Lecture II: Intermittency in planar billiards

Dispersing billiards with cusps and tunnels

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work in progress with N. Chernov and D. Dolgopyat

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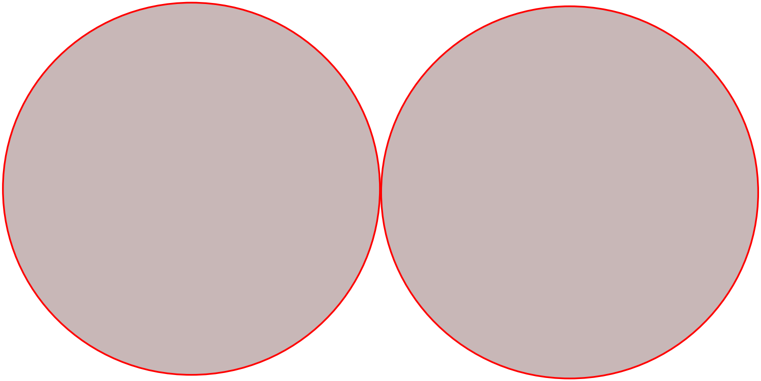
Mathematical Billiards and their Applications
University of Bristol, June 2010

In a nutshell

- Billiards with cusps: slow decay of correlations, non-standard limit theorem;
- Billiards with tunnels: CLT, but variance blows up as $\varepsilon \to 0$.



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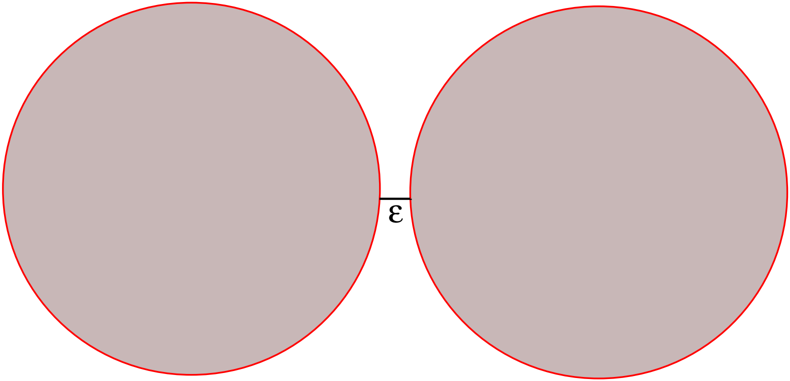
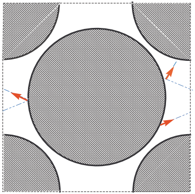


Outline

Billiards

$Q = \mathbb{T}^2 \setminus \bigcup_{k=1}^{K} C_k$ strictly convex scatterers

- Billiard flow : $S^t : \mathcal{M} \to \mathcal{M}$, $(q, v) \in \mathcal{M} = Q \times \mathbb{S}^1$, $|v| = 1$
  Uniform motion within Q, elastic reflection at the boundaries
- Billiard map phase space: $M = \bigcup_{k=1}^{K} M_k$
- $(r, \phi) \in M_k$, r: arclength along ∂C_k, $\phi \in [-\pi/2, \pi/2]$ outgoing velocity angle
- invariant measure $d\mu = c \cos\phi \, dr \, d\phi$



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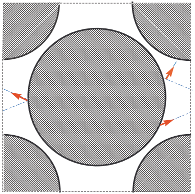

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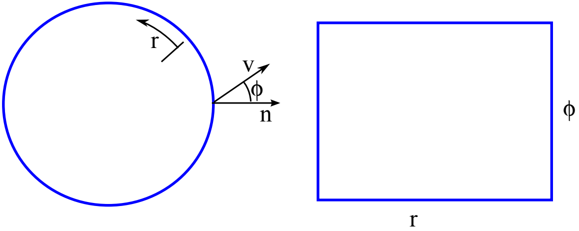

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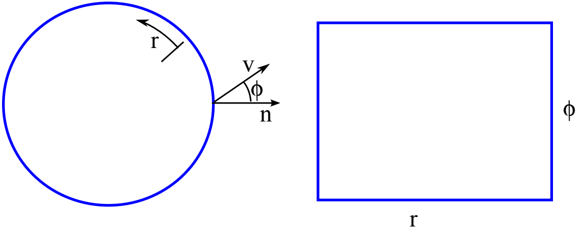

Sinai billiards

C_k are C^3 smooth and disjoint (no corner points);
finite horizon: flight length uniformly bounded from above

- Billiard map is ergodic, K-mixing (Sinai '70)
- EDC: $f, g : M \to \mathbb{R}$ Hölder continuous, $\int f d\mu = \int g d\mu = 0$ let $C_n(f,g) = \mu(f \cdot g \circ T^n)$, then $|C_n(f,g)| \leq C\alpha^n$ for suitable $C > 0$ and $\alpha < 1$
  - Young '98 – tower construction with exponential tails,
  - Chernov & Dolgopyat '06 – standard pairs
- CLT: let $S_n f = f + f \circ T + ... + f \circ T^{n-1}$, then $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \sigma)$ where $\sigma = \int f^2 d\mu + 2\sum_{n=1}^{\infty} C_n(f,f)$.
  Bunimovich & Sinai '81, Chernov '06, Melbourne '06.
- Billiard flow: $F, G : \mathcal{M} \to \mathbb{R}$, $C_t(F,G)$:
  - stretched exponential bound, Chernov '07 (approximate Markov partitions)
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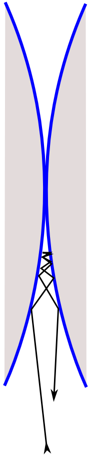
Cusp map



C_1 and C_2 touch tangentially – unbounded series of consecutive reflections in the vicinity of the cusp

- Reháček '95 ergodicity
- Machta '83 numerics and heuristic reasoning for $C_n(f, g) \asymp 1/n$
- Chernov & Markarian '07: $C_n(f, g) \leq C \frac{\log^2 n}{n}$
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Not summable $\Rightarrow$ non-standard limit law?

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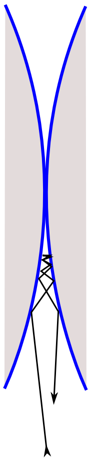

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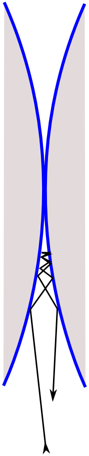

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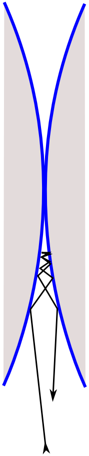

long collision series near the cusp correspond to bounded flow time – flow mixes faster?

Melbourne & B. ’08

- $C_t(F, G)$ decays faster than any polynomial
- S^t admits CLT (almost sure invariance principle)

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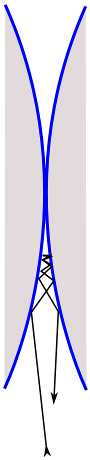

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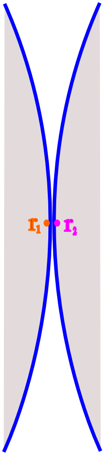
Cusp superdiffusion constant



"Result" (C)

- *Denote by* $r_1 \in C_1$ *and* $r_2 \in C_2$ *the two points that make the cusp.*
- *Let* $I_f = \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi))\rho(\phi) d\phi$ *with* $\rho(\phi) = \frac{\sqrt{\cos\phi}}{\int_{-\pi/2}^{\pi/2} \sqrt{\cos\phi} d\phi}$
- *if* $I_f \neq 0$ *then* $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ *where* $D_f = c^* I_f^2$ *and* c^* *is some numerical constant.*
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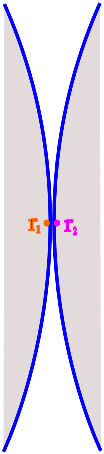

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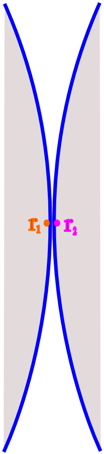

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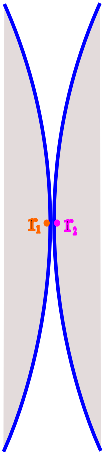

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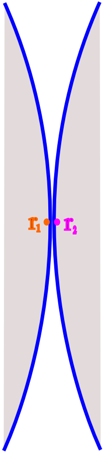

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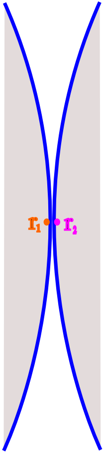

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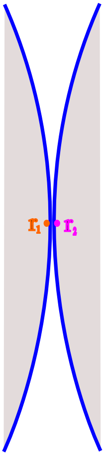

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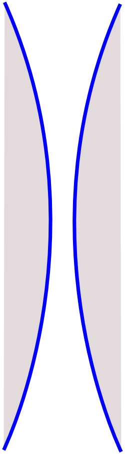
Blow-up of the variance in tunnels



“Result” (T)

Denote be $T_\varepsilon : M \to M$ the billiard map
same phase space, same $f : M \to \mathbb{R}$

- *for fixed $\varepsilon > 0$ this is a Sinai billiard, hence CLT:*
- $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_{f,\varepsilon})$ *with*
- $D_{f,\varepsilon} = D_f |\log \varepsilon| (1 + o(1))$

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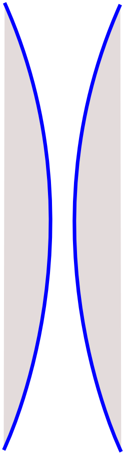

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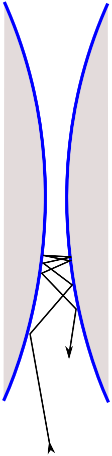

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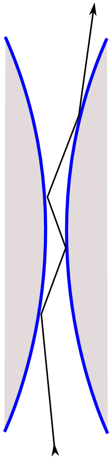

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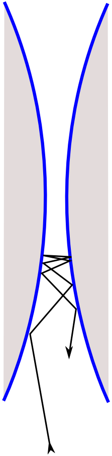

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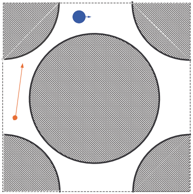
- $D_{f,\varepsilon} = D_f |\log \varepsilon| (1 + o(1))$

Motivation

1. Brownian Brownian motion – Chernov & Dolgopyat '09



$m \ll M$ (separation of time scales)

SDE for large particle:

$dV = \sigma_Q(f)dW$

collisions of the heavy particle with the wall?

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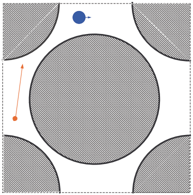

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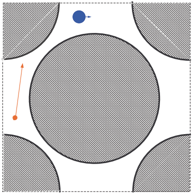

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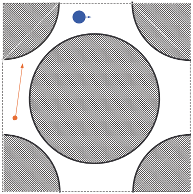

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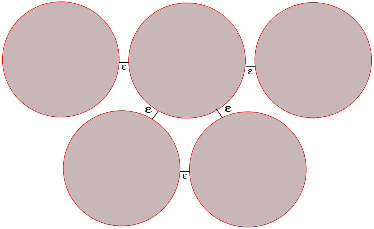
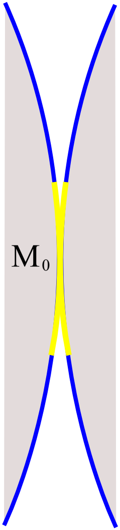


How does the planar diffusion depend on ε?

The first return map



Let $\hat{M} = M \setminus M_0$ where M_0 is a fixed small nbd. of the cusp.

- $\hat{T} : \hat{M} \to \hat{M}$ first return map
- $R : \hat{M} \to \mathbb{N}$ unbounded return time
- $\hat{f}(x) = \sum_{k=0}^{R(x)-1} f(T^k x)$ induced observable

limit law for $\hat{S}_n \hat{f}$ implies limit law for $S_n f$ (eg. Gouëzel '04)

$D_f = \mu(R) D_{\hat{f}} = \frac{D_{\hat{f}}}{\mu(\hat{M})}$

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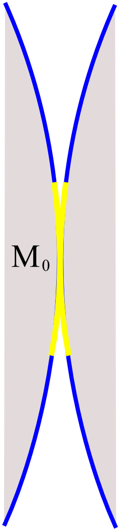

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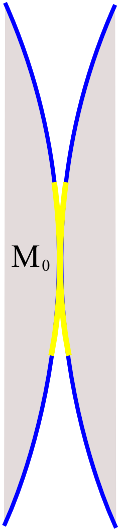

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Fast mixing of the first return map

Lemma (C1)

The map $\hat{T} : \hat{M} \to \hat{M}$ is uniformly hyperbolic and it satisfies the Growth Lemma ("Expansion prevails fractioning")

so that

- Young tower with exponential tails can be constructed
- standard pairs can be coupled at an exponential rate

Hence: EDC for Hölder observables

Lemma (C2)

$|\hat{\mu}(\hat{f} \cdot \hat{f} \circ \hat{T}^n)| \leq Ce^{-\alpha n}$ with $C > 0, \alpha < 1$ for $n \geq 1$
Not for $n = 0$ as $\hat{f}$ is not Hölder and not in L^2

Summarizing: the sequence $\hat{f} \circ \hat{T}^n$ behaves almost like an i.i.d. sequence

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Blow-up of $\hat{f}^2$

- $M_n = \{x \in \hat{M} | R(x) = n\}$ n-cell
- $L_n = \bigcup_{j \le n} M_j$ low cells, $H_n = \bigcup_{j > n} M_j$ high cells

Lemma (C3)

- $\hat{f}|_{M_n} = nI(1 + o(1))$

  (recall $I = c_1 \int_{-\pi/2}^{\pi/2} (f(r_1, \phi) + f(r_2, \phi))\sqrt{\cos(\phi)}d\phi$)
- $\hat{\mu}(H_n) = \frac{c_2}{n^2}(1 + o(1))$ (here c_1, c_2 are numerical constants)
- hence $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}|_{L_n}) = 2 \log n D_{\hat{f}}(1 + o(1))$

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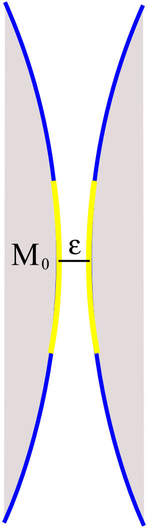
First return map for tunnel



$T_\varepsilon : M \to M$, M_0: same nbd. for any ε,
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Lemma (T1)

The map $\hat{T}_\varepsilon : \hat{M} \to \hat{M}$ satisfies the Growth Lemma and EDC for Hölder observables uniformly in ε.

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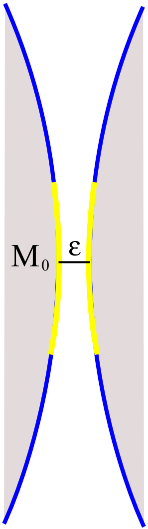

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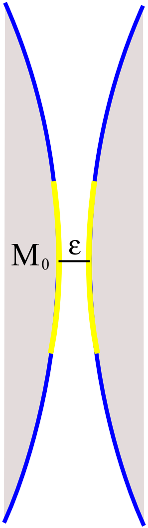

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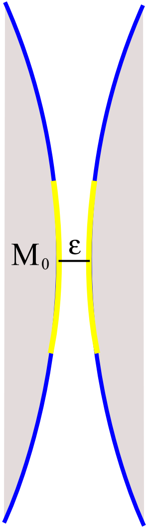

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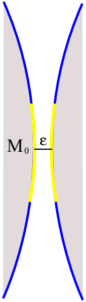

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$\hat{\mu}(\hat{f}_\varepsilon^2) = |\log \varepsilon| D_{\hat{f}}(1 + o(1))$

All these Lemmas require: detailed geometric analysis of the cells M_k (measures, unstable and stable dimensions etc...)

- For cusp, mostly (but not completely) done by Chernov & Markarian
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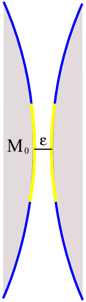

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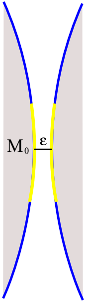

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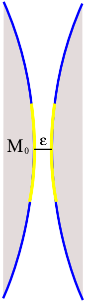

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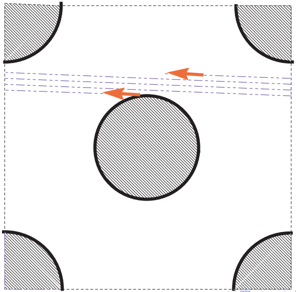
Superdiffusion

- Collision map: growth lemma, Young tower, EDC for Hölder – Chernov 1999
- an observable of particular interest: $\mathbf{L}(x)$ free flight (vector) – neither Hölder, nor in L^2
- $\frac{S_n\mathbf{L}}{\sqrt{n \log n}} \xRightarrow{\mathcal{D}} \mathcal{N}(0, D_{\mathbf{L}})$ – Szász & Varjú 2006

  $D_{\mathbf{L}}$ is determined by the blow-up of the variance – corridor sum



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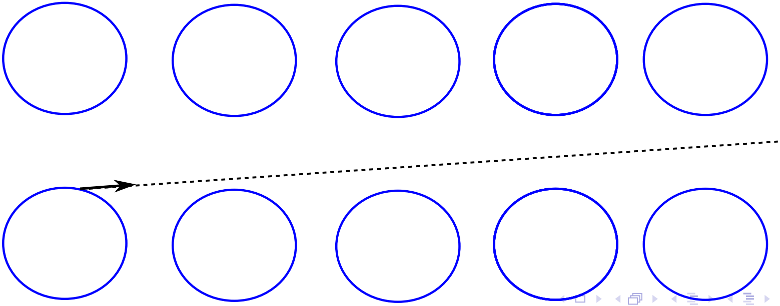

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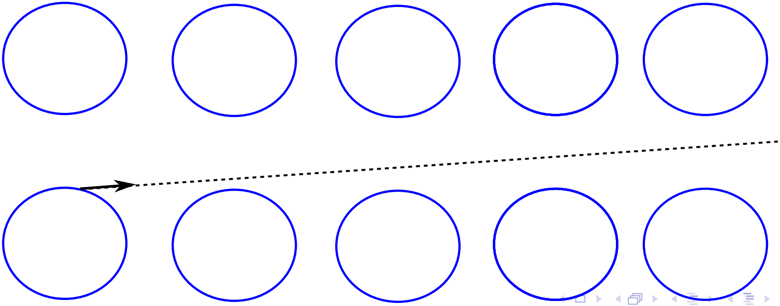
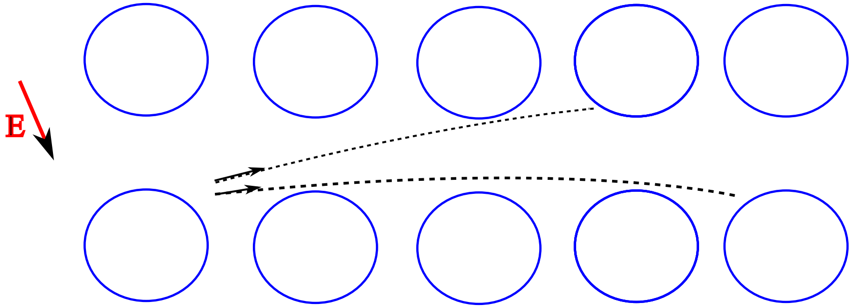


Infinite horizon with field I

- Add field $\mathbf{E}$ transversal to corridors, $|\mathbf{E}| = \varepsilon \ll 1$
- $+$ thermostating: Gaussian $\dot{\mathbf{v}} = \mathbf{E} - \langle \mathbf{E}, \mathbf{v} \rangle \mathbf{v}$
- free flight $\mathbf{L}_\varepsilon \leq \frac{C}{\sqrt{\varepsilon}}$ is bounded, but depends on ε .



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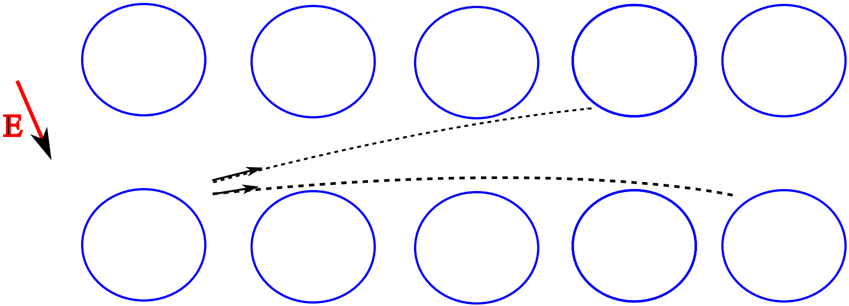

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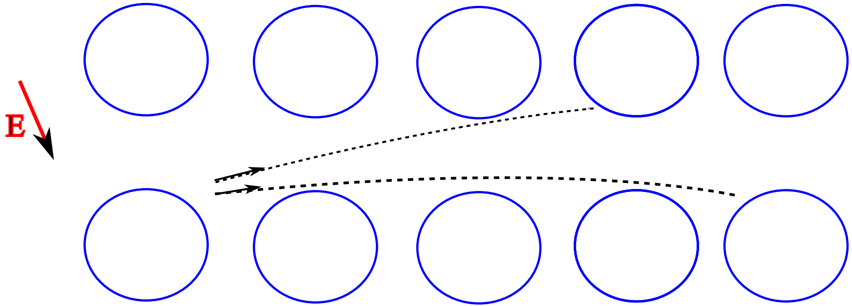
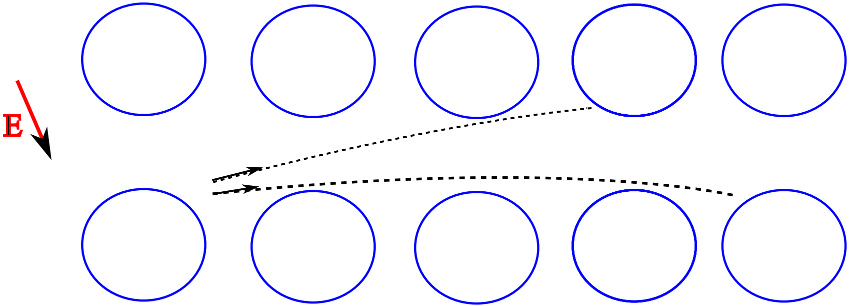


Infinite horizon with field II

Chernov-Dolgopyat 2009:

- SRB measure (non-equilibrium steady state) μ_ε
- current $\mathbf{J} = \mu_\varepsilon(L_\varepsilon) = \frac{1}{2}|\log\varepsilon|\mathbf{D_L}\mathbf{E} + \mathcal{O}(\varepsilon)$
- fluctuations: $\frac{S_n\mathbf{L} - \mathbf{J}n}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_\varepsilon)$ with $D_\varepsilon = |\log\varepsilon| D_{\mathbf{L}}(1 + o(1))$.



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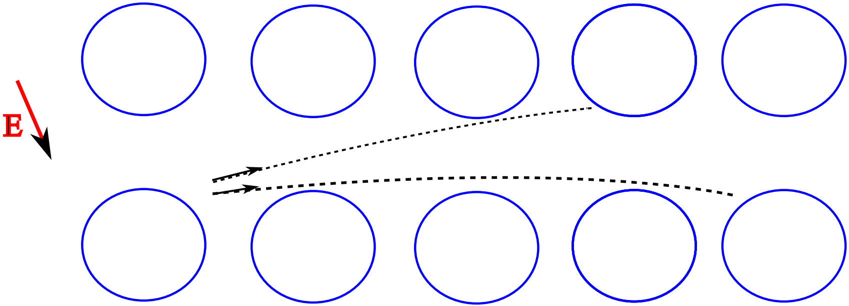

Infinite horizon with field II

Chernov-Dolgopyat 2009:

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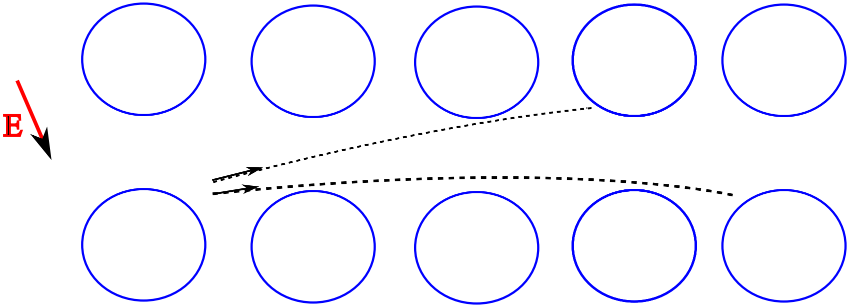
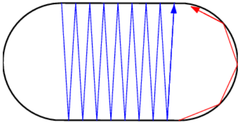
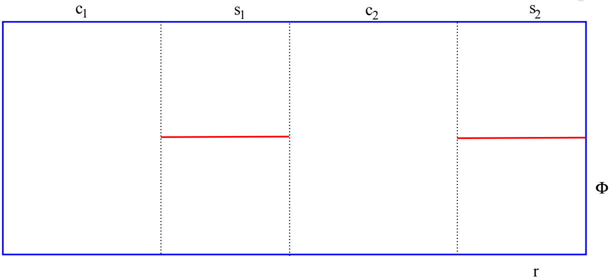


Superdiffusion in the straight stadium I



- Gouëzel & B. 2006. $f : M \to \mathbb{R}$, $\mu(f) = 0$.
- Let $I_f = \int_{s_1 \cup s_2} f(r, \frac{\pi}{2}) dr$.
- if $I_f \neq 0$ then $\frac{S_n f}{\sqrt{n \log n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_f)$ where $D_f = \frac{4+3\log 3}{4-3\log 3} c^* I_f^2$



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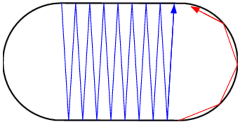

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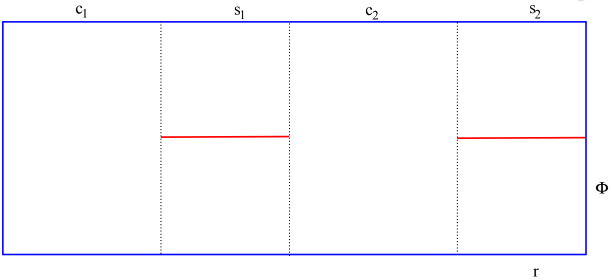

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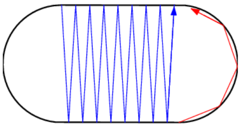

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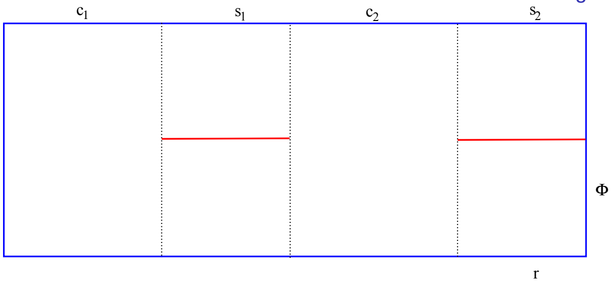
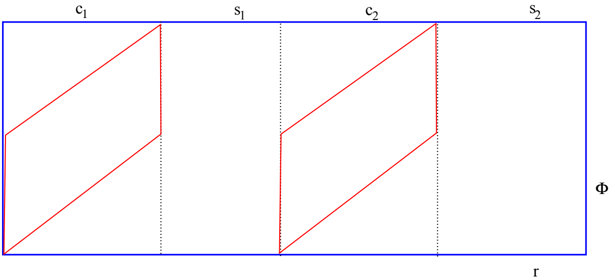


Why $\frac{4+3\log 3}{4-3\log 3}$?

- $\hat{M}$: leaving one of the semicircular arcs.
- in cusp or infinite horizon horizon: $E(R(Tx)|R(x)=K)=c\sqrt{K}(1+o(1))$
- in stadium: $E(R(Tx)|R(x)=K)=\alpha K(1+o(1))$ for some $\alpha<1$, computable $\Longrightarrow$ i.i.d. clusters



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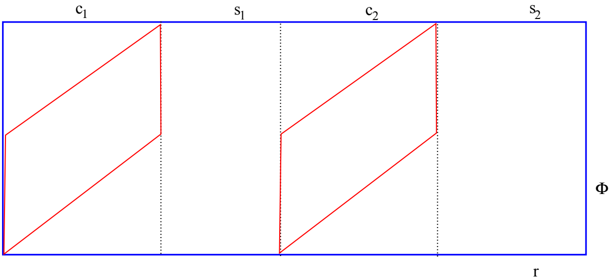

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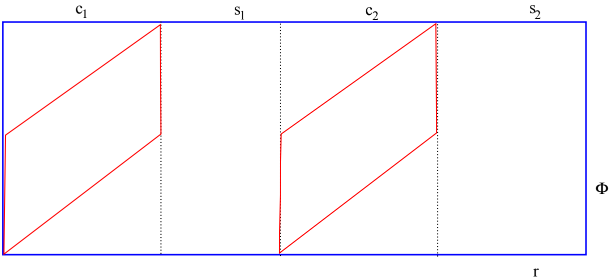

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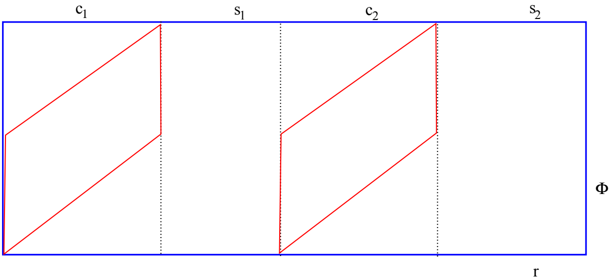

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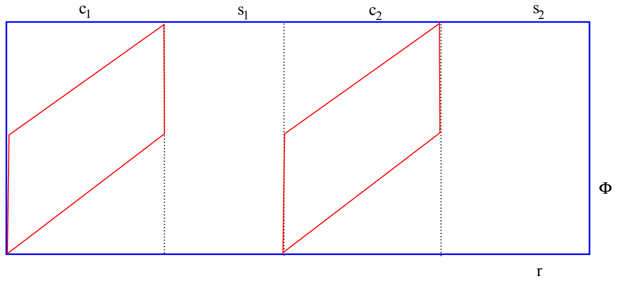

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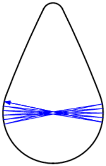
Skewed stadium, squashes



skewed stadia: similar, bouncing $\Longrightarrow$ diametrical

Numerics and heuristic reasoning: Ergodicity for large enough finite c (Halász, Sanders, Tahuilán, B., submitted)

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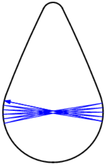

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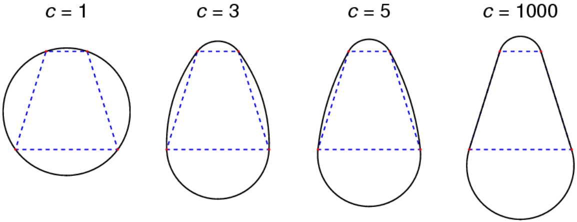

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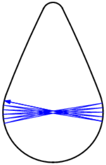

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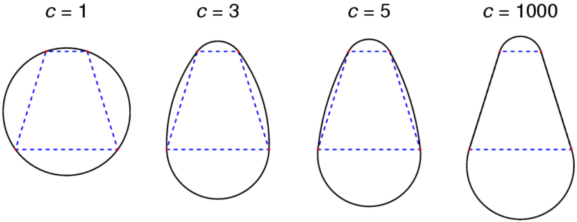

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Corner series

For simplicity assume that C_1 and C_2 are circles of radius 1.

Coordinates: α distance from cusp, $\gamma = \frac{\pi}{2} - \phi$

- while going down the cusp: α decreases, $\gamma : 0 \longrightarrow \frac{\pi}{2}$
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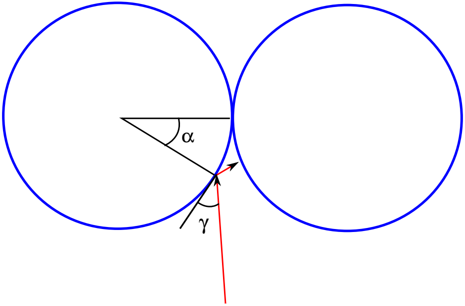

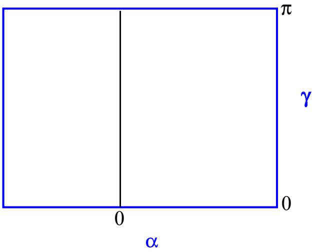

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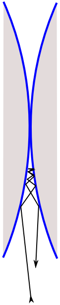

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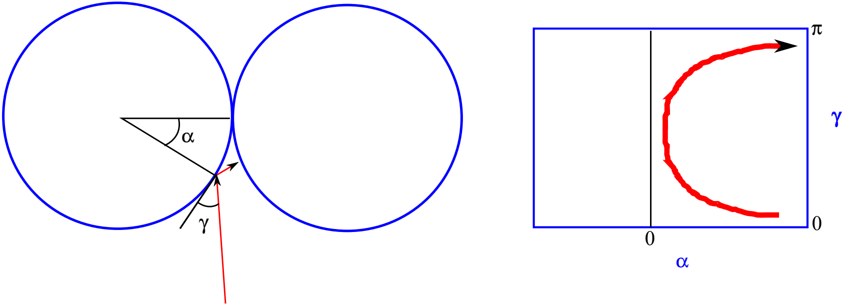
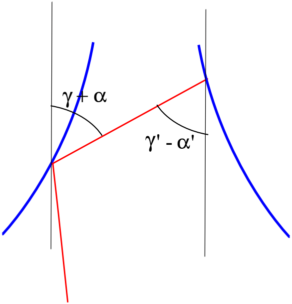


Equations of motion



$$\gamma' - \alpha' = \alpha + \gamma$$

$b = \sin\alpha - \sin\alpha'$;
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$$\sin\alpha' - \sin\alpha = -\frac{2 - \cos\alpha' - \cos\alpha}{\tan(\alpha+\gamma)}$$

- Throughout the corner series: $\alpha \ll 1$, $\alpha < \gamma$;
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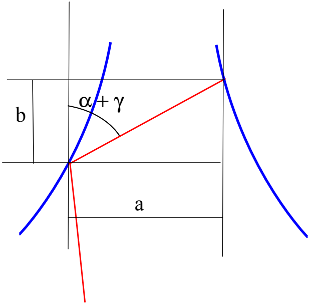

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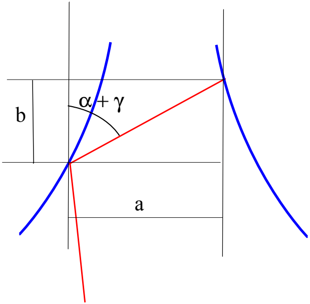

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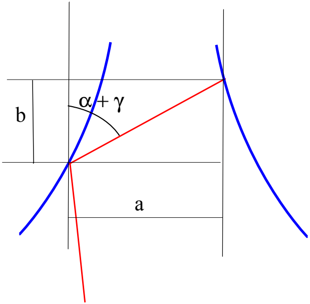

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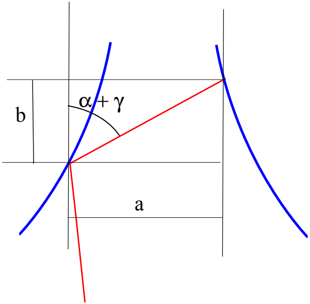

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proportion of time between γ_1 and γ_2 $\asymp \int_{\gamma_1}^{\gamma_2} \sqrt{\sin\gamma}\, d\gamma$.

Recall $I_f = c \int_{-\pi/2}^{\pi/2} (f(r_1,\phi) + f(r_2,\phi))\sqrt{\cos(\phi)}\, d\phi$.

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Corner series for tunnel

Coordinates: α, γ as for cusp

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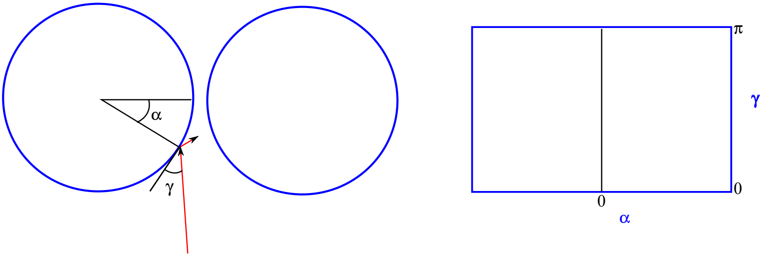
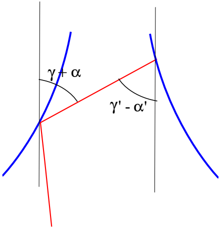


Corner series for tunnel

Coordinates: α, γ as for cusp



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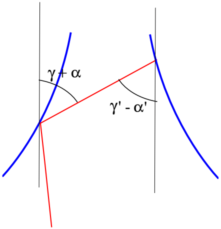

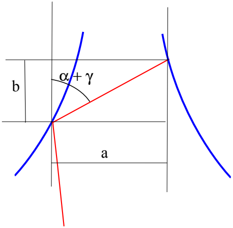

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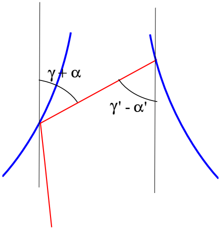

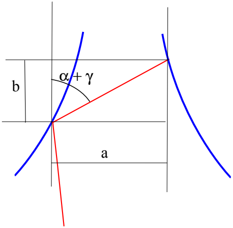

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Flow approximation for tunnel

$$\dot{\gamma} = 2\alpha; \qquad \dot{\alpha} = -\frac{\alpha^2 + \varepsilon}{\tan(\gamma)}.$$

$J = (\alpha^2 + \varepsilon)\sin\gamma$ is first integral, so $\dot{\gamma} = 2\alpha = \pm 2\sqrt{\frac{J}{\sin\gamma} - \varepsilon}$.

Fix some small δ_0. We distinguish three cases:

$$J > \varepsilon/\delta_0, \qquad J < \delta_0\varepsilon \quad \text{and} \quad J/\varepsilon \approx 1.$$

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Cusp case

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$J > \varepsilon/\delta_0$:

- $\alpha > 0$ and $\alpha^2 \gg \varepsilon$ throughout the excursion
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Contribution to the variance: $\hat{\mu}(\hat{f}^2 \cdot \mathbf{1}_{L_{\frac{c}{\sqrt{\varepsilon}}}}) = D_{\hat{f}}|\log\varepsilon|$

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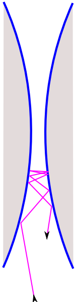

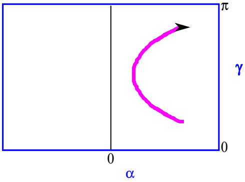

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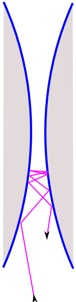

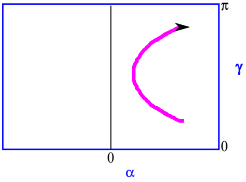

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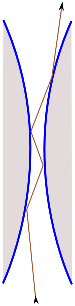

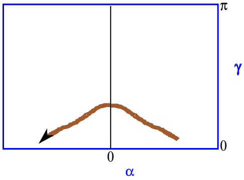

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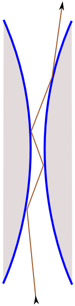

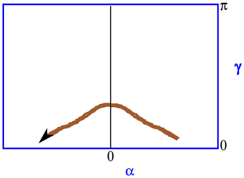

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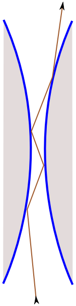

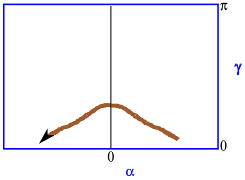

The third case

What is in between?

$\alpha = 0, \gamma = \pi/2$ is a hyperbolic fixed point (period two orbit)

Saddle case: if $J \approx \varepsilon$, R can be arbitrary large, however, it is dominated by the hyperbolic periodic orbit

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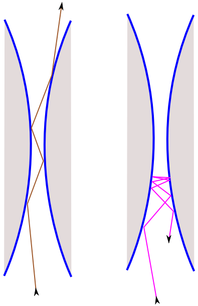

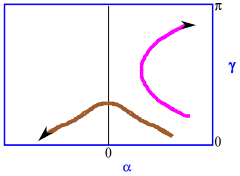

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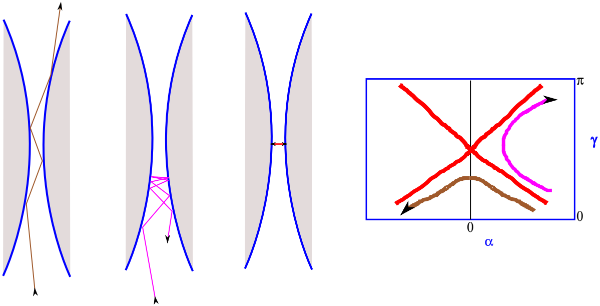

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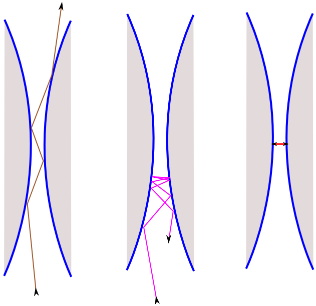

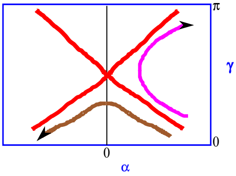

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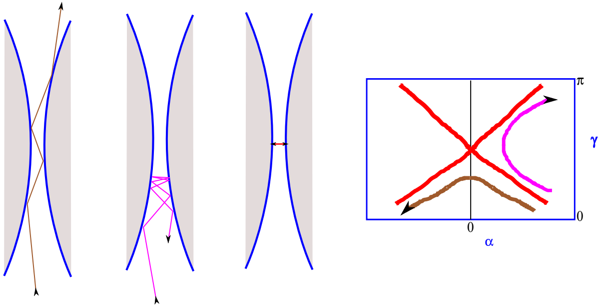

Summary and comparisions

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- Tunnel: $\frac{S_n f}{\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0, D_{f,\varepsilon})$ with $D_{f,\varepsilon} = |\log \varepsilon| D_f (1 + o(1))$

Related models:

1. Infinite horizon Lorentz gas and field of strength ε
2. Stadia what is ε?

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Thank you for your attention!

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