Lecture I: Dispersing billiards in 2D and 3D

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Mathematical Billiards and their Applications
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Plan

1. Today: Dispersing (Sinai) Billiards
   - in $2D$: uniform hyperbolicity, strong ergodic properties
   - in $3D$: similar phenomena, but serious technical complications

2. Tomorrow: Planar billiards with intermittency.
   - billiards with cusps and tunnels: WIP with Chernov and Dolgopyat
   - comparisons: stadia, infinite horizon...
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Outline for Lecture I

Planar dispersing billiards
Results
Phenomena

Dispersing Billiards in 3D
Results
Phenomena

Singularities in 3D dispersing billiards
Unbounded curvature
Example with exponential complexity
Billiards in 2D

\( Q = \mathbb{T}^2 \setminus \bigcup_{k=1}^{K} C_k \) strictly convex scatterers

- **Billiard flow**: \( S^t : \mathcal{M} \rightarrow \mathcal{M} \), \( (q, v) \in \mathcal{M} = Q \times \mathbb{S}^1 \), \( |v| = 1 \)
  Uniform motion within \( Q \), elastic reflection at the boundaries

- **Billiard map** phase space: \( M = \bigcup_{k=1}^{K} M_k \)

- \( (r, \phi) \in M_k \), \( r \): arclength along \( \partial C_k \), \( \phi \in [-\pi/2, \pi/2] \)
  outgoing velocity angle

- invariant measure \( d\mu = c \cos \phi \, dr \, d\phi \)

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![Diagram of billiards in 2D with scatterers and trajectories]
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Sinai billiards in 2D

$C_k$ are $C^3$ smooth and disjoint (no corner points);
finite horizon: flight length uniformly bounded from above

- **Billiard map** is ergodic, K-mixing (Sinai ’70)
- **EDC:** $f, g : M \to \mathbb{R}$ Hölder continuous, $\int fd\mu = \int gd\mu = 0$
  let $C_n(f, g) = \mu(f \cdot g \circ T^n)$, then $|C_n(f, g)| \leq C\alpha^n$ for suitable $C > 0$ and $\alpha < 1$
  - Young ’98 – tower construction with exponential tails,
  - Chernov & Dolgopyat ’06 – standard pairs
  crucial: Growth Lemma on unstable curves

- **CLT:** let $S_nf = f + f \circ T + \ldots + f \circ T^{n-1}$, then
  $\frac{S_nf}{\sqrt{n}} \xrightarrow{D} \mathcal{N}(0, \sigma)$ where $\sigma = \int f^2d\mu + 2 \sum_{n=1}^{\infty} C_n(f, f)$.
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Unstable curves

Neutral (or convex) wavefront $\rightarrow$ Convex front

Definition

U-curve $W$: Trace of a convex front on $M$.

- Increasing in the $r, \phi$ coordinates.
- Invariant and expanding under $T$. In particular:
  $\exists \lambda > 1$ such that $\rho(Tx, Ty) \geq \lambda \rho(x, y), \forall W, \forall x, y \in W$
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Singularities I

Preimages of tangencies: $T$ discontinuous, $S^t$ non-differentiable
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Preimages of tangencies: $T$ discontinuous, $S^t$ non-differentiable
Singularities II

\[ S_n = T^{-n}S_0 \] where \( S_0 \) is the tangency

Discontinuity set for \( T^n \): \( S^{(n)} = \bigcup_{i=0}^{n} S_i \)

- The \( S_n \) are smooth Decreasing curves in the \( r, \phi \) coordinates.
- \( S^{(n)} \) fills \( M \) more and more densely as \( n \) increases.
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Evolution of u-curves

$W$ (sufficiently small) u-curve $TW$

- increases in length
- partitioned by the singularities

Expansion prevails fractioning: “Most” components of $W$ are “long”

How to quantify this?
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The Growth Lemma

- $W$ is small u-curve, $m_W$ Lebesgue measure on $W$.
- $G_\varepsilon$: set of points in $W$ that are at most $\varepsilon$ from the boundary:
  \[ G_\varepsilon = \{ x \in W \mid \rho(x, \partial W) \leq \varepsilon \} \]

- $H_\varepsilon$: set of points in $W$ that will be at most $\varepsilon$ from the boundary.
  \[ H_\varepsilon = \{ x \in W \mid \rho(Tx, \partial(TW)) \leq \varepsilon \} \]

If there were no singularities: $m_W(H_\varepsilon) \leq m_W(G_{\varepsilon/\Lambda})$.

Lemma

There exists a constant $\lambda < \Lambda$, independent of $W$, such that

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Complexity of the singularity set

Definition

\( K_n(x) \), \( n \)-step complexity of a point \( x \in M \): number of different symbolic collision sequences that can be observed in the vicinity of \( x \).

\( n \)-step complexity of the singularity set: \( K_n = \sup_{x \in M} K_n(x) \)
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Subexponential complexity

Subexponential growth of complexity:

\[ \exists C > 0 \text{ and } \lambda < \Lambda \text{ such that } K_n \leq C\lambda^n \]

**Lemma**

*Bunimovich, 1991*: In 2D Sinai billiards (finite horizon, no corner points) \( K_n \) grows at most linearly.
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Billiard dynamics in 3D

- $M$: hemisphere-bundle,
  $\dim M = 4$
- convex fronts – u-manifolds
  $\dim W = 2$
- singularity set – codimension 1
  $\dim S_n = 3$
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History: 3D dispersing billiards

- Sinai & Chernov 1987
  - Ergodicity
  - Local ergodicity theorem – many further applications: semi-dispersing billiards hard ball systems, Simányi
- Chernov, Szász, Tóth & B. 2002
  - Unbounded curvature for $S_n, n \geq 2$
  - Proof of ergodicity reconsidered, algebraic scatterers
- Tóth & B. 2008 – Assuming sub-exponential complexity
  - Growth Lemma, Young tower, EDC, CLT
  - With Bachurin: Growth Lemma implies Ergodicity
  - Counterexample with exponential complexity
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What is responsible for all this...

- **Unbounded expansion** near singularities (highly nonlinear, applies to 2D)
- in 3D **highly anisotropic** expansion near singularities  
  cf. astigmatism
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$W$ is a small u-manifold (2 dimensional)

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\[ 
\begin{array}{c}
H_\varepsilon \\
\end{array} 
\] 

\[ 
\begin{array}{c}
T \\
\end{array} 
\] 

\[ 
\begin{array}{c}
\text{Diagram} \\
\end{array} 
\]
The pathological intersection I

$S_0$: tangency, $S_1 = T^{-1}S_0$, $S_2 = T^{-2}S_0$

- in 2D: $S_1 \cap S_2$ is single point
- in 3D: $S_1 \cap S_2$ has structure, $\dim(S_1 \cap S_2) = 2$

- $S_2$ terminates on $S_1$ typically tangentially,
- transversally in a one dimensional pathological set $P \subset S_1 \cap S_2$
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Pathological intersection II

\( \dim M = 4 \Rightarrow S_1 \cap S_2 \) has structure

I "Typical" case

II "Pathological" case

\( n_1 \perp n_2 \)
Pathological intersection III

I 'TYPICAL CASE'

\[ x_0 \in S_1 \land S_2 \Rightarrow \]
\[ \Rightarrow \text{PERTURB OF ORDER } [\epsilon^1]: x_1 \in S_1 \Rightarrow \]
\[ \Rightarrow \text{PERTURB OF ORDER } [\epsilon^2]: x_2 \in S_2 \]

II 'PATHOLOGICAL CASE'

\[ x_0 \in S_1 \land S_2 \Rightarrow \]
\[ \Rightarrow \text{PERTURB OF ORDER } [\epsilon^1]: x_1 \in S_1 \Rightarrow \]
\[ \Rightarrow \text{PERTURB OF ORDER } [\epsilon^1]: x_2 \in S_2 \]

CONSEQUENCE:

**THERE IS** \( P \subset S_1 \land S_2 \), **PATHOLOGY:**

\( S_2 \): - UNBOUNDED CURVATURE NEAR \( P \)
- NO SMOOTH MANIFOLD STRUCTURE AT \( P \)
Analogy: Whitney Umbrella

\[ W = \{(x,y,z) \mid x^2 + y^2 = 1\} \]

- No manifold structure at \( P \)
- No curvature bound near \( P \)
The example

\( x_0 \in M \text{ singular periodic point} \)

\( P_0 \): plane spanned by \( x \) and the centers of the “small” scatterers

\( P_\varepsilon \parallel P_0 \) of distance \( \varepsilon \) from \( P_0 \)

\( x_\varepsilon \in M \) starting \( \parallel x_0 \) in \( P_\varepsilon \)
The example

\( x_0 \in M \) singular periodic point

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Symbolic sequences in $P_\varepsilon$

Collisions on the “small” scatterers: strong expansion $\implies$
Trajectory may collide at either of the two scatterers from each pair, i.e.
$\varepsilon$-close to $x_\varepsilon$: $2^n$ distinct collision sequences of length $2n$
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1. $\{A B A B A B\}$
2. $\{A B\}$
3. $\{A B A B\}$
4. $\{A B\}$
5. $\{A B A B\}$

$x_0$ twoeps
Back to the 3D example

Orthogonal to $P_0$: moderate expansion $\Rightarrow$

$\forall n$ and $\varepsilon$, $\exists \delta$ such that $T^k x_\delta \in P_{\varepsilon_k}$ for some $\varepsilon_k \leq \varepsilon$
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Summary and outlook

- **2D Sinai billiard maps**: strong ergodic and statistical properties
  - Methods
    - key phenomena: growth of u-curves
    - approaches: Young tower, coupling, ???
  - Applications (Chernov-Dolgopyat)
    - slow-fast systems, eg. Brownian Brownian motion

Open problems
- EDC for the flow

- **3D dispersing billiards**: analogous phenomena, but technically more involved
  - genericity of subexponential (finite) complexity?
  - statistical properties of example with exponential complexity?
  - so far: Young tower. Alternative methods?
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Further reading

N. Chernov & R. Markarian
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N. Chernov & D. Dolgopyat
*Hyperbolic billiards and statistical physics*
in ICM Proceedings, EMS, 2006

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*An Application of Young’s Tower Method: Exponential Decay of Correlations in Multidimensional Dispersing Billiards*
Erwin Schrödinger Institut preprint No. 2084, 2008
Thanks

Thank you for your attention!