Quantum Billiards

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Mathematical Billiard and their Applications

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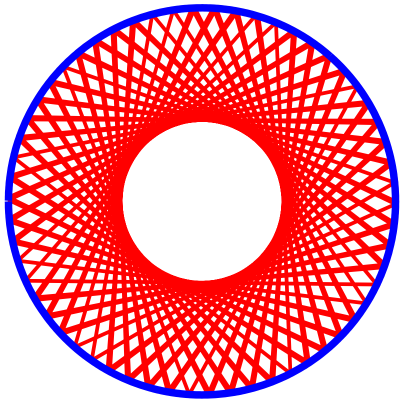
Most pictures are courtesy of Arnd Bäcker

Examples of wavefunctions

circle



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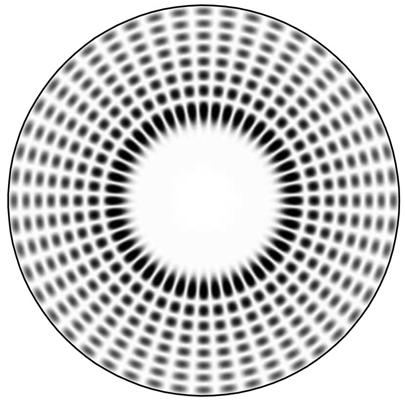
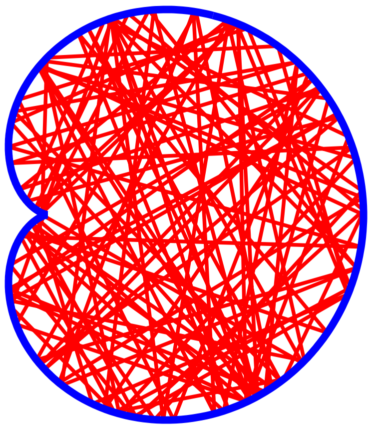


cardioid



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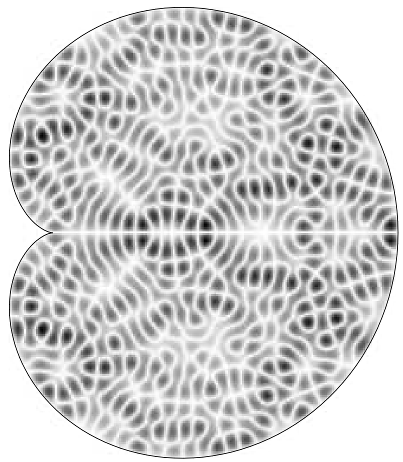
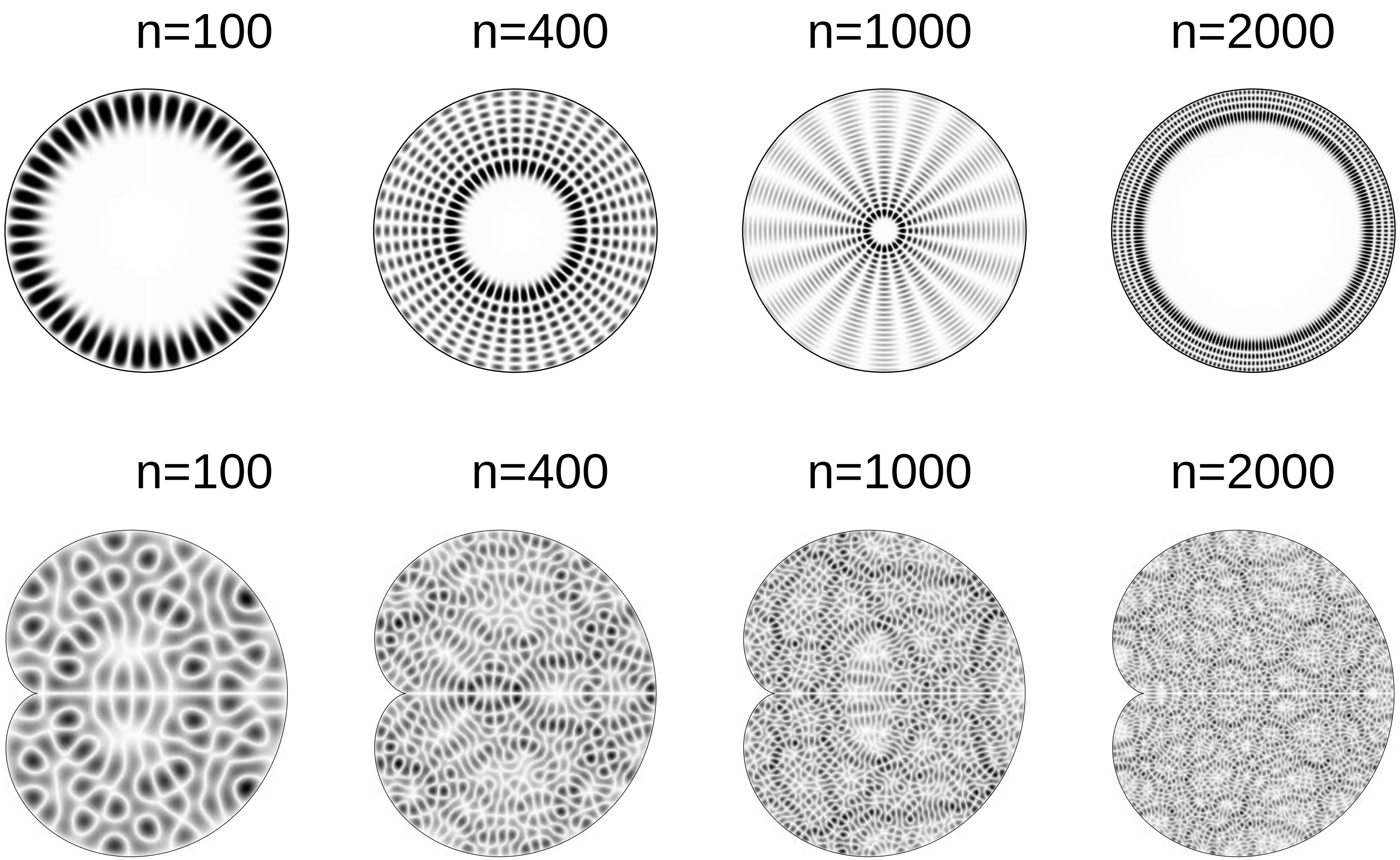


Examples of wavefunctions



Quantum ergodicity

Typical wavefunctions of chaotic billiard systems seem to be uniformly distributed over the area of the billiard. In ergodic systems a typical trajectory explores the energy shell uniformly. A natural analogy for wavefunctions would be that the quantum eigenstates ψ_n of an ergodic system are uniformly distributed over the energy shell. That is

$$\int \mathrm{d}^2\boldsymbol{q}\,|\psi(\boldsymbol{q})|^2\,g(\boldsymbol{q}) \sim \frac{1}{\Omega}\int \mathrm{d}^2\boldsymbol{q}\,\mathrm{d}^2\boldsymbol{p}\,g(\boldsymbol{q})\,\delta(E-H(\boldsymbol{q},\boldsymbol{p}))$$

as $n \to \infty$, where $g(\boldsymbol{q})$ is a bounded continuous function.

This is indeed correct for *almost all* eigenstates. This is the content of the *quantum ergodicity theorem*

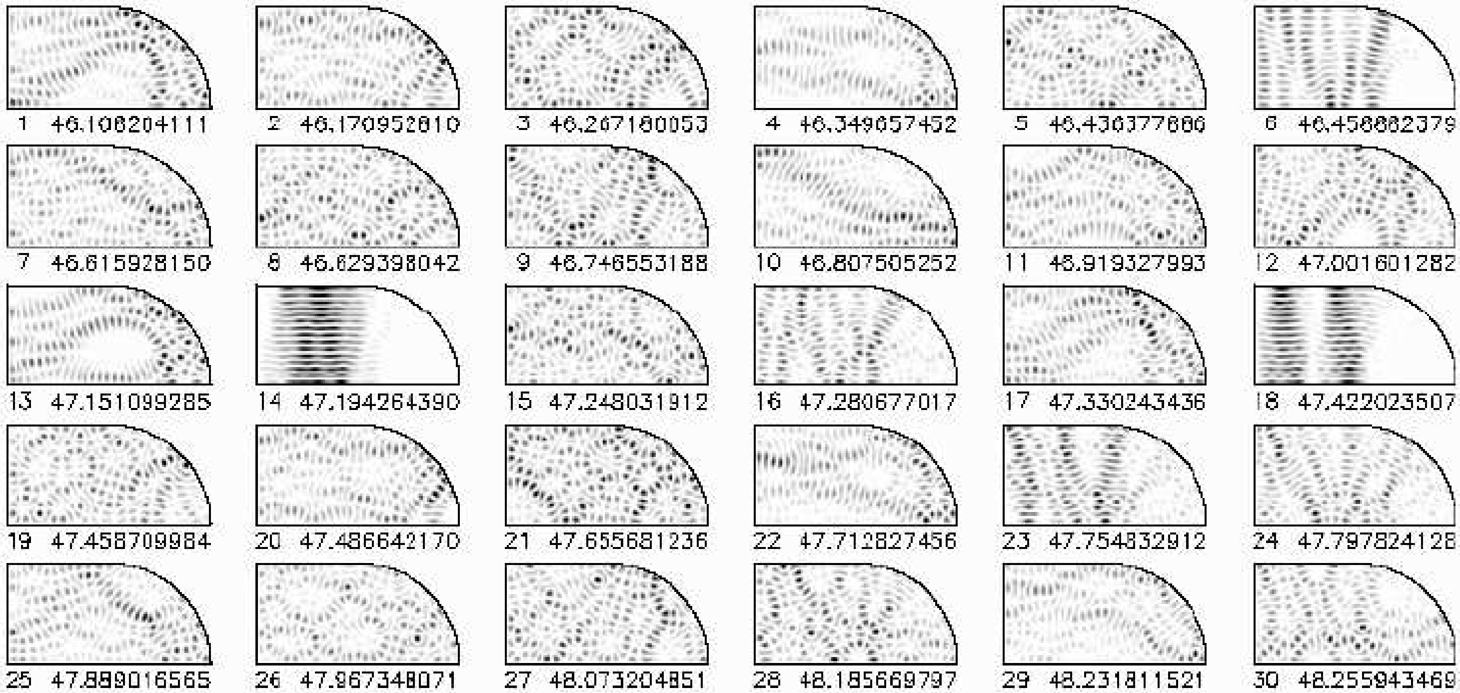
Quantum ergodicity theorem

Quantum ergodicity theorem (Shnirelman, Zelditch, Colin de Verdière): The relation on the previous page holds for subsequences of the eigenstates of density one, i.e. sequences $\{\psi_{j_n} | n = 1, 2, \ldots\}$ such that

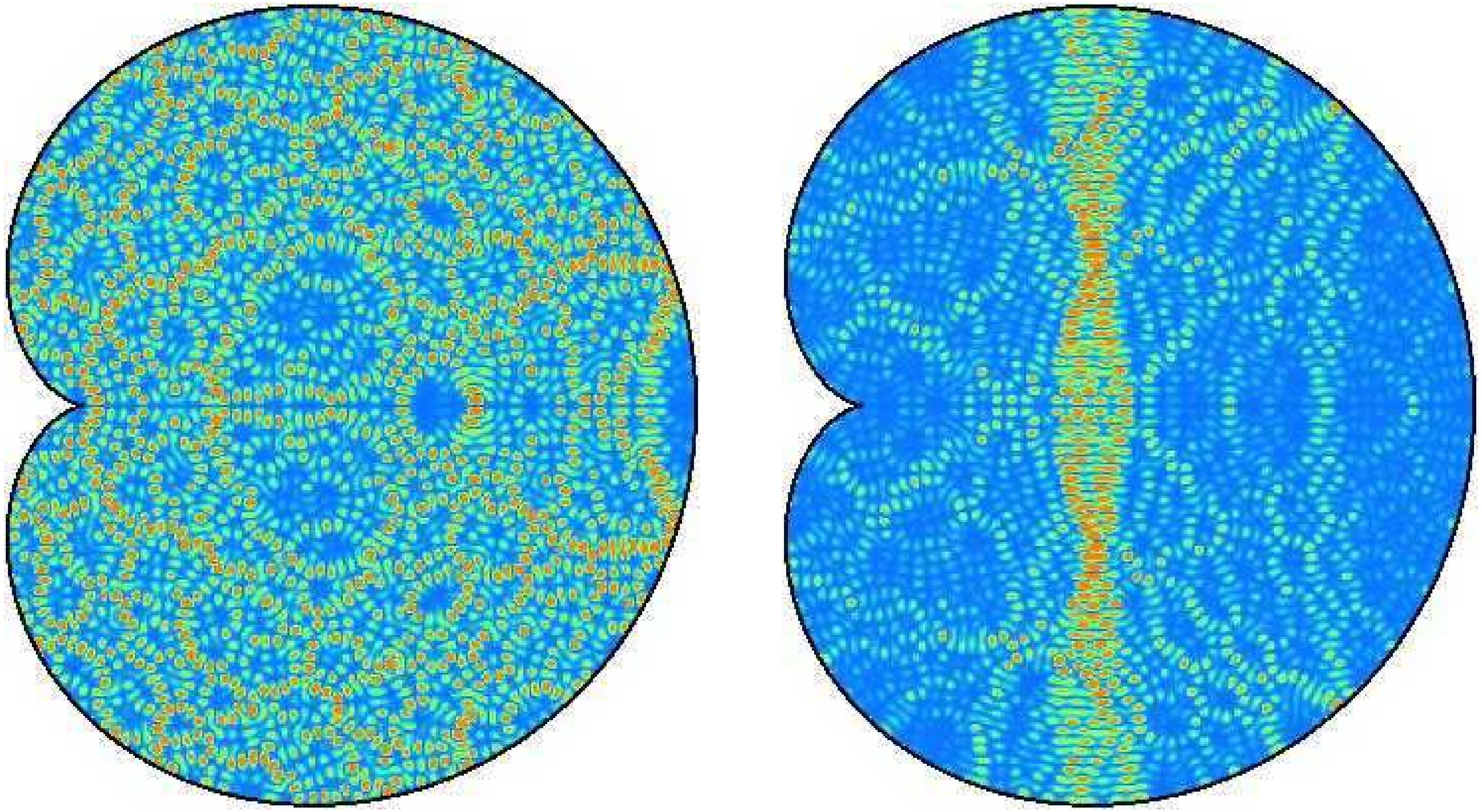
$$\lim_{N\to\infty} \frac{\#\{n \in \mathbb{N} | j_n \leq N\}}{N} = 1$$

The theorem allows for exceptional subsequences of eigenstates which do not approach the uniform distribution in the semiclassical limit. Candidates for such exceptional states are, for example, states that are concentrated on the families of neutral orbits in the stadium or Sinai billiard, the “bouncing ball states”.

Eigenfunctions of the stadium billiard



The states 1816 and 1817



Scars

An issue that is controversially discussed are “scars”, the concentration of wavefunctions along *unstable* periodic orbits (Heller 1984). Numerical calculations show scarring for low-lying states, but it is debated whether scars appear for arbitrarily high quantum numbers.

If there are no exceptional states then *quantum unique ergodicity* holds. This means that the sequence of all eigenstates approaches the uniform distribution on the energy shell in the semiclassical limit.

Rudnick and Sarnak conjectured that quantum unique ergodicity holds for general Riemann surfaces of constant negative curvature, but proofs exist only for certain surfaces with arithmetic properties (Lindenstrauss).

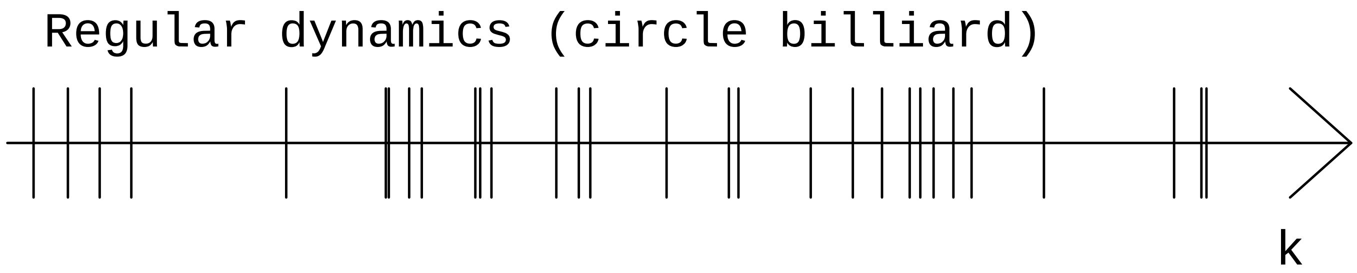
The semiclassical eigenfunction hypothesis

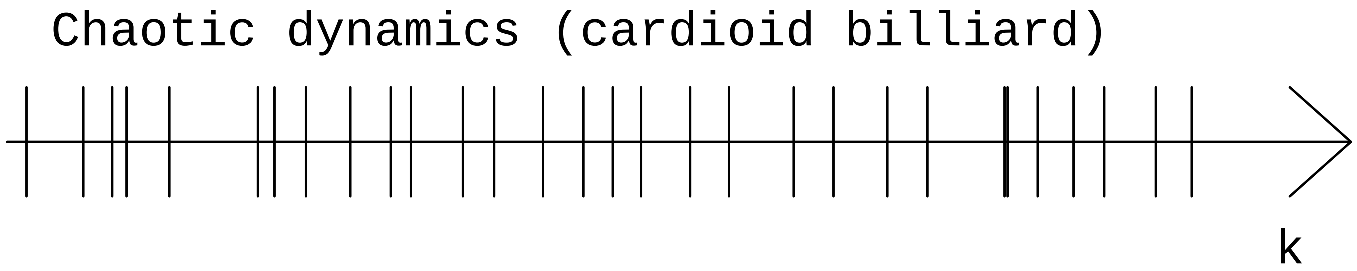
Berry and Voros conjectured that eigenstates concentrate in the semiclassical limit on regions in phase space that are explored by a typical classical trajectory

- In chaotic systems this agrees with quantum ergodicity
- In integrable systems wavefunctions concentrate on tori
- In mixed systems it is expected that wavefunctions appear in two types, those that concentrate on chaotic regions and those that concentrate on invariant tori.

Berry conjectured further, that eigenfunctions of chaotic systems can be represented as a random superposition of plane waves (de Brogly waves) in the semiclassical limit.

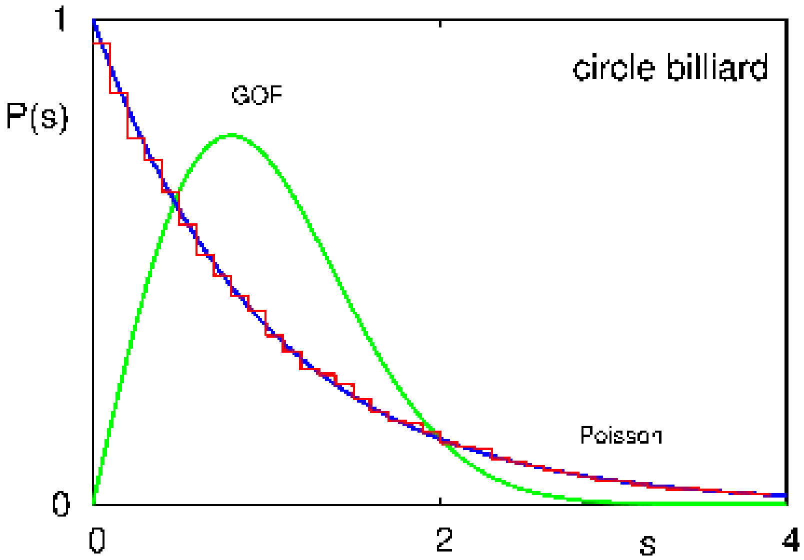
Quantum spectra

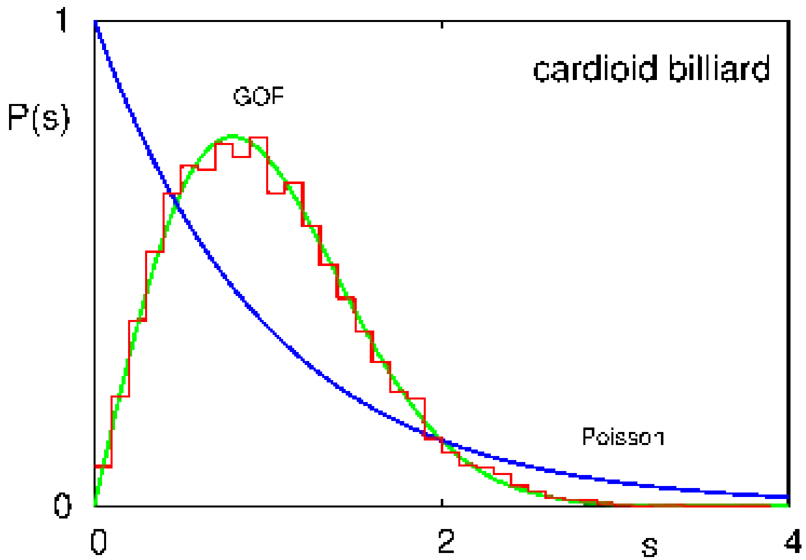




Sample of spectra of a regular (above) and a chaotic system (below). In order to compare different spectra one has to scale them so that the mean spacing between neighbouring levels is one.

The spacings distribution





The distribution of spacings between adjacent energy levels (nearest neighbour spacings distribution) of an integrable and a chaotic system, in comparison with a Poison distribution e^{-s} and the GOE distribution from Random Matrix Theory (RMT).

Conjectures on spectra

Random Matrix Hypothesis (Bohigas, Gianonni, Schmit 1984): The energy eigenvalues of quantum systems with a chaotic classical limit are distributed in the semiclassical limit like eigenvalues of random matrices.

The conjecture applies to systems without discrete symmetries like rotation symmetries or reflection symmetries. Alternatively, it applies to subsets of eigenstates that share the same eigenvalues of these symmetry operators. The random matrix ensemble depends on the remaining symmetries

- Gaussian Orthogonal Ensemble (GOE): real symmetric matrices. Relevant to systems with time-reversal symmetry
- Gaussian Unitary Ensemble (GUE): hermitian matrices. Relevant to systems without time-reversal symmetry
- Gaussian Symplectic Ensemble (GSE): real quaternion matrices. Relevant to systems with time-reversal symmetry and half-integer spin

Further conjectures

Integrable systems (Berry, Tabor): The energy eigenvalues of quantum systems with an integrable classical limit are distributed in the semiclassical limit like independent random numbers (Poison distribution)

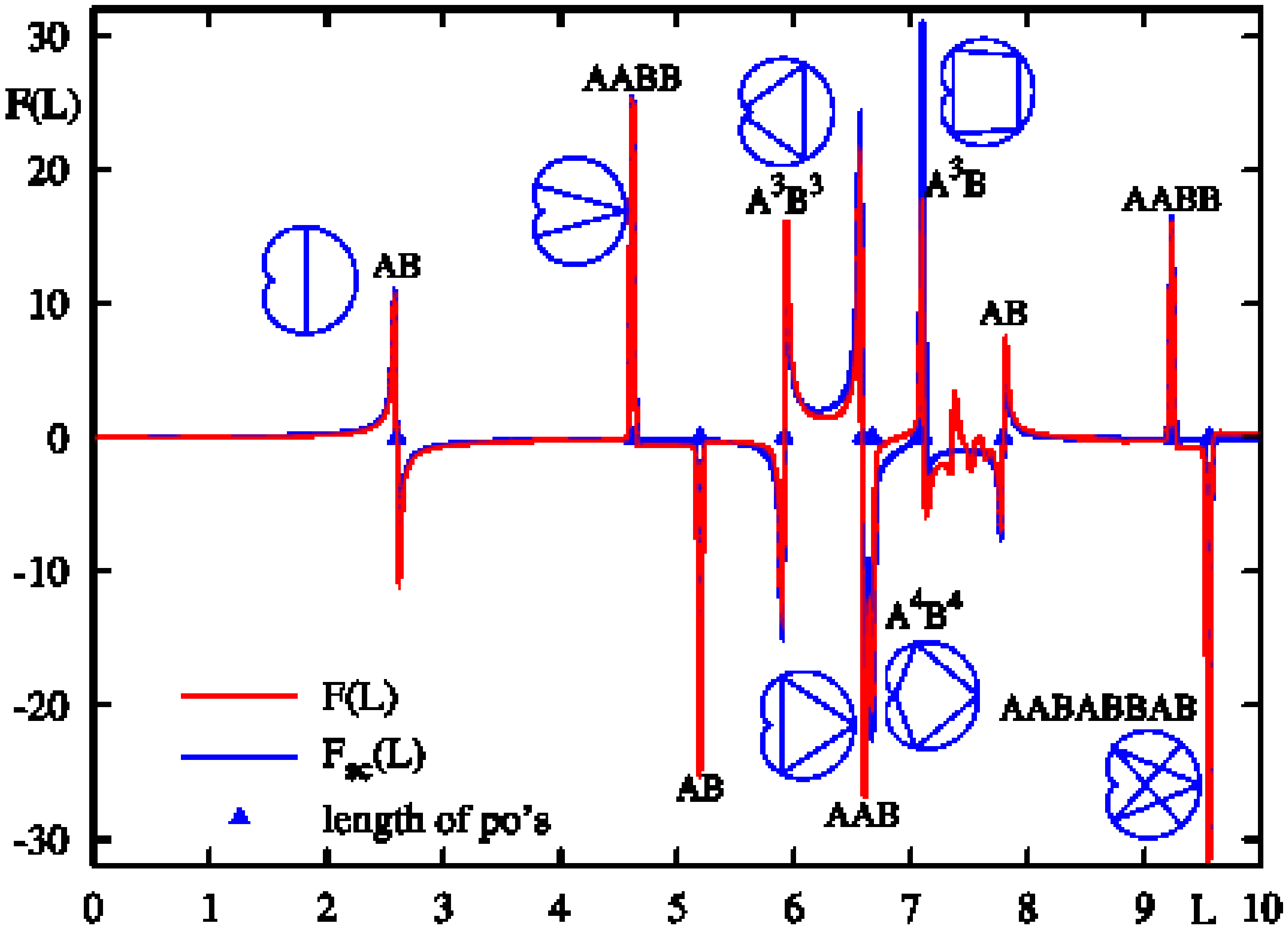
As in the case of chaotic systems there are exceptions. An example is the isotropic harmonic oscillator. The conjecture was proved for some integrable systems (relation to the value distribution of quadratic forms)

Mixed systems (Berry, Robnik): The quantum spectra of mixed systems have a chaotic component (with random matrix distribution) and a regular component (with Poison distribution) in the semiclassical limit. Both components are superposed and their relative weights depend on the size of the chaotic and the regular regions in phase space.

Numerical investigations suggest that this can be seen only for very large level numbers.

Fourier transform

The Fourier transform of the density of states has peaks at the periods of the periodic orbits



Numerical results for the hyperbola billiard

The hyperbola billiard is an unbounded billiard system with boundaries $y = 0$, $y = x$, and $y = 1/x$. The dashed line shows the quantum result, and the full line the semiclassical result