Escape of particles in a time dependent potential well

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We investigate the escape of an ensemble of non interacting particles inside an infinitely deep potential box that contains a time-dependent potential well. The dynamics of each particle is described by a two dimensional non-linear area preserving mapping for the variables energy and time, leading to a mixed phase space. The chaotic sea in the phase space surrounds periodic islands and is limited by a set of invariant spanning curves. When a hole is introduced in the energy axis, the histogram of frequency for the escape of particles, which we observe to be scaling invariant, grows rapidly until it reaches a maximum and then decreases toward zero at sufficiently long times. A plot of the survival probability of a particle in the dynamics as function of time is observed to be exponential for short time, reaching a crossover time and turning to a slower decay regime, due to sticky regions observed in the phase space.

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I. INTRODUCTION

After the seminal result of Buttiker [1] on tunneling through a time dependent potential barrier, interest in the dynamics of a particle in driven potential has markedly increased. This dynamics can be described using both theoretical and experimental approaches and can be considered using either classical or quantum characterization. Several applications have been discussed including ballistic conductance in a periodically modulated channel [2], magneto-transport through heterostructures of GaAs/AlGaAs [3], sequential resonant tunneling in semiconductor super-lattices due to intense electrical field [4], influence of transport in the presence of microwaves [5], anomalous transmission in periodic waveguides [6], trapping in driven barrier [7], characterization of traversal time [8, 9], symmetry breaking and drift of particles in a chain of potential barriers [10], Lyapunov characterization of chaotic dynamics and destruction of invariant tori [11] and many other applications [12, 13].

The classical case of a time-dependent one-dimensional potential is a 1.5-degree of freedom problem. The most common formalism used is the so called mapping description. The phase space exhibits a very intricate mixed structure typical of area preserving dynamics, including a set of islands surrounded by a chaotic sea that is confined by a set of invariant tori (also called invariant spanning curves). In many cases, the chaotic dynamics of the particle inside the driven potential leads to very interesting phenomena including power law distribution for the trapping [14], scattering [15] and critical exponents for the average properties of the chaotic sea [16]. It was also shown that an external periodic field changes the asymptotic populations of the asymmetric energy levels [17] and a stochastic perturbation can lead the particle to experience unlimited energy growth [18].

The mixed form of the phase space leads to non-uniformity [19] and sticky domains [20] that produce anomalous transport. A sticky region traps a particle in the phase space and the escape from this region happens at a very long time after the entrance. This part of the orbit is relatively regular and since the particle spends more time in such a sticky zone than elsewhere, important observables like recurrence times [21] and Lyapunov exponents [22] are directly affected. Stickiness may be quantified in terms of the distribution of recurrence times of a typical orbit in the phase space. For fully chaotic dynamics, the decay is exponential [23] while for mixed phase space it is observed to be described by a power law [24].

Recurrence statistics are also intimately related to the statistics of escape from a region containing an initial distribution of particles into a hole, where again exponential or power law decay can be observed for fully chaotic or mixed dynamics respectively [25]. These also provide useful information for the control of chaotic systems [26]. Escape properties measured as a function of a varying hole provide a further sensitive and non-destructive probe of the dynamics [27]. Very recently this approach has also been successfully applied to a time dependent Hamiltonian system [28].

In this paper we consider the dynamics of a classical particle confined inside an infinitely deep potential box which contains an oscillating square well, with the aim of understanding the escape of particles to a defined hole in the energy coordinate. The Hamiltonian that describes the model is $H(x,p,t) = p^2/(2m) + V(x,t)$ where $V(x,t) = V_0(x) + V_1(x,t)$, and $x$, $p$ and $t$ correspond to the position and momentum coordinates and the time respectively. The potential $V_0(x)$ denotes the integrable part of the Hamiltonian while $V_1(x,t)$ leads to the non-integrable part. As we will see in the next section, the potential $V_1(x,t)$ is controlled by three relevant control parameters. If the amplitude of oscillation of the moving well is fixed to zero, the system is integrable and phase space exhibits only regular dynamics. On the other
hand, if $V_1(x,t) \neq 0$, the phase space becomes mixed thereby exhibiting islands, chaotic seas and invariant spanning curves. Depending on the region of the phase space considered, one can observe fully chaotic dynamics leading to an exponential distribution of the recurrence time. However, when islands are present, they produce a power law decay due to the stickiness.

This paper is organized as follows: In Sec. II we describe the model and the map. The numerical results and discussions are also presented in this section. Final discussions are drawn in Sec. III.

II. THE MODEL, THE MAP AND NUMERICAL RESULTS

We have considered the dynamics of a classical particle confined inside a box of infinitely deep potential which contains a periodically oscillating square well in the middle. A typical sketch of the potential is shown in Fig. 1. The potential $V(x,t)$ is given by

$$V(x,t) = \begin{cases} 
\infty, & \text{if } x \leq 0 \text{ or } x \geq (a + b) \\
V_0, & \text{if } 0 < x < \frac{b}{2} \text{ or } (a + \frac{b}{2}) < x < (a + b) \\
V_1 \cos(\omega t), & \text{if } \frac{b}{2} \leq x \leq (a + \frac{b}{2}) 
\end{cases}$$

where the control parameters $a$, $b$, $V_0$, $V_1$ and $\omega$ are constants. Given the symmetry of the problem, we update the variables of the two dimensional mapping at each entrance of the particle in the oscillating square well. A step by step derivation of the equations of the mapping can be obtained in Ref. [16]. It is appropriate to define dimensionless variables $\delta = V_1/V_0$, $r = b/a$, $e_n = E_n/V_0$, $N_c = \omega/(2\pi) (a/\sqrt{2V_0/m})$ and measure the time in terms of the number of oscillations of the moving well, $\phi = \omega t$. Here $N_c$ corresponds to the number of oscillations that the square well completes in a time $t = a/\sqrt{2V_0/m}$. Given that $N_c$ is proportional to $\omega$, changing $N_c$ implies changing the driving frequency $\omega$. The mapping is written as

$$T : \begin{cases} 
e_{n+1} = e_n + \delta [\cos(\phi_n + i\Delta\phi_a) - \cos \phi_n] \\
\phi_{n+1} = [\phi_n + i\Delta\phi_a + \Delta\phi_b] \mod 2\pi 
\end{cases} \quad (2)$$

where the auxiliary variables are given by

$$\Delta\phi_a = \frac{2\pi N_c}{\sqrt{e_n - \delta \cos \phi_n}}, \quad \Delta\phi_b = \frac{2\pi N_c r}{\sqrt{e_{n+1} - 1}},$$

where $i$ is the smallest integer number which makes the relation $[e_n + \delta(\cos(\phi_n + i\Delta\phi_a) - \cos \phi_n)] > 1$ true.

The determinant of the Jacobian matrix is equal to the unity and the mapping (2) is area preserving. The phase space is mixed and contains both periodic islands, a large chaotic sea and a set of invariant spanning curves that prevent the particle to gain unlimited energy from the moving well. Figure 2 shows a typical plot of the phase space for the control parameters $r = 1$, $\delta = 0.5$ and $N_c = 33.18$, which corresponds to a moderate frequency.
of oscillation (see Ref. [15] for specific details). We must emphasize that an initial condition in the chaotic sea can not penetrate the island nor trespass the invariant spanning curve.

Our numerical results will be obtained as a function of the three control parameters \( N_c \), \( \delta \) and \( r \) as well as the number of iterations \( n \). Let us start with the parameter \( N_c \). As it increases, the auxiliary variables \( \Delta \phi_n \) and \( \Delta \phi_0 \), which depends linearly on \( N_c \), increase too. Such a growth reduces the correlation between \( \phi_{n+1} \) and \( \phi_n \) causing the phase space location of the lowest energy invariant spanning curve to rise. For example, the curves for the control parameters \( N_c = 9 \), \( N_c = 10 \) and \( N_c = 20 \) for fixed \( r = 1 \) and \( \delta = 0.5 \) are shown in Fig. 3(a). A plot of the minimum energy along the invariant spanning curve as function of \( N_c \) is shown in Fig. 3(b). A power law fitting gives \( e_{\text{min}} \propto N_c^{\alpha_1} \), where \( \alpha_1 = 0.654(2) \). We stress that \( \alpha_1 \) is closely related to the critical exponent obtained in [29]. Extensions for the other control parameters can also be made as shown in Fig. 3(c) for \( r \), leading to \( e_{\text{min}} \propto r^{\alpha_2} \) with \( \alpha_2 \cong 0.321(7) \) and Fig. 3(d) for \( \delta \) yielding \( e_{\text{min}} \propto \delta^{\alpha_3} \) with \( \alpha_3 \cong 0.66(1) \). Grouping the three control parameters in a single expression we obtain \( e_{\text{min}} \propto N_c^{\alpha_1} r^{\alpha_2} \delta^{\alpha_3} \).

Let us now introduce a energy window through which the particle can escape. Specifically, the particle escapes from the potential when increasing beyond a given critical energy. We define this critical energy as a fraction of the lowest energy among the invariant spanning curve, as for example \( h = 0.7e_{\text{min}} \) (other values were also used). Then, we start an ensemble of \( 10^7 \) particles with low energy, say \( e_0 = 1.01 \), and \( 10^7 \) different initial phases \( \phi_0 \in [0, 2\pi] \), and let them evolve in time (as we discuss below, other values of initial energy were used too). If, along the orbit, the particle reaches the critical level, we determine that the particle escapes from the potential and a new initial condition is chosen. A histogram of the distribution of escape times (rescaled to 1 for visual purposes) is shown in Fig. 4(a). We see that very few particles escape at very short times. The escape rate increases rapidly and reaches a peak, marking a preferred iteration number, which we denote as \( n_p \), and then decreases again, approaching zero asymptotically. When the initial energy is raised, say \( e_0 = 2 \), \( e_0 = 3 \) and so on, the number of iterations needed for the particle to reach the escape hole decreases as far as the initial energy increases. To illustrate this behavior, Fig. 5 shows three plots of \( n_p \) vs \( e_0 \). One see that as the initial energy increases, the particle needs fewer iterations to reach the hole therefore leading to a decrease on \( n_p \). Our simulations however were inconclusive regarding the variation of the parameter \( \delta \) as shown in Fig. 6(a). By considering the two parameters \( N_c \) and \( r \), we can suppose that

\[
n_p \propto N_c^{\alpha_1} r^{\alpha_2},
\]
where $z_1$ and $z_2$ are exponents. Plots of $n_p$ for a large range of $N_c$ and $r$ are shown in Fig. 6(b,c). A power law gives that $z_1 = 1.26(3)$ and $z_2 = 0.57(2)$. The plots shown in Fig. 6(b,c) make us to suppose that the $n_p$ is scaling invariant with respect to $N_c$ and $r$. It means that a proper rescaling to the $n$ axis, i.e., $n \to n/N_c^{z_1}$ for fixed $r$ and $\delta$, will collapse all the curves into a single and universal curve, as shown in Fig. 4(b). Similar procedure for $n \to n/r^{z_2}$, considering fixed $N_c$ and $\delta$, collapses all the curves into a single one as shown in Fig. 4(c).

Let us now discuss the behavior of the survival probability, which we define as

$$P = \frac{1}{N} \sum_{j=1}^{N} N_{\text{surv}}(n),$$

where the summation is taken along the ensemble of $N$ different initial conditions and $N_{\text{surv}}(n)$ is the number of initial conditions that do not have enough energy to escape through the hole until a time $n$. When Eq. (4) is evaluated in a fully chaotic dynamics its behavior is an exponential [27] while for a mixed phase space where periodic orbits exist, the exponential decay turns into a power law [25].

We obtain the behavior of $P$ as a function of $n$ for different positions of the hole, as shown in Fig. 7(a,b). We see that the initial behavior, as shown in Fig. 7(a), is clearly an exponential decay $P \propto \exp(\gamma n)$ until the curve reaches a crossover $n_x$ and changes to a slower decay indicating the existence of sticky regions in the phase space. The slope of the exponential decay is plotted as function of the position of the hole, as shown in Fig. 8. Figure 7(b) shows a merger, for small $n$, of all curves plotted in 7(a) after a rescaling $n \to n/h^{2\alpha_2}$, confirming a scaling invariance of the survival probability for small $n$. Similar overlap was observed for different combination of control parameters as well as different initial energies.

The decay of the survival probability in Fig. 8 is almost independent of the parameter $N_c$. This indicates that for the range of (fixed) $h$ shown, which is in the strongly chaotic region corresponding to $e \lesssim 0.7 e_{\text{min}}$ (see Fig. 2), the transport of energy with $n$ is apparently almost independent of $N_c$.

If we assume that this transport is similar to normal Brownian diffusion, we predict that the number of collisions required to diffuse on average to an energy $h$ is proportional to $h^2$. For $h = 0.7 e_{\text{min}}$ as in figures 4 and 6 this means we expect that $n_p$ is proportional to $e_{\text{min}}^2$ which in turn is proportional to $N_c^2 \alpha_1$ (Fig. 3). However, due to the existence of sticky regions including mainly the islands, the transport is not exactly normal Brownian diffusion. Hence, we can expect that $z_1 \approx 2\alpha_1$ and $z_2 \approx 2\alpha_2$. Such diffusive laws also predict that the distribution of escape times follows a universal curve when scaled appropriately with $N_c$ and $r$ as shown in Fig. 4.

At longer times and when the particle reaches higher positions in the phase space where islands exist, diffusion is not Brownian. The stickiness surrounding the elliptic
III. SUMMARY AND CONCLUSIONS

We studied the problem of a classical particle confined inside a box of infinitely deep potential containing a periodically oscillating square well. The transport of energy was found to be independent of the driving frequency in the low energy, strongly chaotic regime. A histogram of escape frequency at short times was measured and characterized as invariant under a scaling consistent with normal diffusion of energy as a function of the number of collisions $n$. The survival probability plotted as function of $n$ was exponential initially and, after a crossover, followed a slower decay at late times. Both the exponential and the slower decay showed evidence of the stickiness arising from islands in the mixed phase space.

For this model we have used the escape properties with a variable hole to elucidate the dynamics at a level not visible in a phase space plot such as Fig. 2. The escape rate, which focuses on unusual trajectories at long times, exhibits features not accessible to average properties of trajectories. The anomalous dependence of an exponential escape rate with the hole parameter is of particular interest and deserves further study in more general contexts: theoretical and experimental, classical and quantum mechanical.

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