_0(z_{0,i}r/a)\rangle} \equiv \frac{\int_0^a f(r)J_0(z_{0,i}r/a)rdr}{\int_0^a J_0^2(z_{0,i}r/a)rdr}$$

which can be found (at least numerically).

Note that the orthogonality result of Bessel functions is, ensured by SL theory is

$$\langle J_0(z_{0,i}r/a), J_0(z_{0,j}r/a)\rangle \equiv \int_0^a J_0(z_{0,i}r/a)J_0(z_{0,j}r/a)rdr = 0, \qquad i \neq j$$

General Character of the Solution

For the diffusion equation in 1d, with $u = 0$ at the endpoints, any initial condition gets smoother and smoother (as a function of x) as t increases, and for large t looks like a single hump of a sine curve decreasing exponentially with time.

The picture here is very similar. The later terms in the series (7.25) are wiggly, but $\to 0$ faster as t increases. So the solution as a function of r gets smoother as t increases and its shape approaches a single hump of J_0, decreasing exponentially with time, more or less equivalent to the fundamental mode of the wave problem.

