Lecture I: Dispersing billiards in 2D and 3D

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Mathematical Billiards and their Applications
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Plan

1. Today: Dispersing (Sinai) Billiards
   - in $2D$: uniform hyperbolicity, strong ergodic properties
   - in $3D$: similar phenomena, but serious technical complications
2. Tomorrow: Planar billiards with intermittency.
   - billiards with cusps and tunnels: WIP with Chernov and Dolgopyat
   - comparisions: stadia, infinite horizon...

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Outline for Lecture I

Planar dispersing billiards
- Results
- Phenomena

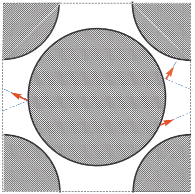
Dispersing Billiards in 3D
- Results
- Phenomena

Singularities in 3D dispersing billiards
- Unbounded curvature
- Example with exponential complexity

Billiards in 2D

$Q = \mathbb{T}^2 \setminus \bigcup_{k=1}^{K} C_k$ strictly convex scatterers

- Billiard flow : $S^t : \mathcal{M} \to \mathcal{M}$, $(q, v) \in \mathcal{M} = Q \times \mathbb{S}^1$, $|v| = 1$ Uniform motion within Q, elastic reflection at the boundaries
- Billiard map phase space: $M = \bigcup_{k=1}^{K} M_k$
- $(r, \phi) \in M_k$, r: arclength along ∂C_k, $\phi \in [-\pi/2, \pi/2]$ outgoing velocity angle
- invariant measure $d\mu = c \cos\phi \, dr \, d\phi$



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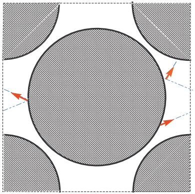

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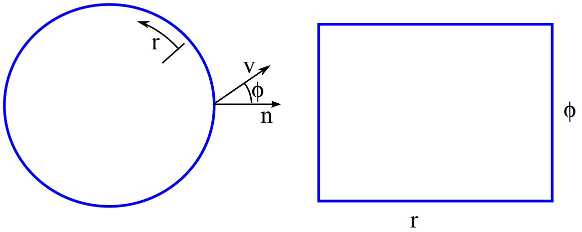

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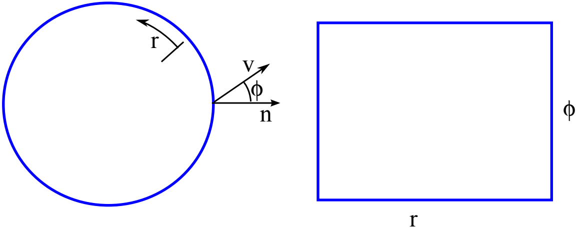

Sinai billiards in 2D

C_k are C^3 smooth and disjoint (no corner points);
finite horizon: flight length uniformly bounded from above

- Billiard map is ergodic, K-mixing (Sinai '70)
- EDC: $f, g : M \to \mathbb{R}$ Hölder continuous, $\int f d\mu = \int g d\mu = 0$ let $C_n(f,g) = \mu(f \cdot g \circ T^n)$, then $|C_n(f,g)| \leq C\alpha^n$ for suitable $C > 0$ and $\alpha < 1$
  - Young '98 – tower construction with exponential tails,
  - Chernov & Dolgopyat '06 – standard pairs

  crucial: Growth Lemma on unstable curves
- CLT: let $S_n f = f + f \circ T + ... + f \circ T^{n-1}$, then $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma)$ where $\sigma = \int f^2 d\mu + 2 \sum_{n=1}^{\infty} C_n(f,f)$.
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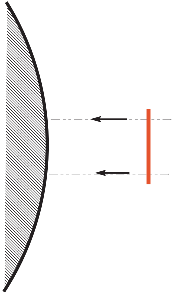
Unstable curves

Neutral (or convex) wavefront → Convex front

Definition

U-curve W: Trace of a convex front on M.

- Increasing in the r, ϕ coordinates.
- Invariant and expanding under T. In particular: $\exists \Lambda > 1$ such that $\rho(Tx, Ty) \geq \Lambda \rho(x, y)$, $\forall W, \forall x, y \in W$



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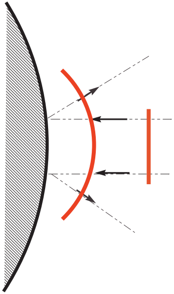

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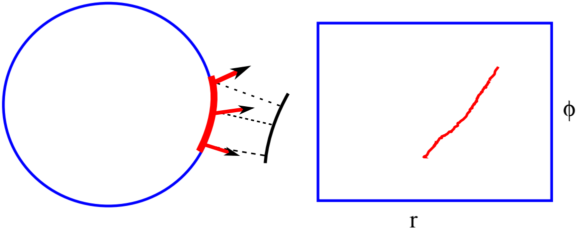

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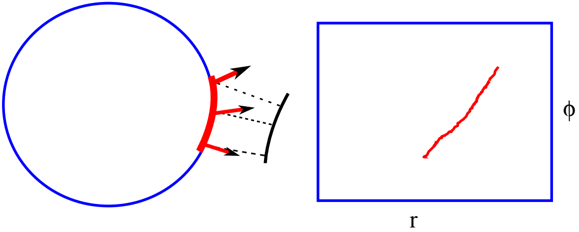
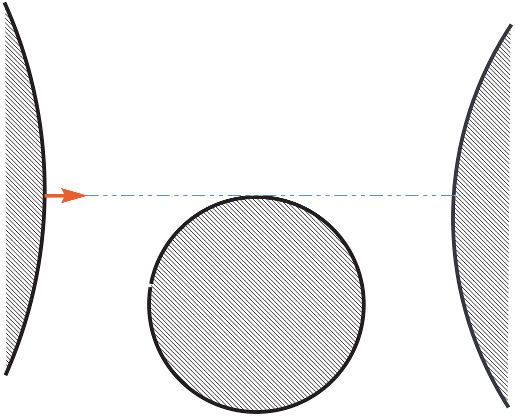


Singularities I

Preimages of tangencies: T discontinuous, S^t non-differentiable



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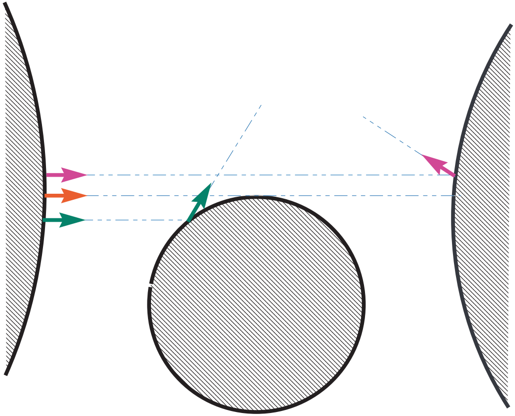
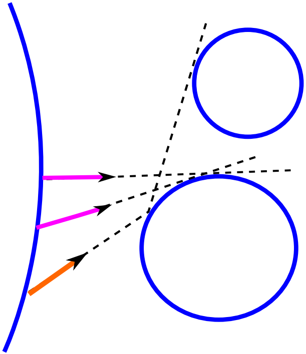


Singularities II

$S_n = T^{-n} S_0$ where S_0 is the tangency

Discontinuity set for T^n: $S^{(n)} = \cup_{i=0}^{n} S_i$

- The S_n are smooth Decreasing curves in the r, ϕ coordinates.
- $S^{(n)}$ fills M more and more densely as n increases.



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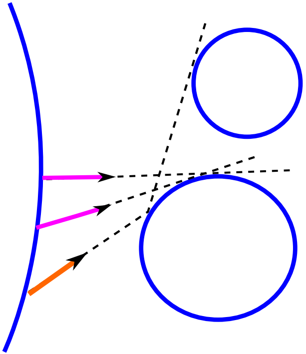

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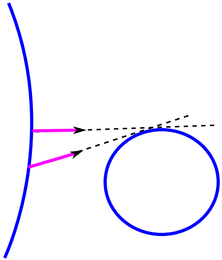

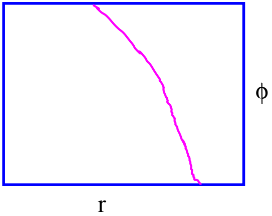

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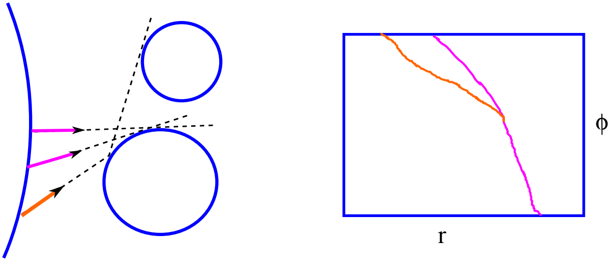


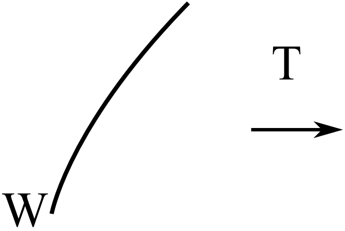
Evolution of u-curves

W (sufficiently small) u-curve TW

- increases in length
- partitioned by the singularities

Expansion prevails fractioning: "Most" components of W are "long"

How to quantify this?



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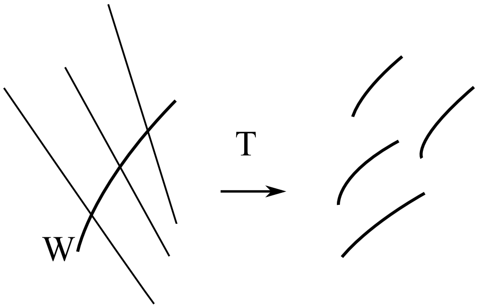

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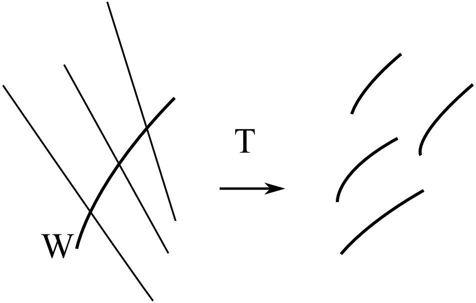

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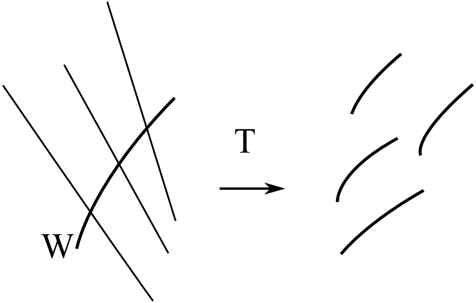

The Growth Lemma

- W is small u-curve, m_W Lebesgue measure on W.
- G_ε: set of points in W that are at most ε from the boundary:

$$G_\varepsilon = \{x \in W \mid \rho(x, \partial W) \le \varepsilon\}.$$

- H_ε: set of points in W that will be at most ε from the boundary.

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If there were no singularities: $m_W(H_\varepsilon) \le m_W(G_{\varepsilon/\Lambda})$.

Lemma

There exists a constant $\lambda < \Lambda$, independent of W, such that

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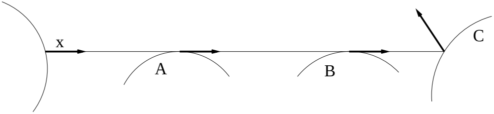
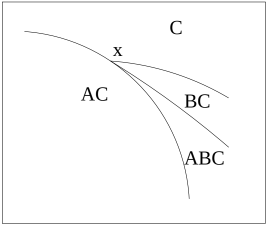
Complexity of the singularity set

Definition

$K_n(x)$, n-step complexity of a point $x \in M$: number of different symbolic collision sequences that can be observed in the vicinity of x.

n-step complexity of the singularity set: $K_n = \sup_{x \in M} K_n(x)$





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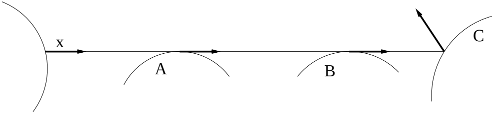
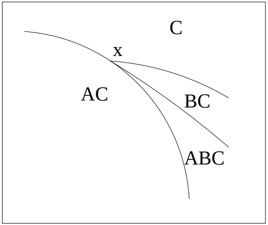




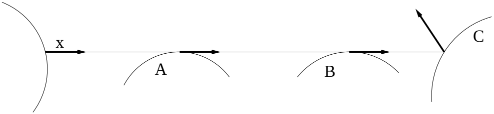
Subexponential complexity

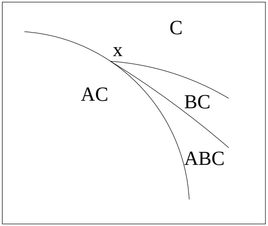
subexponential growth of complexity:

$$\exists C > 0 \text{ and } \lambda < \Lambda \text{ such that } K_n \le C\lambda^n$$

Lemma

Bunimovich, 1991: In 2D Sinai billiards (finite horizon, no corner points) K_n grows at most linearly.





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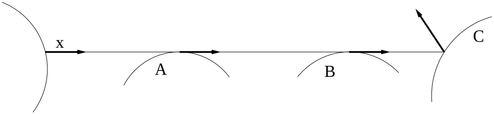
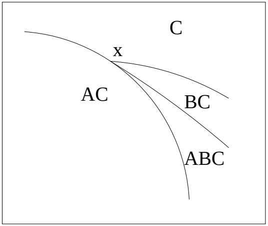
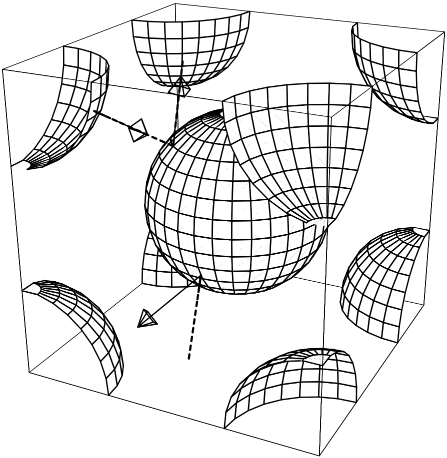




Billiard dynamics in 3D



- M: hemisphere-bundle, $\dim M = 4$
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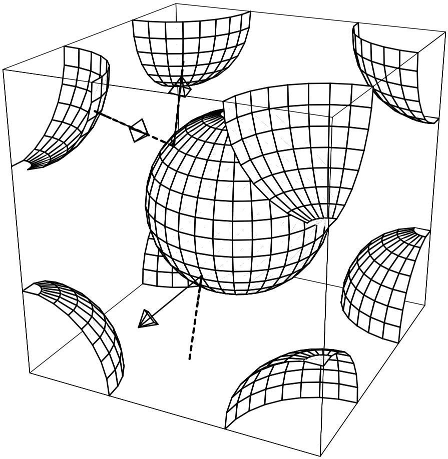
Billiard dynamics in 3D



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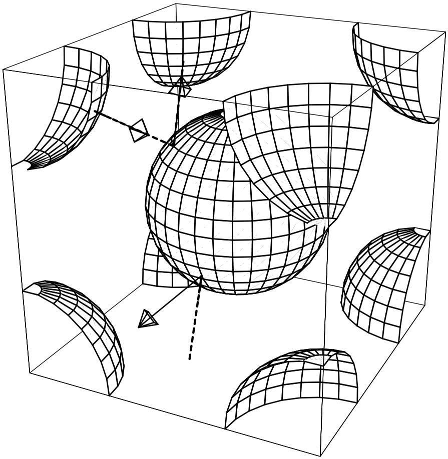

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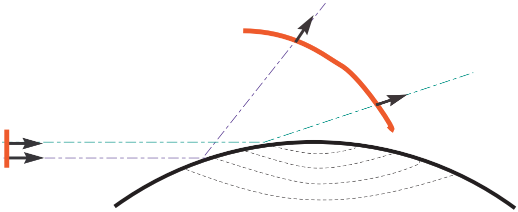
What is responsible for all this...

- Unbounded expansion near singularities (highly nonlinear, applies to 2D)
- in 3D highly anisotropic expansion near singularities cf. astigmatism



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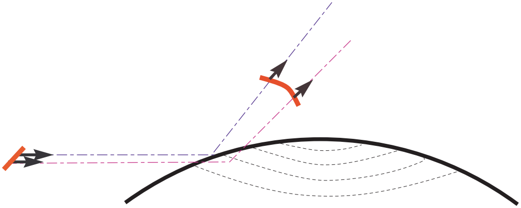


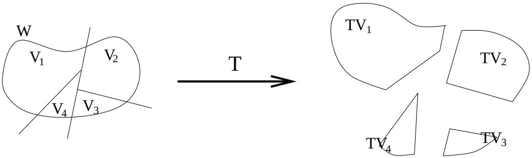
Growth Lemma

W is a small u-manifold (2 dimensional)

$$G_\varepsilon = \{x \in W \,|\, \rho(x, \partial W) \le \varepsilon\}\,.$$

$$H_\varepsilon = \{x \in W \,|\, \rho(Tx, \partial(TW)) \le \varepsilon\}\,.$$

$m_W(H_\varepsilon) \le \lambda\, m_W(G_{\varepsilon/\Lambda})$ with $\lambda < \Lambda$.



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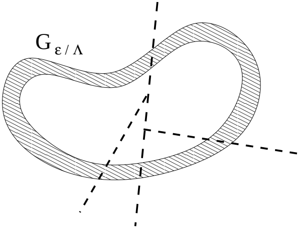

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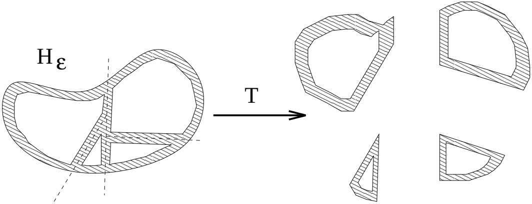

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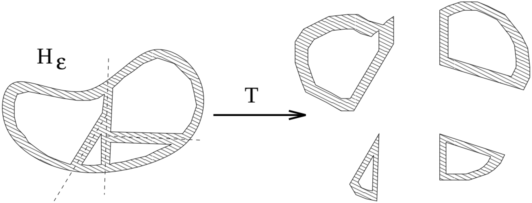

The pathological intersection I

S_0: tangency, $S_1 = T^{-1}S_0$, $S_2 = T^{-2}S_0$

- in 2D: $S_1 \cap S_2$ is single point
- in 3D: $S_1 \cap S_2$ has structure, $\dim(S_1 \cap S_2) = 2$
- S_2 terminates on S_1 typically tangentially,
- transversally in a one dimensional pathological set $P \subset S_1 \cap S_2$

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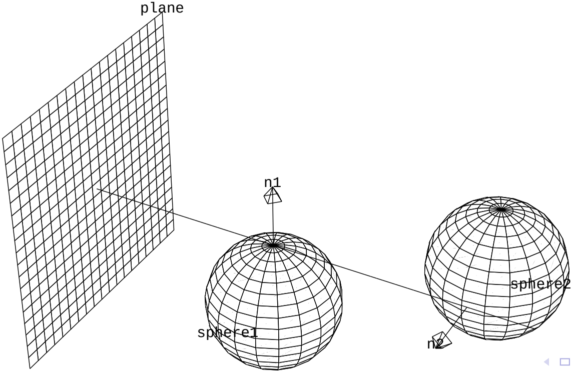
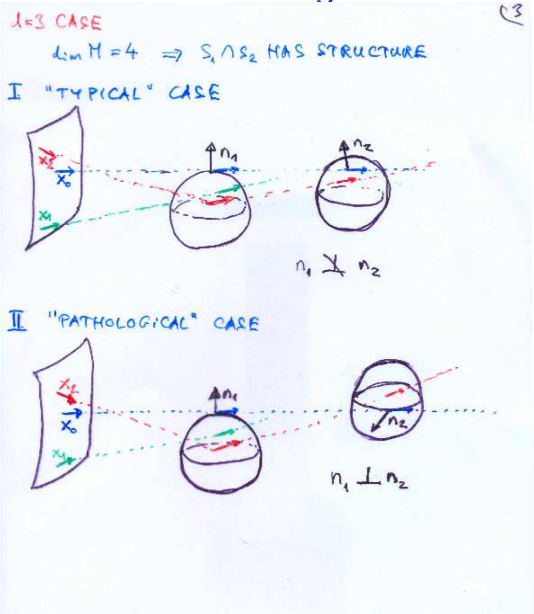


Pathological intersection II

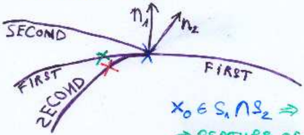


Pathological intersection III

4.

WHAT DO I SEE FROM THE STARTING POINT?

I "TYPICAL CASE"



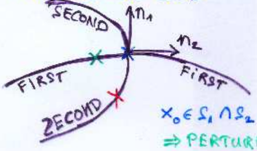
TANGENTIAL INTERSECTION

$x_0 \in S_1 \cap S_2 \Rightarrow$

$\Rightarrow$ PERTURB OF ORDER $\boxed{\varepsilon}$: $x_1 \in S_1 \Rightarrow$

$\Rightarrow$ PERTURB OF ORDER $\boxed{\varepsilon^2}$: $x_2 \in S_2$

II "PATHOLOGICAL CASE"



TRANSVERSAL INTERSECTION

$x_0 \in S_1 \cap S_2 \Rightarrow$

$\Rightarrow$ PERTURB OF ORDER $\boxed{\varepsilon}$: $x_1 \in S_1 \Rightarrow$

$\Rightarrow$ PERTURB OF ORDER $\boxed{\varepsilon}$: $x_2 \in S_2$

CONSEQUENCE:

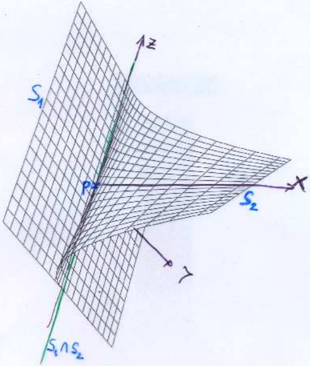
THERE IS $P \subset S_1 \cap S_2$, PATHOLOGY:

S_2:
- UNBOUNDED CURVATURE NEAR P
- NO SMOOTH MANIFOLD STRUCTURE AT P

Analogy: Whitney Umbrella

(8

ANALOGY: WHITNEY-UMBRELLA



(HALF OF THE) WHITNEY-UMBRELLA $W = \{(x,y,z) \mid xz^2 = y^2\}$

- NO MANIFOLD STRUCTURE AT P
- NO CURVATURE BOUND NEAR P

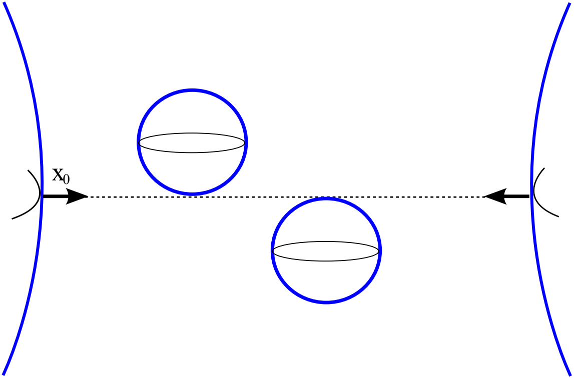
The example

$x_0 \in M$ singular periodic point

P_0: plane spanned by x and the centers of the “small” scatterers

$P_\varepsilon \parallel P_0$ of distance ε from P_0

$x_\varepsilon \in M$ starting $\parallel x_0$ in P_ε



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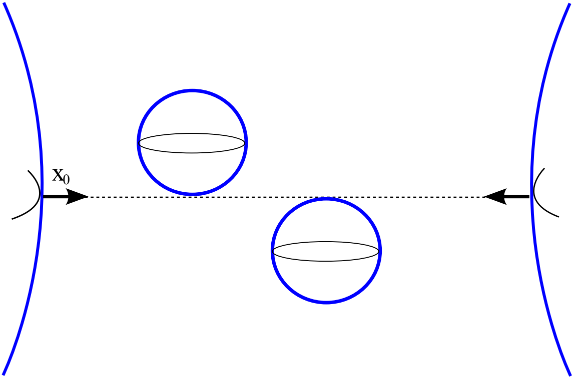

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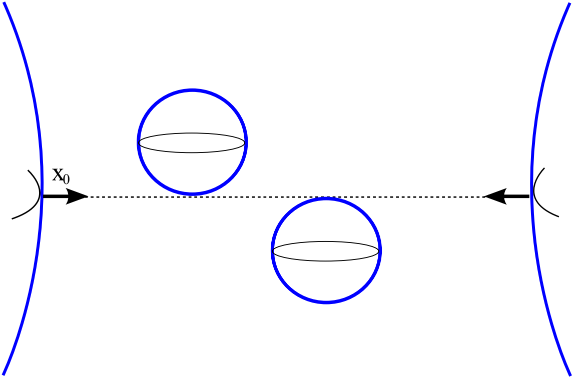
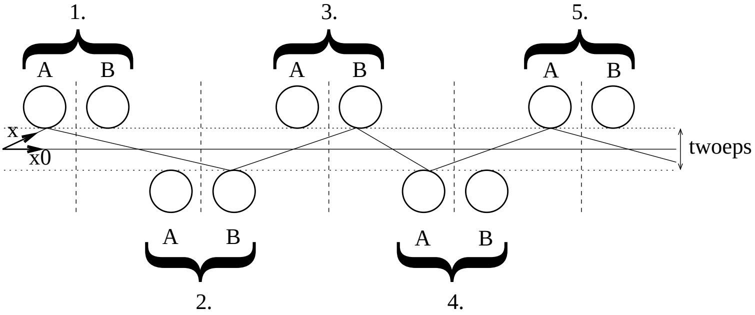


Symbolic sequences in P_ε

Collisions on the "small" scatterers: strong expansion $\Longrightarrow$

Trajectory may collide at either of the two scatterers from each pair, i.e.

ε-close to x_ε: 2^n distinct collision sequences of length $2n$



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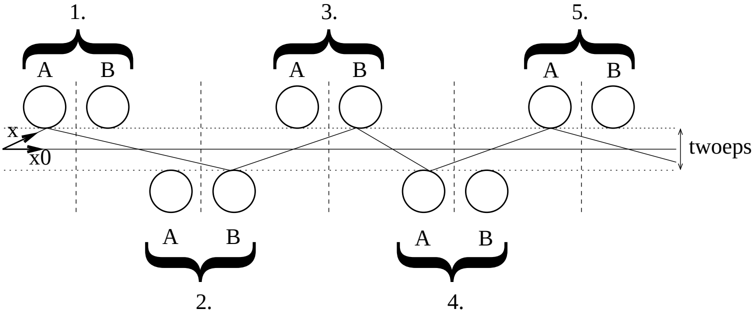

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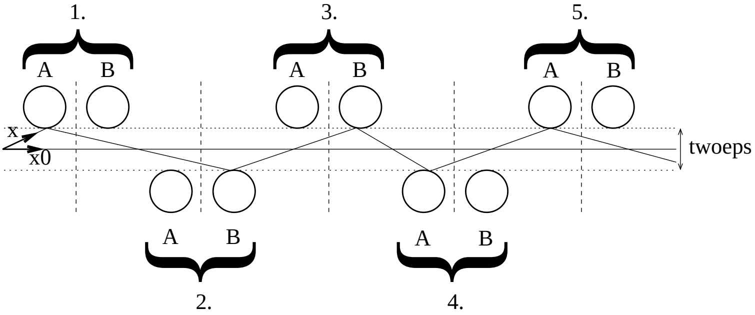
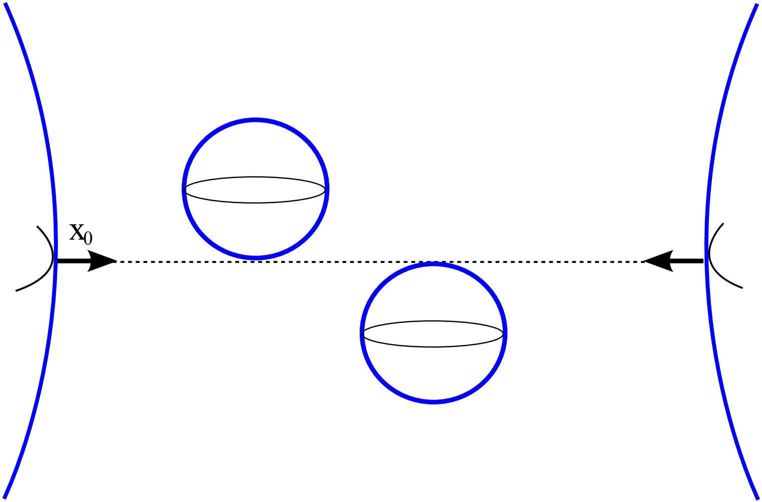


Back to the 3D example

Orthogonal to P_0: moderate expansion $\Longrightarrow$

$\forall n$ and ε, $\exists \delta$ such that $T^k x_\delta \in P_{\varepsilon_k}$ for some $\varepsilon_k \leq \varepsilon$



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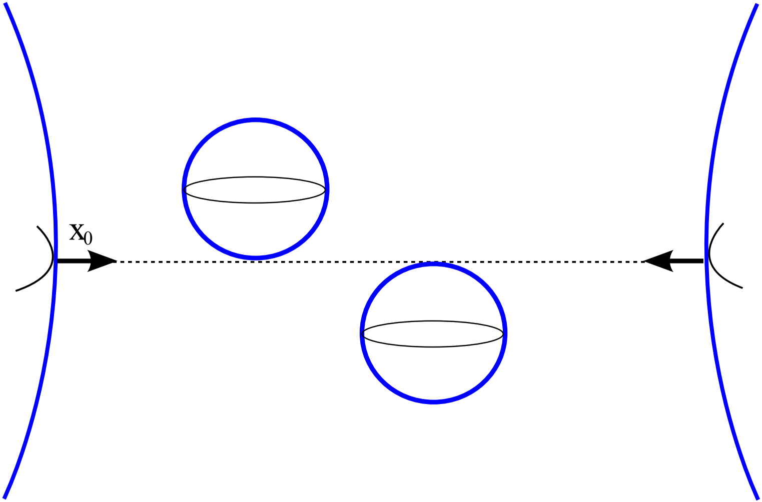

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  Methods
  - key phenomena: growth of u-curves
  - approaches: Young tower, coupling, ???

  Applications (Chernov-Dolgopyat) slow-fast systems, eg. Brownian Brownian motion

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Further reading

- N. Chernov & R. Markarian
  Chaotic billiards
  Mathematical Surveys and Monographs, **127**, AMS, 2006
- N. Chernov & D. Dolgopyat
  Hyperbolic billiards and statistical physics
  in ICM Proceedings, EMS, 2006
- P. Bálint & I.P. Tóth
  An Application of Young's Tower Method: Exponential Decay of Correlations in Multidimensional Dispersing Billiards
  Erwin Schrödinger Institut preprint No. 2084, 2008

Thanks

Thank you for your attention!