Escape of particles in a time dependent potential well

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We investigate the escape of an ensemble of non interacting particles inside an infinitely deep potential box that contains a time-dependent potential well. The dynamics of each particle is described by a two dimensional non-linear area preserving mapping for the variables energy and time, leading to a mixed phase space. The chaotic sea in the phase space surrounds KAM islands and is limited by a set of invariant spanning curves. When a hole is introduced in the energy axis, the histogram of frequency for the escape of particles, which we observe to be scaling invariant, grows rapidly until it reaches a maximum and then decreases toward zero for sufficiently long time. A plot of the survival probability of a particle in the dynamics as function of time is observed to be exponential for short time, reaching a crossover time and turning to a power law, due to sticky regions observed in the phase space.

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\textbf{I. INTRODUCTION}

After the seminal result of Buttiker [1] for tunneling on a time dependent potential barrier, interest in the dynamics of a particle in driven potential has markedly increased. This dynamics can be described using both theoretical and experimental approaches and can be considered using either classical or quantum characterization. Several applications have been discussed including ballistic conductance in a periodically modulated channel [2], magneto-transport through hetero-structures of GaAs/AlGaAs [3], sequential resonant tunneling in semiconductor super-lattices due to intense electrical field [4], influence of transport in the presence of microwaves [5], anomalous transmission in periodic waveguides [6], trapping in driven barrier [7], characterization of traversal time [8, 9], symmetry breaking and drift of particles in a chain of potential barriers [10], Lyapunov characterization of chaotic dynamics and destruction of invariant tori [11] and many other applications [12, 13].

The classical case of a time-dependent one-dimensional potential is indeed a 1\textsuperscript{st} degree of freedom problem. The most common formalism used is the so called mapping description. The phase space exhibits a very intricate mixed structure typical of area preserving dynamics, namely Kolmogorov-Arnold-Moser (KAM) islands surrounded by a chaotic sea that is confined by a set of invariant tori (also called invariant spanning curves). In many cases, the chaotic dynamics of the particle inside the driven potential leads to very interesting phenomena including power law distribution for the trapping [14], scattering [15] and critical exponents for the average properties of the chaotic sea [16]. It was also shown that an external periodic field changes the asymptotic populations of the asymmetric energy levels [17] and a stochastic perturbation can lead the particle to experience unlimited energy growth [18].

The mixed form of the phase space leads to non-uniformity [19] and sticky domains [20] that produce anomalous transport. A sticky region traps a particle in the phase space and the escape from this region happens after a very long time after the entrance. This part of the orbit is relatively regular and since the particle spends more time in such a sticky zone than elsewhere, important observables like recurrence times [21] and Lyapunov exponents are directly affected [22]. Stickiness may be quantified in terms of the distribution of recurrence times of a typical orbit in the phase space. For fully chaotic dynamics, the decay is exponential [23] while for mixed phase space it is observed to be described by a power law [24].

Recurrence statistics are also intimately related to the statistics of escape from a region containing an initial distribution of particles into a hole, where again exponential or power law decay can be observed for fully chaotic or mixed dynamics respectively [25]. These also provide useful information for the control of chaotic systems [26]. Escape properties measured as a function of a varying hole provide a further sensitive and non-destructive probe of the dynamics [27]. Very recently this approach has also been successfully applied to a time dependent Hamiltonian system [28].

In this paper we consider the dynamics of a classical particle confined inside an infinitely deep potential box which contains an oscillating square well, with the aim of understanding the escape of particles to a defined hole in the energy coordinate. The Hamiltonian that describes the model is $H(x, p, t) = \frac{p^2}{2m} + V(x, t)$ where $V(x, t) = V_0(x) + V_1(x, t)$, and $x$, $p$ and $t$ correspond to the position and momentum coordinates and the time respectively. The potential $V_0(x)$ denotes the integrable part of the Hamiltonian while $V_1(x, t)$ leads to the non-integrable part. As we will see in the next section, the potential $V_1(x, t)$ is controlled by three relevant control parameters. If the amplitude of oscillation of the moving well is fixed to zero, the system is integrable and phase
space exhibits only regular dynamics. On the other hand, if $V_1(x,t) \neq 0$, the phase space becomes mixed thus exhibiting KAM islands, chaotic seas and invariant spanning curves. Depending on the region of the phase space considered, one can observe fully chaotic dynamics leading to an exponential distribution of the recurrence time. However, when islands are present, they produce a power law decay due to the stickiness.

This paper is organized as follows: In Sec. II we describe the model and the map. The numerical results and discussions are also presented in this section. Final discussions are drawn in Sec. III.

II. THE MODEL, THE MAP AND NUMERICAL RESULTS

We have considered the dynamics of a classical particle confined inside a box of infinitely deep potential which contains a periodically oscillating square well in the middle. A typical sketch of the potential is shown in Fig. 1. The potential $V(x,t)$ is given by

$$V(x,t) = \begin{cases} 
\infty, & \text{if } x \leq 0 \text{ or } x \geq (a + b) \\
V_0, & \text{if } 0 < x < \frac{b}{2} \text{ or } (a + \frac{b}{2}) < x < (a + b) \\
V_1 \cos(\omega t), & \text{if } \frac{b}{2} \leq x \leq (a + \frac{b}{2}) 
\end{cases}$$

(1)

where the control parameters $a$, $b$, $V_0$, $V_1$ and $\omega$ are constants. Given the symmetry of the problem, we update the variables of the two dimensional mapping at each entrance of the particle in the oscillating square well. A step by step derivation of the equations of the mapping can be obtained in Ref. [16]. It is appropriate to define dimensionless variables $\delta = V_1/V_0$, $r = b/a$, $e_n = E_n/V_0$, $N_c = \omega/(2\pi) (a/\sqrt{2V_0/m})$ and measure the time in terms of the number of oscillations of the moving well, $\phi = \omega t$. Here $N_c$ corresponds to the number of oscillations that the square well completes in a time $t = a/\sqrt{2V_0/m}$. Then the mapping is written as

$$T : \begin{cases} 
eq n + \delta [\cos(\phi_n + i\Delta \phi_a) - \cos \phi_n] \\
\phi_{n+1} = [\phi_n + i\Delta \phi_a + \Delta \phi_b] \mod 2\pi 
\end{cases}$$

(2)

where the auxiliary variables are given by

$$\Delta \phi_a = \frac{2\pi N_c}{\sqrt{e_n - \delta \cos(\phi_n)}}, \quad \Delta \phi_b = \frac{2\pi N_c r}{\sqrt{e_{n+1} - 1}},$$

where $i$ is the smallest integer number which makes the relation $|e_n + \delta (\cos(\phi_n + i\Delta \phi_a) - \cos(\phi_n))| > 1$ true.

The determinant of the Jacobian matrix is equal to the unity and the mapping (2) is area preserving. The phase space is mixed and contains both KAM islands, a large chaotic sea and a set of invariant spanning curves that prevent the particle to gain unlimited energy from the moving well. Figure 2 shows a typical plot of the phase space for the control parameters $r = 1$, $\delta = 0.5$ and $N_c = 33.18$, which corresponds to a moderate frequency of oscillation (see Ref. [15] for specific details). We must emphasize that an initial condition in the chaotic sea can not penetrate the island nor trespass the invariant spanning curve.
FIG. 3: (a) Position of the lowest energy invariant spanning curves for the control parameters $N_c = 9$, $N_c = 10$ and $N_c = 20$. (b) Plot of $e_{\text{min}} \times N_c$. The control parameters used were $r = 1$ and $\delta = 0.5$.

From now on we will keep fixed the control parameter $r = 1$, which denotes the balanced case, and $\delta = 0.5$. The numerical results will be obtained as a function of $N_c$, which describes essentially the frequency of oscillation of the moving well. Results of other variations of $r$ and $\delta$ will be published elsewhere. As the parameter $N_c$ rises, the auxiliary variables $\Delta \phi_a$ and $\Delta \phi_b$, which depends linearly on $N_c$ increase. Such a growth reducing the correlation between $\phi_{n+1}$ and $\phi_n$. This reduction makes the cos functions (see first equation of map (2)) work more randomly causing the phase space location of the lowest energy invariant spanning curve to rise. For example, the curves for the control parameters $N_c = 9$, $N_c = 10$ and $N_c = 20$ are shown in Fig. 3(a). A plot of the minimum energy along the invariant spanning curve as function of $N_c$ is shown in Fig. 3(b). A power law fit gives $e_{\text{min}} \propto N_c^{\alpha_1}$, where $\alpha_1 = 0.654(2) \approx 2/3$. We stress that $\alpha_1$ is closely related to the critical exponent used in the characterization of a phase transition from integrability to non-integrability [29].

Let us now introduce a energy window through which the particle can escape, specifically, the particle escapes from the potential when increasing beyond a given critical energy. We define this critical energy as $h = 0.7 e_{\text{min}}$. We start an ensemble of $10^7$ particles with low energy, say $e_0 = 1.01$, and $10^7$ different initial phases $\phi_0 \in [0, 2\pi)$, and let them evolve in time. If along the orbit, the particle reaches the critical level, we determine that the particle escape from the potential and a new initial condition is chosen. A histogram of the distribution of escape times (rescaled to 1 for visual purposes) is shown in Fig. 4(a). A plot of the number of iterations for different control parameters $N_c$, labeled in the figure with $r = 1$ and $\delta = 0.5$. (b) Different histograms of escape frequency merge to a single universal function, confirming the scaling invariance.

We also see that when parameter $N_c$ grows, the $n_p$ also grows, leading us to suppose that

$$n_p \propto N_c^z,$$

where $z$ is an exponent. A plot of $n_p$ for a large range of $N_c$ is shown in Fig. 5 where a power law gives that $z = 1.26(3)$. This behavior makes us to suppose that the $n_p$ is scaling invariant. It means that a proper rescaling to the $n$ axis, i.e. $n \rightarrow n/N_c^z$, will collapse all the curves into a single and universal curve, as shown in Fig. 4(b).

Let us now discuss the behavior of the survival probability, which we define as

$$P = \frac{1}{N} \sum_{j=1}^{N} N_{\text{surv}}(n),$$

where the summation is taken along the ensemble of $N$ different initial conditions and $N_{\text{surv}}(n)$ is the number of initial conditions that do not have enough energy to escape through the hole until a time $n$. When Eq. (4)
FIG. 5: Plot of $n_p \times N_c$. A power law gives that $z = 1.26(3)$. The control parameters used were $r = 1$ and $\delta = 0.5$.

FIG. 6: (a) Plot of $P \times n$ for different positions of the hole, namely: $h = 4$, $h = 6$, $h = 10$ and $h = 12$. The control parameters used were $N_c = 33.18$, $r = 1$ and $\delta = 0.5$. The dashed lines were obtained via an exponential fit. (b) Merger, for small $n$, of all curves plotted in 6(a) after a rescaling $n \rightarrow n/h^{2.5}$, confirming a scaling invariance of the survival probability for small $n$.

FIG. 7: Slope of the exponential decay $\gamma$ as function of $h$ for three different values $N_c = 33.18$, $N_c = 66.36$ and $N_c = 99.54$ for $r = 1$ and $\delta = 0.5$. The straight lines are the best fit for each $N_c$.

is evaluated in a fully chaotic dynamics its behavior is an exponential [27] while for a mixed phase space where periodic orbits exist, the exponential decay turns into a power law [25].

We obtain the behavior of $P$ as a function of $n$ for different positions of the hole, as shown in Fig. 6(a,b). We see that the initial behavior, as shown in Fig. 6(a), is clearly an exponential decay $P \propto \exp(\gamma n)$ until the curve reaches a crossover $n_p$ and turns into a power decay, indicating the existence of sticky regions in the phase space. The slope of the exponential decay is plotted as a function of the position of the hole, as shown in Fig. 7. Figure 6(b) shows a merger, for small $n$, of all curves plotted in 6(a) after a rescaling $n \rightarrow n/h^{2.5}$, confirming a scaling invariance of the survival probability for small $n$.

The decay of the survival probability in Fig. 7 is almost independent of the parameter $N_c$. This indicates that for the range of (fixed) $h$ shown, which is in the strongly chaotic region corresponding to $e \lesssim 0.7 e_{\min}$ (see Fig. 2), the transport of energy with $n$ is apparently almost independent of $N_c$.

If we assume that this transport is similar to normal Brownian diffusion, we predict that the number of collisions required to diffuse on average to an energy $h$ is proportional to $h^2$. For $h = 0.7 e_{\min}$ as in figures 4 and 5 this means we expect that $n_p$ is proportional to $e_{\min}^2$ which in turn is proportional to $N_c^{2\alpha}$ (Fig 3). Hence we have $z = 2\alpha + 1$ which is within the numerical uncertainties. Such a diffusive law also predicts that the distribution of escape times follows a universal curve when scaled appropriately with $N_c$ as in Fig. 4.

At longer times, however, diffusion is not Brownian. The stickiness surrounding the elliptic islands leads to the power law decays observed at the longest times in Fig. 6. Interestingly, even in the intermediate time regime, where decay is approximately exponential, there is a subtle deviation from a normal diffusive law: The exponents in Fig. 7 are significantly different from $-2$. Thus the stickiness corresponding to very small elliptic islands, many of which are invisible in Fig 2, has the effect of enhancing long time correlations, even in an apparently strongly chaotic regime.

III. SUMMARY AND CONCLUSIONS

We studied the problem of a classical particle confined inside a box of infinitely deep potential containing a peri-
odically oscillating square well. The transport of energy was found to be independent of the driving frequency in the low energy, strongly chaotic regime. A histogram of escape frequency at short times was measured and characterized as invariant under a scaling consistent with normal diffusion of energy as a function of the number of collisions \( n \). The survival probability plotted as function of \( n \) was exponential initially and, after a crossover, followed a power law at late times. Both the exponential and the power law decay showed evidence of the stickiness arising from islands in the mixed phase space.

For this model we have used the escape properties with a variable hole to elucidate the dynamics at a level not visible in a phase space plot such as 2. The escape rate, which focuses on unusual trajectories at long times, exhibits features not accessible to average properties of trajectories. The anomalous dependence of an exponential escape rate with the hole parameter is of particular interest and deserves further study in more general contexts: theoretical and experimental, classical and quantum mechanical.

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