

Escape and transport for an open bouncer: Stretched exponential decays

*C. P. Dettmann¹, School of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom
and*

*E. D. Leonel, Departamento de Estatística, Matemática Aplicada e Computação, IGCE
Univ Estadual Paulista, Av.24A, 1515 - Bela Vista - CEP: 13506-900 - Rio Claro - SP - Brazil*

Abstract

We consider time-dependence of dynamical transport, following a recent study of the stadium billiard in which classical transmission and reflection probabilities were shown to exhibit exponential or algebraic decay depending on the choice of the lead or “hole”, raising the question of whether this feature is due to special properties of the stadium. The system considered here is much more general, having a generic mixed phase space structure, time-dependence of the dynamics, and Fermi acceleration (trajectories with unbounded velocity). We propose an efficient numerical scheme for this model, observe escape and transport effects including similar asymmetry, and also clear stretched exponential decays.

1 Introduction

Consider a dynamical system that is “opened” by introducing one or more holes in phase space; if the trajectory reaches one of these, it “escapes” and is no longer considered. The initial condition could be inside the system, in which case we speak of an escape problem, and ask for quantities such as the “survival” probability of remaining within the system for a given time, given an initial condition chosen randomly with respect to a probability measure supported within the system. Alternatively, the initial condition could be located at one of the holes, in which case we speak of a transport problem. We seek similar probabilities, now depending on whether the trajectory returns to the initial hole (reflection) or reaches another hole first (transmission).

The open dynamical system could refer to a physical particle or ray approximation of a wave escaping from a container, or more abstractly a representation of a metastable state in phase space. Open dynamical systems provide a useful description of microlasers [38], acoustics [29, 39], fluid dynamics [36], chemical reaction dynamics [15] and astronomy [37], as well as experiments involving electrons in semiconductors [31] or cold atoms confined by laser beams [35]. They also yield a useful means to investigate quantum chaos [30], including random matrix approaches [33] and the fractal Weyl conjecture [28, 34, 38], as well as Poincaré recurrence [2], control [6] and nondestructive measurement of chaotic systems [4].

Here we consider the bouncer model, in which a particle falls under gravity except for collisions with a sinusoidally vibrating plate. For certain

¹Corresponding author, email: Carl.Dettmann@bris.ac.uk

phases of oscillation (in other words a periodic collection of time intervals) the plate becomes absorbing, corresponding to a hole in the phase space. Our purpose is to discuss the escape and transport problems for this system, as a second and far more general test case for the recently noted phenomenon of asymmetric transport [12], and for discussion of outstanding issues related to open dynamical systems with time-dependence, non-compact phase space and generic mixed phase space structure. In the process we make the first observation of stretched exponential decays of survival probabilities, in contrast to the typical exponential or algebraic decays for strongly chaotic [4, 14, 19] and most other (regular [3], mixed [2] or intermittent [11]) open systems respectively.

For the transport problem we follow [12], in which it was found for the stadium billiard that the transmission and reflection probabilities for trajectories entering the billiard through one of two holes decayed at long times exponentially or algebraically depending on the choice of entrance and exit hole. The stadium billiard consists of two parallel straight sides and two semicircular ends, with the usual billiard dynamics in which a particle moves in straight lines except for mirror-like reflections from the boundary. It is an ergodic and chaotic system [10], described as intermittent as the chaos is interspersed with long stretches of regular motion almost perpendicular to the straight sides, called “bouncing ball orbits.” The intermittency means that the survival probability for the escape problem (uniform initial conditions within the billiard) decay as a constant divided by the time, where the constant can be written explicitly in terms of the geometrical parameters [11]. Thus the exponential transport properties found in [12] for one hole in a straight segment and one in a curved segment are unexpected; the mechanism is that orbits from the hole in the curved segment are blocked by the other hole from coming too close to the bouncing ball orbits, and so have properties determined by the chaos characterising the rest of the phase space.

The stadium billiard is however a very special chaotic system, being completely ergodic, a property which is easily broken by perturbations [5]. The bouncer model considered here is much more general; in common with generic Hamiltonian systems it has a mixed phase space consisting of elliptic islands and (for certain parameter values) a large ergodic component with chaotic properties. The question of escape and transport in systems with generic mixed phase space consisting of elliptic islands and chaotic sea(s) has been previously considered by many authors [2]. The general consensus is that time dependence of both escape and transport probabilities is eventually algebraic (although often apparently exponential at short times), but many different power laws have been proposed and observed numerically [1]. Claims of universal power law exponents have also been made [9]. We do not observe a universal law, perhaps due to the small number of prominent islands considered in our system (for fixed velocity range; see below). The understanding of these exponents, and the question of a universal law are interesting, important but also very challenging problems, we will postpone them to future work. What we can say about the system considered here is that clear stretched exponential decay is observed, at least for finite times and in some situations; this is discussed below.

In Section 2 we define the bouncer model and propose a new numerical approach for efficient solution of its transcendental equations. In Sec. 3 we consider the escape problem, leading on to the transport problem in Sec. 4. Concluding discussions are in Sec. 5.

2 The bouncer model and its numerical simulation

The bouncer model described above and defined below is also a time-dependent system and exhibits Fermi acceleration, so the phase space is not compact or of finite invariant measure. Fermi acceleration is a phenomenon in which a particle gains unbounded energy from a moving infinitely heavy plate (with usually periodic, specifically sinusoidal motion considered). The original motivation was the origin of high energy cosmic rays [16]. The particle returns to the plate either due to another plate which is fixed, the Fermi-Ulam model; a gravitational field, the bouncer model considered here; or a combination of the two [23, 24]. These problems are generalisations of billiards due to the hard collisions, however they are time-dependent and (in the original form) one dimensional. The phase of the oscillation plays the role of a second dimension, so that the collision map is two-dimensional, as for two-dimensional billiards such as the stadium. Fermi acceleration is observed for the bouncer model, but not in the Fermi-Ulam or combined models for periodic oscillations of the plate. Time-dependent two-dimensional billiards have also been much studied, and can exhibit Fermi acceleration; see [17, 21, 26, 27].

At this point we make some remarks about the chaotic properties of systems with Fermi acceleration such as the bouncer model. In the Fermi acceleration regime, recurrence and ergodicity in a single unbounded component of phase space are likely, but not proven to our knowledge. The largest Lyapunov exponent as calculated for typical points in this component are likely to be positive for the discrete but zero in the continuous version of the dynamics, as the chaos-inducing collisions are rare when the particle has high velocity. Properties such as mixing are not uniquely defined for systems with infinite invariant measure. We will describe the large component of phase space below as “strongly chaotic”, but it should be recalled that this has a more limited meaning for systems such as the bouncer model.

We now define the bouncer model. A plate moves periodically with vertical position $y_0(t) = \epsilon \cos wt$, and a particle with position $y(t)$ makes elastic collisions, returning to the plate due to a constant gravitational force $-g$. We can scale the time to set $w = 1$ and the position to set $g = 1$, leaving only a single parameter ϵ ; here we mostly do not consider air resistance or an inelastic restitution coefficients, although these have been considered in the literature [20, 22, 25]. The natural dynamical variables are the phase $\phi_n = t_n \pmod{2\pi}$ of the collision at time t_n and the velocity $v_n = \dot{y}(t_n)$ immediately after this collision. The dynamics is

given implicitly by

$$\epsilon \cos \phi_{n+1} = \epsilon \cos \phi_n + v_n \Delta_n - \frac{\Delta_n^2}{2} \quad (1)$$

$$v_{n+1} = -(v_n - \Delta_n) - 2\epsilon \sin \phi_{n+1} \quad (2)$$

where $\Delta_n = t_{n+1} - t_n = \phi_{n+1} - \phi_n + 2\pi k$ with $k \in \{0, 1, 2, \dots\}$ is the time difference between collisions and takes its smallest possible positive value when there is more than one positive solution.

It is worth mentioning one of the few exact solutions of this model, which also determines the main phase space structure. It is $\phi_n = 0$, $v_n = \pi m$, $\delta_n = 2\pi m$ for some $m \in \{1, 2, 3, \dots\}$, giving a family of fixed points of the collision map. Since it corresponds to the top (or for $\epsilon < 0$, the bottom) of the cycle, a linear perturbation of the phase leads only to a quadratic perturbation of the height y and may be neglected for a linear stability analysis. It is easy to show that linear perturbations $(\delta\phi, \delta v)$ satisfy

$$\begin{pmatrix} \delta\phi_{n+1} \\ \delta v_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2\epsilon & 1 - 4\epsilon \end{pmatrix} \begin{pmatrix} \delta\phi_n \\ \delta v_n \end{pmatrix} \quad (3)$$

where the matrix has complex eigenvalues corresponding to an elliptic fixed point in the region $0 < \epsilon < 1$, in particular only when the particle collides at the top of the cycle and the forcing is not too strong. For $\epsilon < 0$ or $\epsilon > 1$ the orbit is hyperbolic.

Now, let us consider numerical simulation of the bouncer model. The dynamical equation is transcendental. The most common approach considered in the literature to simulate this system rapidly is to simplify it using the Holmes method which uses a constant height for the plate, but time-varying impulse provided to the particle; see [24]. Here we consider the exact equations, using the following efficient numerical method, which is based on the approach used by one of the authors in a different billiard problem [13]. The free flight motion is exactly solvable for a known time step. The time step is chosen using a rigorous lower bound for the actual time step, thus the first collision is never overstepped (modulo round off error). While in this work we consider only the original bouncer model, the numerical method is discussed in the more general context of an additional frictional force, which is considered elsewhere in the literature [20, 22].

We know $\ddot{y}_0(t) \leq \epsilon$, thus the second derivative of the displacement of the particle above the plate $d(t) = y(t) - y_0(t)$ satisfies $\ddot{d}(t) \geq -1 - \epsilon$ in the absence of friction, where the 1 corresponds to gravity. More generally we consider a frictional force $-m\eta(v)$ given by a monotonically increasing function $\eta(v)$ with $\eta(0) = 0$. During a free flight the velocity is decreasing, so the acceleration of the particle is becoming less negative and we have $\ddot{d}(t) \geq -1 - \eta(v(0)) - \epsilon$ for $t > 0$ within the same free flight. Now we can integrate this inequality twice, using the initial conditions $v(0)$ and $y(0)$ and find in the general case

$$d(t) > d(0) + \dot{d}(0)t - [1 + \eta(v(0)) + \epsilon] \frac{t^2}{2} \quad (4)$$

The RHS is positive at $t = 0$, thus a rigorous lower bound for the time step is given by the solution of the quadratic equation RHS=0. Numerically,

the form of the solution is chosen depending on the sign of the linear term, to minimise subtraction error as usual. The larger of the two solutions of the quadratic is chosen, corresponding to the collision in the future. In the event that roundoff error gives a displacement that is slightly negative, the use of the larger solution also prevents a spurious collision with the lower side of the plate.

Let us assume we are close to a collision, for which the actual time step required is $\delta t \ll 1$. The lower bound computed by the algorithm is as shown above. An upper bound for $d(t)$ is the same expression, except that in the acceleration term, $\eta(v(0))$ is replaced by $\eta(v(t)) > 0$ and ϵ is replaced by $-\epsilon$. Thus the displacement $d_0(t)$ is determined to within a bounded constant times t^2 , as is the time step in the generic case when the linear term in the quadratic equation is not too small. Thus the algorithm typically converges quadratically, similar to Newton’s method, for example. In practice, the approach to the collision to a tolerance close to machine precision takes around five steps, and is then followed by the collision map transformation of the velocity.

The bouncer model exhibits a structure very similar to that of the Chirikov standard map [8] with $K \approx 4\epsilon$; see [23, 24]. This is a 2D map that can be reduced to a torus, being periodic in both directions. The bouncer is clearly periodic in ϕ , and it is also “nearly” periodic in v . Recall the family of periodic orbits at $\phi = 0$ and $v = \pi m$; clearly both these and their stability is periodic in v with period π . The map as a whole is not periodic, as can be seen for free flights starting and ending at different y values, but this becomes less significant at large v , and even at small v is strongly noticeable; see Fig. 1.

The bouncer model exhibits a series of transitions as the parameter is varied, analogous to those of the standard map. The case $\epsilon = 0$ is regular. For small ϵ the velocity is bounded due to invariant curves. Near $\epsilon \approx 0.24$ (corresponding to $K = 0.971\dots$) invariant spanning curves at all but the lowest velocities are disrupted, leading to unbounded velocity (Fermi acceleration) of some orbits. For $\epsilon > 1$ the fixed points studied above become unstable (initially bifurcating into elliptic period 2 orbits) and for large ϵ the motion is almost entirely chaotic. At some parameter values $K > 2\pi$, the standard map also has “accelerator modes”, which are orbits in which velocity increases linearly with discrete time; these can be stable and surrounded by elliptical islands, so yielding anomalous diffusion of velocity for the chaotic component [18]. The parameter $\epsilon = 0.6$ of Fig. 1 shows clear elliptic islands around the fixed points, coexisting with Fermi acceleration of the chaotic orbits; this ϵ is below the lowest accelerator modes, so diffusion of velocity in discrete time is expected to be normal to a good approximation.

3 The escape problem

Now we discuss the bouncer model as an open system allowing escape, leading on to the transport problem in the next section. In open dynamics, we choose a distribution (more precisely, probability measure) of initial conditions, allow the dynamics to evolve with escape, so that any

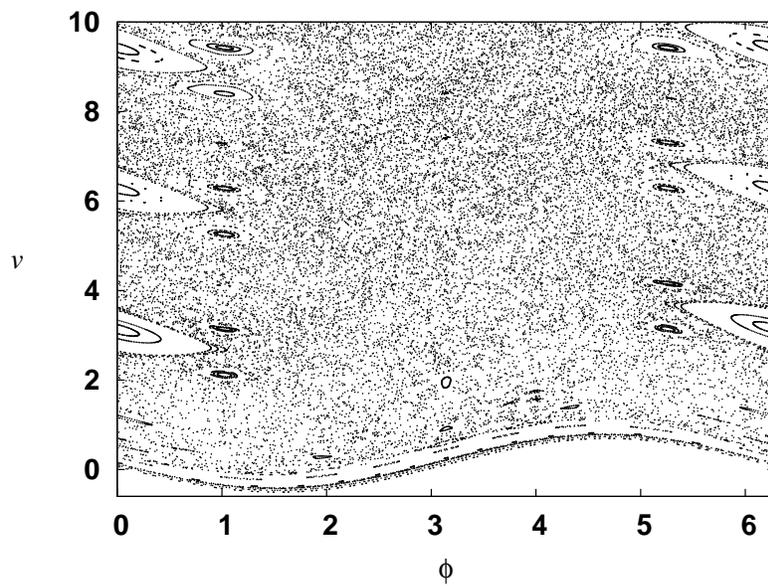


Figure 1: The phase space of the bouncer model for the parameter value $\epsilon = 0.6$.

trajectory reaching a subset of phase space called the ‘hole(s)’ leaves the system and is no longer considered. After some time (or number of collisions) we ask what fraction of initial conditions are still in the system, the survival probability $P(t)$.

The initial measure can be uniform (which is natural where this is an invariant measure, as for bounded Hamiltonian systems), or a more general measure supported in a general subset of phase space, or on the hole. If the initial measure is supported on the hole, we talk about the gap or residence time distribution. We naturally consider more than one hole, and ask for the reflection probabilities (trajectory returns to the same hole) and transmission probabilities (escape through a different hole); these are discussed in the next section.

The bouncer model is time-dependent, leading to the question of how to consider open time-dependent dynamical systems; general theories of these systems do not appear in the literature to our knowledge, although aspects of many specific physical examples have been studied. In the time-dependent case, either the initial probability measure or the survival probability function needs to include information about the initial time as well as phase space point. Here we have a periodic system, where the phase can be considered an additional phase variable, and it is natural to consider the initial conditions distributed uniformly over some interval in ϕ as well as in v . In a quasiperiodically forced system, it would be natural to extend the phase space to include a finite number of additional phases. For a randomly forced system, the probability measure would naturally incorporate that of the forcing. However for a general deterministic but aperiodic forced system, the best approach is likely to be considering the survival probability as a function of both initial and final times.

Another issue here, which is common to all non-ergodic systems, and in particular those with mixed phase space, is that the results depend on the support of the measure of initial conditions. So, if the initial conditions include an elliptic region not connected with the hole, the survival probability will tend to a constant as $t \rightarrow \infty$, giving the probability of choosing such an initial condition. Numerically, this washes out other time-dependence of $P(t)$ and also slows the simulation, as the elliptic trajectories need to be simulated for the full duration of the time interval. Ref. [2] refers to initial measures that “touch” the elliptic island, but this is likely to be problematic due to the fractal boundary of these islands. Most simulations in the literature therefore use initial conditions in a (largely) strongly chaotic region of phase space, and we do this also, drawing 10^8 uniformly from $2 < v < 4$ and $3.85 < \phi < 4.15$, and the parameter $\epsilon = 0.6$ as in Fig. 1; any other reasonably large set in the chaotic region and any other smooth density would give the same results, at least at long times.

The hole is chosen to be a subset of ϕ only, ie vertical strips in the phase plot of Fig. 1. In general we could choose any subset of the phase space. The choice of intervals in ϕ corresponds to the plate being removed at certain parts of its periodic motion (except for very low velocities, in which case multiple collisions are possible), and in more general terms, a hole with few control parameters typically accessing both regular and chaotic parts of phase space, which is the most likely situation experimentally. This choice of hole location also allows the non-escaping orbits to

access the Fermi acceleration. We avoid here the situation where the hole overlaps the set of initial conditions; this belongs to the next section on transport.

The results for the escape are shown in Fig. 2, using a hole of size $h = 0.3$ and centred at various locations both in and away from the main elliptic islands. For a hole placed over part of the elliptic islands, decay is initially slower, as it takes some time for the dynamics to “find” regions close to the elliptic islands, faster at intermediate times, as the hole covers sticky orbits which would slow decay, and eventually a slow power law, as islands not covered by the hole become significant. Where the hole is in the chaotic sea, decay is generally slowed by the available large elliptic islands. Note that a hole partially covering an island may in fact destroy all orbits surrounding that island; here we see effective destruction of islands using holes quite a lot smaller than the islands concerned.

We now focus on the intermediate behaviour in the case where the hole largely blocks orbits near the elliptic islands. When measured in discrete time, this is exponential as expected for strongly chaotic systems. When escape is measured in continuous time (right plots), the rapid decay is instead close to a stretched exponential, with \sqrt{t} in the exponent. This can be explained by the Fermi acceleration, as follows: A typical chaotic orbit lasting for n collisions has survival probability e^{-n/n_0} where $1/n_0$ is the discrete escape rate. This orbit diffuses in velocity, with a typical velocity proportional to \sqrt{n} , and hence time $t_0 \approx n\sqrt{n}$ on average. The tail of the distribution of time as a function of n follows a typical diffusion process, decaying as $e^{-c_1(t/t_0)^2}$, thus the probability of surviving for n collisions and taking a time t is proportional to the product of these exponentials. For a given time t , this is maximised when the exponent $-n/n_0 - c_1 t^2/n^3$ is maximum; differentiating this and setting it equal to zero we find the largest contribution from $n \approx (3c_1 n_0 t^2)^{1/4}$, with probability $e^{-c_2 \sqrt{t}}$ and c_2 determined from the other constants. The algebraic decays in continuous time are, however, the same as in discrete time; this is because it is dominated by orbits near the elliptic island, which do not accelerate, and in fact have almost constant velocity.

4 Time-dependent transmission and reflection probabilities

Recent work [12] has discussed reflection and transmission probabilities in an intermittent system, the stadium billiard, as a function of time, and found a striking asymmetry, namely that the reflection probability from one hole decayed algebraically, while that from the other decayed exponentially, despite the ergodicity of the closed system. The intermittency in the stadium arises from a single family of parabolic “bouncing ball” orbits. The bouncer model considered here is a significantly more complicated system. While in the Fermi acceleration regime there is a large ergodic component corresponding to the chaotic sea, intermittency arises from the much more detailed and generic mechanism of a hierarchy of elliptic (“KAM”) islands. One of the main aims of this paper is to de-

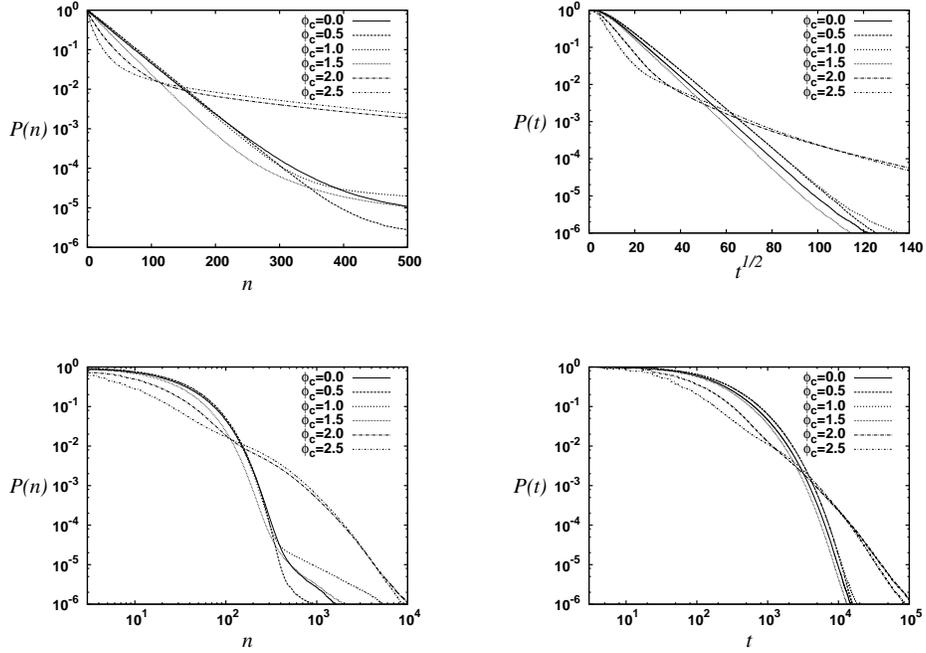


Figure 2: Survival probability for the escape problem, for holes of size $h = 0.3$ at different locations (compare with Fig. 1). Holes centred at $\phi = 0, 0.5, 1, 1.5$ are located over or near the large elliptic islands, while holes centred at $\phi = 2, 2.5$ are almost entirely in the chaotic sea. The two plots on the left are for discrete time, showing primarily exponential decay in the former case, and algebraic in the latter. Those on the right are for continuous time, showing stretched exponential decay; note the \sqrt{t} on the axis. The algebraic decay has the same power as for discrete time.

termine whether the same asymmetric effect can occur in this much more general setting.

In the stadium of [12], the mechanism of asymmetric transport is related to a splitting of the ergodic phase space due to the hole in the intermittent region. We note the hole in the straight segment of the stadium also allows escape of orbits with large angles, hence far from the bouncing ball orbits, but this does not alter the effect. The mechanism is that the hole “traps” the intermittent orbits, so that while they may take some time to escape (due to the intermittency), they can escape only through the same hole, and not another hole placed far from the intermittent region.

Holes near $\phi = 0$ will cover both elliptic islands and chaotic sea, while holes near $\phi = \pi$ will cover mostly chaotic sea. Of course any elliptic island completely covered by a hole cannot lead to slow escape. We must also choose a measure of initial conditions. As we are studying transport, these will be supported on one of the holes. The measure is chosen to be uniform in the (ϕ, v) coordinates, between $v = 2$ and $v = 4$, as in the escape case. Note that the initial measure is supported on a strict subset of the hole. This is because both dynamics and holes have infinite measure in this system, so some choice needs to be made about the initial measure; the results depend in principle on both the holes (position, size and shape) and on the choice of initial measure. The dependence on the initial measure here facilitates design of transport problems with specified properties, and is a strength of this approach. Here our choice is similar to in the escape section.

For the transport problem we consider $P_{ij}(t)$, the probability of remaining until at least time t and eventually escaping through hole j given that the particle enters through hole i ; note the slight difference in notation from [12]. Here $i, j \in \{1, 2\}$ label the two holes. If the particle always eventually escapes (as would follow from Poincaré’s recurrence theorem if the phase space were of finite invariant measure, and is probably still true here), then we have

$$\sum_j P_{ij}(0) = 1 \tag{5}$$

For the numerical simulations shown in Fig. 3, 10^8 initial conditions are chosen for each hole. The first hole is in the region of the elliptic islands around $\phi = 0$. The second hole is centred at π ; this covers some of the smaller islands, and is far from the large islands around $\phi = 0$, so for practical purposes it is in the strongly chaotic region. First we consider the effect of changing the hole size; see Fig. 3. This figure shows very clearly the effect we are seeking, namely a very strong $P_{11}(t)$ for hole sizes $h = 0.4$ and $h = 1$. For these parameter values (and many not shown), the hole in the elliptic region effectively traps the long-lived orbits near the smaller islands while reducing the lifetime of orbits near the large island. In other words, orbits near these small islands cannot reach other parts of phase space without passing through this hole. On the other hand, a hole that is too small ($h = 0.1$) does not effectively trap these orbits, and one that is too large ($h = 2.2$) allows all orbits to escape within a finite time.

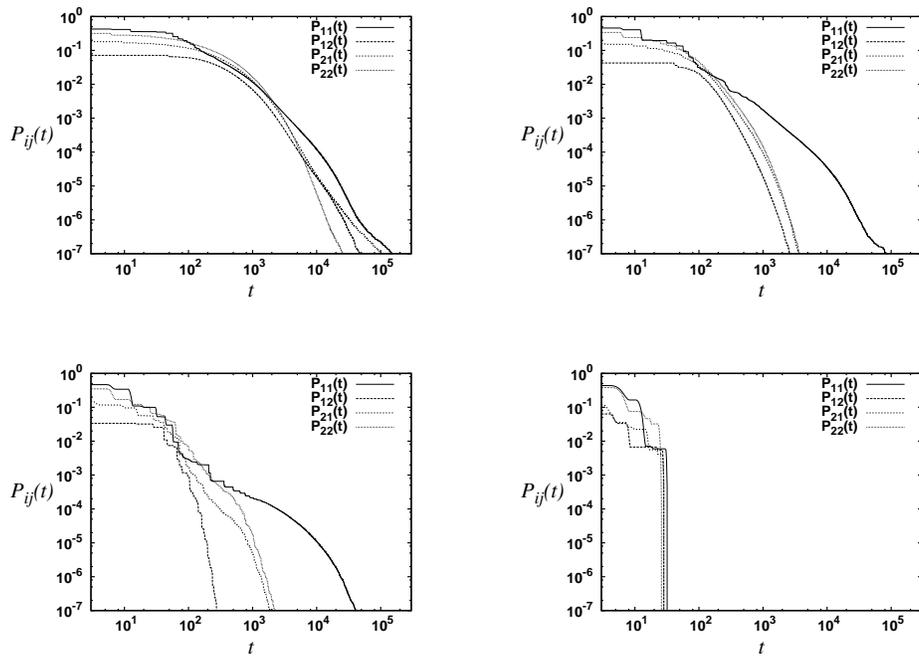


Figure 3: The transmission and reflection probabilities $P_{ij}(t)$ for four different hole sizes h : 0.1 (top left), 0.4 (top right), 1 (bottom left) and 2.2 (bottom right), as a function of continuous time t . Both holes are the same size. Hole 1 is centred at $\phi = 0$ and hole 2 is centred at $\phi = \pi$.

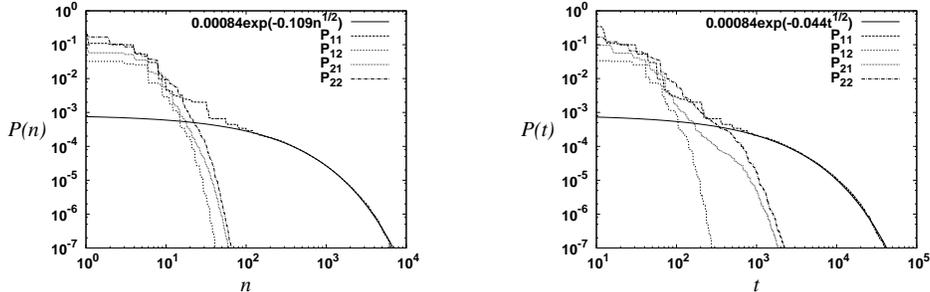


Figure 4: The transmission and reflection probabilities $P_{ij}(t)$ for the hole size 1, compared for discrete time (left) and continuous time (right), and also showing fits to the stretched exponential behaviour in both cases.

Another feature we can see, particularly in the plot for hole size 1, is that the long lived orbits are well approximated by a stretched exponential, $ae^{b\sqrt{t}}$. This goes against conventional wisdom (see the previous section) in which a power is expected in generic mixed systems, however even in papers such as [2, 15], curves intermediate between power and exponential behaviour can be found (but not much remarked upon). A possible explanation is that in situations involving a single large island, the stretched exponential law is visible, while for more generic cases with many islands of comparable size, the effects combine to generate a more uniform power. However, in contrast to the escape problem, we don't at present have an explanation for why it should be a stretched exponential, or whether the power in the exponential is exactly $1/2$.

Fig. 4, compares transport properties of the hole of size $h = 1$ in discrete and continuous time, and also shows stretched exponential fits for the reflection probability of hole 1 in the region of the elliptic islands. The main difference here are the transporting orbits, which decay with many fewer collisions; these can reach large velocities. The discrete time plots for the other values of h are even more similar to those of continuous time given in Fig. 3, and are not shown.

5 Conclusion

In contrast to the stadium billiard considered in [12], the bouncer model exhibits a generic mixed phase space structure, time-dependent dynamics and Fermi acceleration. The properties of escape and transport in mixed phase space have been considered previously, and remain a challenging problem for the future. Here we note that generic mixed phase space allows the asymmetric transport similar to that observed in the stadium, but with different functional forms, and in particular a clear stretched exponential was observed.

The fact that the dynamics is time-dependent affected the definition of

the initial measure and survival probabilities, albeit in a fairly straightforward way, that of increasing the phase space dimension by one. Further approaches for more involved time-dependence were noted in the section on escape.

The Fermi acceleration provided the most interesting and also challenging problems for the study of escape and transport. For the escape problem, it resulted in a different functional form of the survival probability between the discrete and continuous time problems, exponential in the first case and stretched exponential in the second; however the form of the stretched exponential (involving \sqrt{t}) could be deduced from the properties of the velocity diffusion process. More fundamentally, a hole of infinite measure does not lend itself to a unique measure for the initial conditions; this ambiguity and dependence on initial measure can be used to advantage, however, in the flexible design of the problem. This issue deserves further study, both from a numerical and a rigorous point of view.

One or more of these issues apply to very many types of physical systems, and we hope that this work will provide a starting point for future studies, both theoretical and experimental, discussed in the introduction. Many such systems are dissipative, and relevant measures on chaotic regions of phase space are typically fractal. Insight into escape and transport scenarios is also important for the corresponding quantum problems.

Acknowledgements

This paper was written during a visit of CPD to EDL's group, funded by CNPq. EDL thanks FAPESP, CNPq and Fundunesp, Brazilian agencies.

References

- [1] E. G. Altmann, A. E. Motter and H. Kantz, *Phys. Rev. E* **73**, 026207 (2006).
- [2] E. G. Altmann and T. Tél, *Phys. Rev. E* **79**, 016204 (2009).
- [3] L. A. Bunimovich and C. P. Dettmann, *Phys. Rev. Lett.*, **94**, 100201 (2005).
- [4] L. A. Bunimovich and C. P. Dettmann, *EPL* **80**, 40001 (2007).
- [5] L. A. Bunimovich and A. Grigo, *Commun. Math. Phys.* **293**, 127-143 (2010).
- [6] H. Buljan and V. Paar, *Phys. Rev. E* **63**, 066205 (2001).
- [7] L. A. Bunimovich and A. Yurchenko, *Israel J. Math* **182**, 229-252 (2011).
- [8] B. V. Chirikov, *Phys. Rep.* **52**, 263-379 (1979).
- [9] G. Cristadoro and R. Ketzmerick, *Phys. Rev. Lett.* **100**, 184101 (2008).
- [10] N. I. Chernov and R. Markarian *Chaotic billiards* (Amer. Math. Soc., 2006).

- [11] C. P. Dettmann and O. Georgiou, *Physica D* **238**, 2395-2403 (2009).
- [12] C. P. Dettmann and O. Georgiou, *Phys. Rev. E* **83** 036212 (2011).
- [13] C. P. Dettmann, G. P. Morriss and L. Rondoni, *Phys. Rev. E* **52**, R5746-R5748 (1995).
- [14] M. Demers and L.-S. Young, *Nonlinearity* **19**, 377-397 (2006).
- [15] G. S. Ezra, H. Waalkens and S. Wiggins, *J. Chem. Phys.* **130**, 164118 (2009).
- [16] E. Fermi, *Phys. Rev.* **75**, 1169 (1949).
- [17] V. Gelfreich and D. Turaev, *J. Phys. A: Math. Theor.* **41**, 212003 (2008).
- [18] R. Ishizaki, T. Horita, T. Kobayashi and H. Mori, *Prog. Theor. Phys.* **85**, 1013-1022 (1991).
- [19] G. Keller and C. Liverani, *J. Stat. Phys.* **135**, 519-534 (2009).
- [20] G. A. Luna-Acosta, *Phys. Rev. A* **42**, 7155-7162 (1990).
- [21] E. D. Leonel and L. A. Bunimovich, *Phys. Rev. Lett.* **104**, 224101 (2010).
- [22] E. D. Leonel and A. L. P. Livorati, *Physica A* **387**, 1155-1160 (2008).
- [23] A. J. Lichtenberg, M. A. Lieberman and R. H. Cohen, *Physica D* **7**, 291-305 (1980).
- [24] E. D. Leonel and P. V. E. McClintock, *J. Phys. A: Math. Gen.* **38** 823-839 (2005).
- [25] E. D. Leonel, D. F. M. Oliveira and R. E. de Carvalho, *Physica A* **386**, 73-78 (2007).
- [26] A. Y. Loskutov, A. B. Ryabov and L. G. Atkinshin, *J. Exp. Theor. Phys.* **89**, 966 (1999).
- [27] A. Y. Loskutov, A. B. Ryabov and L. G. Atkinshin, *J. Phys. A: Math. Theor.* **33**, 7973 (2000).
- [28] W. Lu, S. Sridhar and M. Zworski, *Phys. Rev. Lett.* **91**, 154101 (2003).
- [29] F. Mortessagne, O. Legrand and D. Sornette, *Chaos* **3**, 529-541 (1993).
- [30] S. Nonnenmacher, *Nonlinearity* **21**, T113-T121 (2008).
- [31] K. Nakamura and T. Harayama, *Quantum chaos and quantum dots* (Oxford University Press, 2004).
- [32] L. D. Pustyl'nikov, *Trans. Moscow Math. Soc.* **2**, 1 (1978).
- [33] C. Poli, B. Dietz, O. Legrand, F. Mortessagne and A. Richter, *Phys. Rev. E* **80**, 035204R (2009).
- [34] J. Sjöstrand, *Duke. Math. J.* **60**, 1-57 (1990).
- [35] F. Saif, *Phys. Rep.* **419**, 207-258 (2005).
- [36] I. Tuval, J. Schneider, O. Piro and T. Tél, *Europhys. Lett.* **65**, 633-639 (2004).

- [37] H. Waalkens, A. Burbanks and S. Wiggins, *Mon. Not. R. Astro. Soc.* **361**, 763-775 (2005).
- [38] J. Wiersig and J. Main, *Phys. Rev. E* **77**, 036205 (2008).
- [39] R. Weaver and M. Wright (Eds.) *New directions in linear acoustic and vibration* (Cambridge University Press, 2010).