Experimental Mathematics (published online)
DOI: 10.1080/10586458.2024.2423180

A billiard in an open circle and the Riemann zeta function

Leonid A. Bunimovich[1] and Carl P. Dettmann[2]
[1]School of Mathematics, Georgia Institute of Technology, USA
[2] School of Mathematics, University of Bristol, UK

November 19, 2024

Abstract

We consider a dynamical billiard in a circle with one or two holes in the boundary, or q symmetrically placed holes. It is shown that the long-time survival probability, either for a circle billiard with discrete or with continuous time, can be written as a sum over never-escaping periodic orbits. Moreover, it is demonstrated that in both cases the Mellin transform of the survival probability with respect to the hole size has poles at locations determined by zeros of the Riemann zeta function and, in some cases, Dirichlet L functions.

1 Introduction

The consideration of open dynamical systems, in which the dynamics continues only until the system reaches a "hole," a specified subset of phase space, was introduced by Pianigiani and Yorke in 1979 [Pia79] and has been actively studied since then. Although a dynamical system of interest can be chosen in an extremely general manner, the goal was to study open billiards [Pia79] To quote:

> Picture an energy-conserving billiard table with smooth obstacles so that all trajectories are unstable with respect to the initial data. Now suppose that a small hole is cut in the table so that the ball can fall through. We would like to investigate the statistical behavior of such phenomena.

There have been many mathematical results relating to open dynamics since then; see, for example, Ref. [Hay20] and references therein.

Physicists have also studied open billiards experimentally, in which the billiard balls are atoms [Fri01, Mil01] (see [Alt13] for a review of the extensive physics literature). These experiments provided impetus for the present authors to compare the effects of having one or two holes, both in a circular billiard [Bun05] and in chaotic systems [Bun07]. What was assumed to be seemingly the simplest case of a billiard in a circle turned out to be very complicated, and highlighted a number of interesting phenomena, including connections with the Riemann hypothesis. We tried to rigorously justify these results [Bun06], but some considerations were not sufficiently clear to the reviewers, and this was not published. The paper [Bun05] continues to receive citations and we continue to receive questions about it, so the present contribution is an effort to give a clear and rigorous exposition of what is known, what is conjectured, or remains an open question in [Bun05].

We note that in the intervening years physicists have performed experiments to measure the Riemann zeros [Cre15, He21]. One of us used methods similar to Ref. [Bun05] to consider open spherical billiards [Det14, Det21]. Ref. [Det17] gives a rigorous account of the scaling limit in which the product of hole size and time is constant, while the latter tends to infinity, for integrable systems. This includes, as an example, the elliptical billiard, which is a generalization of the circle, and is thus a rigorous justification of the limiting function presented in Fig. 3 of [Bun05].

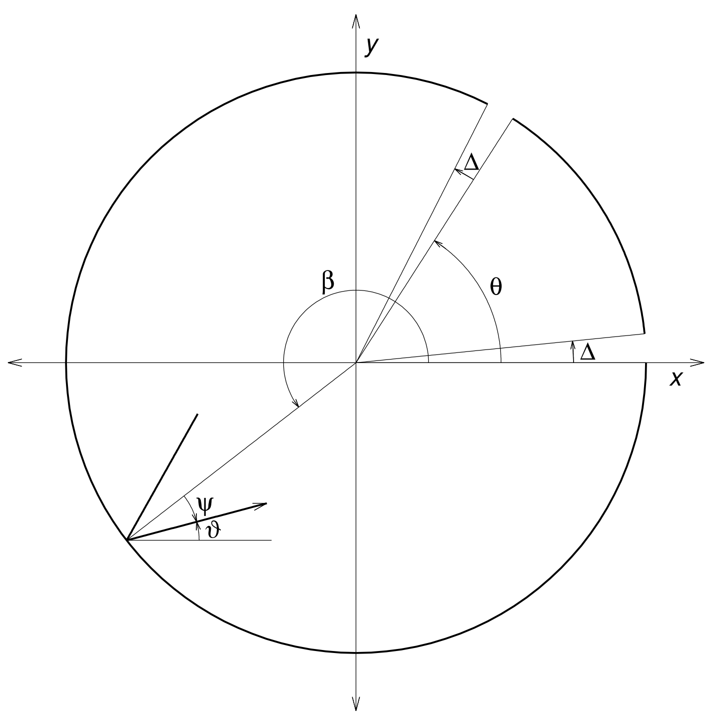

Figure 1: Geometry of the circular billiard

In mathematical literature the studies of open billiards turned out to be intimately related to limiting Poisson distributions of the process of recurrences to a hole when the size of the hole is shrinking to zero. Most extensive mathematical studies of open dynamical systems (particularly open billiards) are related to proving the Poisson and other limit laws (see, e.g., the papers [Su22] and [Bun23] and references therein). In physics studies, especially experimental

j	$\mathcal{P}_{j,\theta}(\Delta)$	a_j	α_j	$C_j(x)$	$K_j(x)$
1	$\lim_{N\to\infty} N\mu_M(\mathfrak{N}_N)$	π	$\frac{2\pi}{3}$	$\cot\frac{\pi}{2x}$	1
$2'$	$\lim_{t\to\infty} t\mu_M(\mathcal{N}_t)$	π	$\frac{\pi^2}{3}$	$x-\delta_{x,1}$	$\frac{\pi}{2}\delta_{x,0}$
$2''$	$\lim_{N\to\infty} N\mu_{\mathcal{M}}(\mathrm{proj}^{-1}\mathfrak{N}_N)$	2	$\frac{2\pi}{3}$	$x-\delta_{x,1}$	$\frac{\pi}{2}\delta_{x,0}$
3	$\lim_{t\to\infty} t\mu_{\mathcal{M}}(\mathrm{proj}^{-1}\mathcal{N}_t)$	2	$\frac{32}{9}$	$3\cot\frac{\pi}{2x}-\cot\frac{3\pi}{2x}$	$3-3^{2x-1}$

Table 1: Definitions of the four limiting survival probabilities and related quantities, labelled by $j\in\{1,2',2'',3\}$. The indices $j\in\{2',2''\}$ are interpreted as 2 when they appear in a formula such as 2^j and $\sin^j x$. These involve the set $\mathfrak{N}_N$ not escaping for discrete time N, the set $\mathcal{N}_t$ not escaping for continuous time t, invariant measures of the map μ_M and of the flow $\mu_{\mathcal{M}}$, and the projection $\mathrm{proj}:\mathcal{M}\to M$. The corresponding definitions will be given in Sec. 2.

ones, holes are at the boundary of billiard tables. Therefore. if one shrinks such hole to a point, the resulting limiting set has a finite size in the angle coordinate, which makes a rigorous mathematical analysis much more involved [Bun24].

We now outline the results of the present paper. The geometry of the circular billiard with two holes (one hole if $\theta=0$) is defined as in Fig. 1. Definitions are given in more detail in Sec. 2.

First, we prove that the surviving set for sufficiently late times consists of neighborhoods of periodic orbits. More specifically,

Theorem 1. *Let $N>\frac{4\pi}{\Delta}$. Then every connected component B_i, $i=1,2,\dots,m$, $m=m(\Delta)$ of the set $\mathfrak{N}_N$ of orbits never escaping until N collisions either (a) contains a unique segment $I_i=\{(\beta,\psi),\quad \beta_{i,1}<\beta<\beta_{i,2}\}$ consisting of never escaping periodic orbits, or (b) contains only creeping orbits.*

The angles β and ψ are defined in Sec. 2 below and illustrated in Fig. 1. Creeping orbits are those that have collisions along an arc of the circle without crossing either of the holes. That their contribution is negligible is established in the following lemma:

Lemma 1 (Creeping orbits)**.** *(a) In discrete time, the contribution of creeping orbits to survival probability, for initial conditions distributed with respect to the map or flow invariant measures, is $o(N^{-1})$. (b) In continuous time, creeping orbits escape in bounded time, specifically, the set $\mathcal{N}_t$ of orbits that never escape until time t contains no creeping orbits for $t>\min(5\sqrt{3},\frac{8\pi}{\Delta})$.*

Theorem 1 and Lemma 1 are proved in Sect. 3.

We find the contribution for each periodic orbit to the survival probability, defined in each of four ways. These four definitions are for initial conditions distributed with respect to the invariant measures of the billiard map and of the billiard flow, and for measuring escape with respect to (a number of) collisions and to (continuous) time. Survival probabilities are denoted $\mathcal{P}_{j,\theta}(\Delta)$ and are defined in Tab. 1. We define $\theta'=\theta\pmod{\frac{2\pi}{n}}$ and $x_+=\max(x,0)$. For two integers m and n the greatest common divisor is denoted (m,n) and the

Kronecker delta function, equal to one if $m = n$, otherwise zero, is denoted $\delta_{m,n}$. Then, we have

Theorem 2. *The limiting survival probabilities defined in Tab. 1 are given by*

$$\mathcal{P}_{j,\theta}(\Delta) = \frac{2^j a_j}{16\pi^2} \sum_{m,n} n \left[\left(\frac{2\pi}{n} - \theta' - \Delta \right)_+^2 + (\theta' - \Delta)_+^2 \right] \sin^j \frac{\pi m}{n} \tag{1}$$

The sum is over $1 \leq m < n$ *with* $(m,n) = 1$*. The indices* $j \in \{2', 2''\}$ *are interpreted as* 2 *when they appear in a formula such as* 2^j *and* $\sin^j x$*. Thus, in full we have*

$$\lim_{N\to\infty} N\mu_M(\mathfrak{N}_N) = \frac{1}{8\pi} \sum_{m,n} n \left[\left(\frac{2\pi}{n} - \theta' - \Delta \right)_+^2 + (\theta' - \Delta)_+^2 \right] \sin \frac{\pi m}{n} \tag{2}$$

$$\lim_{t\to\infty} t\mu_M(\mathcal{N}_t) = \frac{1}{4\pi} \sum_{m,n} n \left[\left(\frac{2\pi}{n} - \theta' - \Delta \right)_+^2 + (\theta' - \Delta)_+^2 \right] \sin^2 \frac{\pi m}{n} \tag{3}$$

$$\lim_{N\to\infty} N\mu_{\mathcal{M}}(\mathrm{proj}^{-1}\mathfrak{N}_N) = \frac{1}{2\pi^2} \sum_{m,n} n \left[\left(\frac{2\pi}{n} - \theta' - \Delta \right)_+^2 + (\theta' - \Delta)_+^2 \right] \sin^2 \frac{\pi m}{n} \tag{4}$$

$$\lim_{t\to\infty} t\mu_{\mathcal{M}}(\mathrm{proj}^{-1}\mathcal{N}_t) = \frac{1}{\pi^2} \sum_{m,n} n \left[\left(\frac{2\pi}{n} - \theta' - \Delta \right)_+^2 + (\theta' - \Delta)_+^2 \right] \sin^3 \frac{\pi m}{n} \tag{5}$$

Theorem 2 is proved in Sec. 4. Note that the sum is finite since both terms in the square brackets vanish if $n \geq \frac{2\pi}{\Delta}$. As $\Delta \to 0$ the number of terms increases without bound. We now perform Möbius and Mellin transforms to represent the limiting behaviour of the survival probability as a contour integral involving an infinite sum over n together with a sum over its divisors. Furthermore, if $\theta = 2\pi r/q$, where $0 \leq r < q$ are coprime integers, then we can express the integrand in terms of Dirichlet L-functions (of which the Riemann zeta function is a special case) involving a double sum in each of q terms:

Theorem 3. *The limiting survival probabilities are given by*

$$\mathcal{P}_{j,\theta}(\Delta) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} P(\Delta, s) F_j(\theta, s) ds \tag{6}$$

$$P(\Delta, s) = \frac{\Delta^{-s}(2\pi)^s}{s(s+1)(s+2)} \tag{7}$$

$$F_j(\theta, s) = a_j \sum_{n=1}^{\infty} \frac{\Theta(n,\theta,s)}{n^{s+1}} \sum_{d|n} \mu(d) C_j \left(\frac{n}{d} \right) \tag{8}$$

$$\Theta(n,\theta,s) = (1 - \{\frac{n\theta}{2\pi}\})^{s+2} + \{\frac{n\theta}{2\pi}\}^{s+2} \tag{9}$$

where {} indicates fractional part, $C > 1$, and $C_j(x)$ is defined in Tab. 1. Furthermore, when $\theta = 2\pi r/q$ we have

$$F_j\left(\frac{2\pi r}{q}, s\right) = a_j \sum_{\bar{c}=0}^{q-1} \sum_{\bar{d}=0}^{q-1} \Theta\left(\bar{c}\bar{d}, \frac{2\pi r}{q}, s\right) \tilde{C}_j(q, \bar{c}, s) \tilde{D}(q, \bar{d}, s) \tag{10}$$

$$\tilde{C}_j(q, \bar{c}, s) = \sum_{c=\bar{c} \pmod{q}} \frac{C_j(c)}{c^{s+1}} \tag{11}$$

$$= \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \left[\frac{\sum_\chi \bar{\chi}(\bar{c}') L(s+2k, \chi)}{b'^{s+2k} \phi(q'')} - \delta_{\bar{c},1}\right]$$

$$\tilde{D}(q, \bar{d}, s) = \sum_{d=\bar{d} \pmod{q}} \frac{\mu(d)}{d^{s+1}} \tag{12}$$

$$= \frac{\mu(b)}{b^{s+1}\phi(q')} \sum_\chi \frac{\bar{\chi}(\bar{d}')}{L(s+1, \chi) \prod_{p|b} (1 - \chi(p) p^{-s-1})}$$

In Eq. (11), $b' = (\bar{c}, q)$, $q'' = q/b'$, $\bar{c}' = \bar{c}/b'$, Dirichlet characters χ are taken with modulus q'', the first sum is over $c \geq 1$, B_{2k} are Bernoulli numbers [DLMF, (24.2.1)] and $K_j(x)$ is defined in Tab. 1. In Eq. (12), $b = (\bar{d}, q)$, $q' = q/b$, $\bar{d}' = \bar{d}/b$, characters are taken with modulus q', and the first sum is over $d \geq 1$.

The two parts of Theorem 3 are proved in Sections 5 and 6 respectively. Note that the one hole case $\theta = 0$ corresponds to $r = 0$ and $q = 1$. Observe also that Θ remains invariant if $\bar{c} \to q - \bar{c}$ or $\bar{d} \to q - \bar{d}$, but these change the signs of all odd characters in $\tilde{C}_j$ or $\tilde{D}$. Thus, no odd L-functions appear in F_j. The number of characters and L-functions modulo q is given by $\phi(q)$, the number of integers in $[1, q]$ that are coprime to q. When $q \in \{1, 2\}$ there is only a single, even, character, and when $q \in \{3, 4, 6\}$ there are two, one of which is odd. One even character for each q is the principal character, for which the L-function can be written in terms of the Riemann zeta function. This then implies that only the Riemann zeta function appears in F_j for $q \in \{1, 2, 3, 4, 6\}$.

These expressions suggest the possibility of connections with the (generalized) Riemann hypothesis: The Riemann hypothesis claims that the complex zeros of $\zeta(s)$ lie on the line $\Re(s) = 1/2$, and the generalized Riemann hypothesis says that the complex zeros of $L(s, \chi)$ lie on the same line. Thus, these statements are related to the locations of the poles of $\tilde{D}(q, \bar{d}, s)$ arising from $L(s+1, \chi)$ in the denominator, with residues contributing to the contour integral.

We then analyze the pole structure of the integrand of Eq. (6) in Section 7; see also Fig. 2. We present the above expressions for all $q \leq 6$, using Mathematica symbolic algebra for $4 \leq q \leq 6$, in Section 8. The residues of the poles are presented in Tab. 2, including exact values from Eqs. (95,106), other exact values using symbolic algebra and numerical values where there is a complicated sum over k.

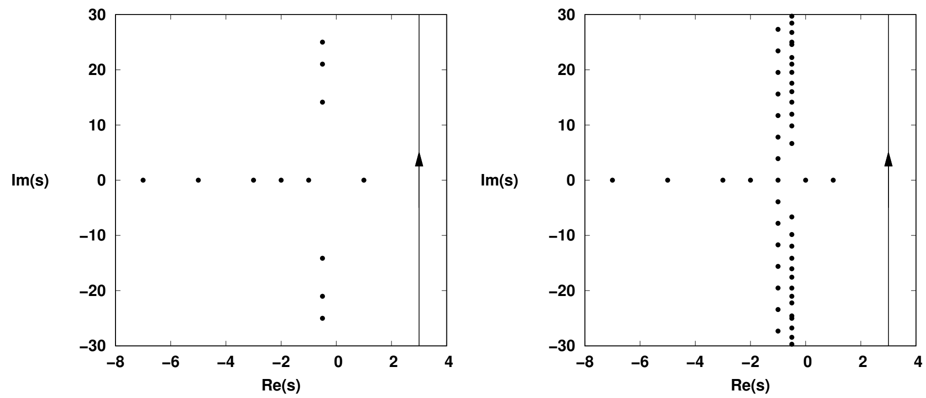

Figure 2: The contour of the Mellin representation of $\mathcal{P}_{j,\theta}(\Delta)$, Eq. (6), and poles of the integrand for $q = 1$ (left) and $q = 5$ (right). Note the scales on the real and imaginary axes. See Sec. 7.7 and Tab. 2.

In the simplest cases, one hole or two symmetric holes, we have

$$F_j(\frac{2\pi r}{q}, s) = a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \mathcal{F}_{r,q}(s,k) \tag{13}$$

$$\mathcal{F}_{0,1}(s,k) = \frac{\zeta(s+2k) - 1}{\zeta(s+1)} \tag{14}$$

$$\mathcal{F}_{1,2}(s,k) = \frac{\zeta(s+2k)}{\zeta(s+1)2^{s+2k}} \tag{15}$$

In Section 9 we consider a different scenario, that of q symmetrically placed holes, where the special case $q = 2$ is given above. We show

Theorem 4. *For the case of $q \geq 2$ symmetrically placed holes, $\mathcal{F}_{r,q}(s,k)$ in Eq. (13) is replaced by*

$$\mathcal{F}_q^{\rm sym}(s,k) = \frac{\zeta(s+2k)}{\zeta(s+1)q^{s+2k}} \tag{16}$$

The locations of the poles are the same as $q = 2$ of the two hole case above. The residues follow immediately from those of the $q = 2$ case where only the $k = 0$ term is relevant, that is, $j \in \{2', 2''\}$ or $s = 1$. In the latter case, the pole comes only from the $\zeta(s+2k)$ term in the numerator, for $k = 0$. For other poles, see the descriptions given in Sect. 7.

We have successfully related the survival probabilities to a contour integral with poles at locations related to the zeros of Dirichlet L functions, including the Riemann zeta function. To continue, we propose the following conjecture:

Conjecture 1. *Consider a circular billiard with one or two holes, using notation as defined above. The function $F_j(\theta, s)$ is meromorphic (for $\theta \in \pi\mathbb{Q}$ this follows from Theorem 3). Furthermore*

$$\mathcal{P}_{j,\theta}(\Delta) = \sum_{s^*} \mathrm{Res}_{s=s^*} \;\; P(\Delta, s) F_j(\theta, s) \tag{17}$$

with the series converging absolutely. Here, s^* are the locations of the poles.

To show this, one would need to infer the existence of a sequence of contours with integral tending to zero and which, in the limit, encloses all the poles (see Fig. 2). The main difficulty is in putting a lower bound on $\zeta(s+1)$ or $L(s+1, \chi)$ that is, in the denominator, uniform on the part of the contour in or near the critical strip. Far from the critical strip, we can bound the zeta function, for example, to show that the sum over the poles on the real axis converges; see Lemma 2 in Sect. 10. Numerical simulations in support of Conj. 1, testing absolute convergence of the sum in Eq. (17), and showing that when the leading term is subtracted from the numerical survival probability, the remainder for small Δ is qualitatively given by the next few real poles, are given in Sect. 11.

Assuming Conj. 1 the limit $\Delta \to 0$ may be taken to give

$$\mathfrak{P}_{j,0} = \lim_{\Delta \to 0} \Delta \mathcal{P}_{j,0}(\Delta) = \mathrm{Res}_{s=1} P(\Delta, s) F_j(\theta, s) = \begin{cases} \frac{4}{\pi} & j = 1 \\ 2 & j = 2' \\ \frac{4}{\pi} & j = 2'' \\ \frac{64}{3\pi^2} & j = 3 \end{cases} \tag{18}$$

for the one hole case, using Eq. (95).

The absence of the poles at $s = -2$ for the case $q = 2$, $j \in \{2', 2''\}$ would then imply that $\mathcal{P}_{2',\pi}(\Delta)$ and $\mathcal{P}_{2'',\pi}(\Delta)$ contain only odd powers of Δ, together with the conjugate complex pairs on the critical line; this is not obvious from Eqs (56,58).

In case of two holes the leading ($s = 1$) residue is exactly half of the one-hole residue, for all values of θ we calculated. This is a special case of Ref. [Det17] (see Remark 2.2 and Section 5) where it is proved that the leading order of the survival probability is independent of the position of the holes. Note that the limits in this paper and in Ref. [Det17] are different; the latter has first $\Delta \to 0$ keeping $S = \Delta t$ constant, and then $S \to \infty$. Thus we can write the leading order survival probabilities for $j = 3$ as

$$\mathfrak{P}_{3,\theta} = \lim_{\Delta \to 0} \lim_{S \to \infty} S \mu_{\mathcal{M}}(\mathrm{proj}^{-1} \mathcal{N}_{S/\Delta}) \tag{19}$$

$$\mathfrak{P}_{3,\theta}^{[\mathrm{Det17}]} = \lim_{S \to \infty} \lim_{\Delta \to 0} S \mu_{\mathcal{M}}(\mathrm{proj}^{-1} \mathcal{N}_{S/\Delta}) \tag{20}$$

If these limits commute (as we expect), we have for all $0 < \theta < 2\pi$, not only for rational multiples of π:

$$\mathfrak{P}_{j,\theta} = \frac{1}{2} \mathfrak{P}_{j,0} \tag{21}$$

The Riemann hypothesis (RH) states that the complex zeros of $\zeta(s)$ all lie on the line $\Re(s) = \frac{1}{2}$. Under Conjecture 1 and RH we can use Theorem 3 including Eqs (13–15) to then bound corrections to the leading order behaviour, and relate the one and two hole cases. For example we have

Theorem 5. *Assume RH and Conjecture 1. Then*

$$\lim_{\Delta\to 0} \Delta^{\delta-1/2} \left[\mathcal{P}_{j,0}(\Delta) - \mathrm{Res}_{s=1} \; P(\Delta,s)F_j(0,s)\right] = 0 \tag{22}$$

$$\lim_{\Delta\to 0} \Delta^{\delta-1/2} \left[\mathcal{P}_{j,0}(\Delta) - 2\mathcal{P}_{j,\theta}(\Delta)\right] = 0 \tag{23}$$

for every $\delta > 0$*, each* $j \in \{1, 2', 2'', 3\}$*, and* $\theta = \frac{2\pi}{q}$ *with* $q \in \{2, 3, 4, 6\}$*.*

Proof. This follows from Tannery's theorem, allowing interchange of the limit and sum over s^* in the expressions obtained by applying Conjecture 1 to Theorem 3. In particular, the summand

$$\Delta^{\delta-1/2} P(\Delta,s)F_j(\theta,s) \tag{24}$$

is bounded, uniform in Δ, for all $s^* \neq 1$ assuming RH. Conjecture 1 also ensures absolute convergence of the sum. □

For $q = 5$ and $q > 6$ there is a pole at $s = 0$ that would affect the above expressions, and also L-functions leading to poles on the same critical axis, the real part of s equal to $-1/2$, so this would require to deal with the generalized Riemann hypothesis.

On the other hand, it seems likely that Conj. 1 together with Eq. (22) or (23) should imply RH, but this does not appear to the authors to be completely obvious.

We conclude this section with some remarks on the extent to which the claims of Ref. [Bun05] can be rigorously justified. We have indeed shown that the long-time survival probabilities of the open circular billiard with one hole, two holes, can be written as a sum over non-escaping periodic orbits. We have further related these expressions to zeta and Dirichlet L-functions. However, connection to the Riemann hypothesis seems only possible at this time with the aid of Conjecture 1.

In addition to Ref. [Bun05] we have shown that the map and flow measures, and thus the escape with respect to the number of collisions and with respect to a continuous time, lead to different powers of sine in the periodic orbits contributions. The calculation presented in Ref. [Bun05] has $\sin^2 \frac{\pi m}{n}$ which corresponds to the "mixed" cases of invariant measure for a billiard map but with continuous time, or to invariant measure for the flow with discrete time. The more natural cases of the discrete-time map and of the continuous-time flow correspond to $\sin \frac{\pi m}{n}$ and $\sin^3 \frac{\pi m}{n}$, respectively. These cases are more involved, but it turned out that we could extend the analysis and present the new results here. We have also added the case of q symmetric holes.

In the future, it would be interesting to prove some of the above conjectures, including Conjecture 1, the commutation of limits in Eqs. (19,20) and showing that RH may be derived from Conjecture 1 and statements like Eqs. (22,23). Our techniques may be applied to find similarly precise asymptotics for escape in other open integrable billiards, such as the square. Finally, it would be good to verify at least the leading order escape rates, Eq. (18) in a physical experiment.

Acknowledgements

The authors thank Andrew Booker and Joseph Najnudel for helpful discussions, and an anonymous reviewer for many helpful suggestions. LAB is grateful for the support of the US National Science Foundation under Grant No. DMS-2054659.

2 Definitions and notations

Consider a billiard on the unit disk D, that is, a dynamical system generated by the motion of a point particle with a constant speed within D with elastic collisions (angle of incidence equals angle of reflection) from its boundary. Without any loss of generality we assume that the particle's speed is identically one, and therefore its velocity is completely defined by the angle ϑ it makes with the horizontal direction $-\pi < \vartheta \leq \pi$. The phase space of the billiard flow S^t, $-\infty < t < \infty$ is denoted $\mathcal{M} = (x, y, \vartheta)$ with $(x, y) \in D$ and ϑ the angle between the direction of motion and the x-axis. See Fig. 1.

Let $M = \{(\beta, \psi) : -\pi < \beta \leq \pi, -\frac{\pi}{2} \leq \psi \leq \frac{\pi}{2}\}$, where $\beta \in \partial D$. The billiard flow S^t induces the billiard map $T : M \to M$ defined as

$$T(\beta, \psi) = (\beta + \pi - 2\psi, \psi) \tag{25}$$

where ψ is the angle between the outward trajectory and the inner normal at $\beta \in \partial D$, and all the angles in Eq. (25) are taken modulo 2π. The natural projection $M \to \partial D$ we denote by π_β. Therefore, $\pi_\beta(\beta, \psi) = \beta$, where $(\beta, \psi) \in M$.

Another useful projection we denote as $\mathrm{proj} : \mathcal{M} \to M$ to give the point corresponding to the previous collision. Explicitly, this is

$$\mathrm{proj}(x, y, \vartheta) = (\vartheta + \arcsin L \pm \pi, \arcsin L) \tag{26}$$

where $L = x \sin\vartheta - y \cos\vartheta$ is the angular momentum of the particle. Let also $\mathrm{proj}^{-1} A \subseteq \mathcal{M}$ denote the preimage of $A \subseteq M$ under proj.

The normalised invariant measures of the map μ_M and flow $\mu_{\mathcal{M}}$ are, as usual for planar billiards, respectively:

$$d\mu_M = \frac{1}{2|\partial D|} \cos\psi \, d\beta \, d\psi = \frac{1}{4\pi} \cos\psi \, d\beta \, d\psi \tag{27}$$

$$d\mu_{\mathcal{M}} = \frac{1}{2\pi|D|} \, dx \, dy \, d\vartheta = \frac{1}{2\pi^2} \, dx \, dy \, d\vartheta \tag{28}$$

It is well known that orbits of the billiard in a circle are periodic with a period n, if

$$\psi = \psi_{m,n} \equiv \frac{\pi}{2} - \frac{m}{n}\pi \tag{29}$$

where m and n are coprime integers and $0 < m < n$. On the other hand, the orbit is everywhere dense in ∂D if ψ is incommensurable with π.

Suppose that two holes $H_1 = \{\beta : 0 < \beta < \Delta\}$ and $H_2 = \{\beta : \theta < \beta < \theta + \Delta\}$, $0 \leq \theta \leq \pi$, $\Delta > 0$, are placed at the boundary ∂D. If $\theta > \pi$ we interchange the holes to obtain $\theta \leq \pi$. Consider now a new dynamical system, an open billiard in ∂D with holes H_1 and H_2. In this open billiard any orbit (β_0, ψ_0) moves under the billiard map Eq. (25) until it hits one of the holes H_1 and H_2. When the orbit hits $H_1 \cup H_2$ it "disappears" (escapes).

When $\theta < \Delta$ the holes overlap and we obtain a single hole of size $\theta + \Delta$. When $\theta = \Delta$ there is again a single hole of size 2Δ except for a single point at its center; the point has zero measure and thus does not affect the survival probability. So, without loss of generality we may consider either the case of two holes $\Delta < \theta \leq \pi$, or one hole, where $H_1 = H_2$, $\theta = 0$.

Obviously, almost all (with respect to the measure μ) orbits will eventually escape. The only orbits that never escape are periodic orbits that never hit $H_1 \cup H_2$. Denote $\widehat{H}_i = \{(\beta, \psi) : \beta \in H_i\}$, $i = 1, 2$. Thus, $\pi_\beta \widehat{H}_i = H_i$, $i = 1, 2$.

Let $N(\beta_0, \psi_0)$, $(\beta_0, \psi_0) \in M$ be a (minimal) number of reflections from the boundary after which the orbit $T^n(\beta_0, \psi_0) = (\beta_n, \psi_n)$, $n = 1, 2, \ldots$ escapes from the circle. If the orbit of (β_0, ψ_0) never escapes, we set $N(\beta_0, \psi_0) = \infty$.

The time between collisions $\tau : M \to \mathbb{R}$ is $\tau(\beta, \psi) = 2\cos\psi$, so $\tau \circ \mathrm{proj}(x, y, \vartheta) = 2\sqrt{1 - L^2}$. The survival time t and a number of collisions N (implicitly functions of (β_0, ψ_0)) are thus related by $N = \lceil t/(2\cos\psi) \rceil$,where $\lceil x \rceil$ is the ceiling function, giving the least integer greater than or equal to its argument. The set $\mathfrak{N}_N \subset M$ surviving for N collisions is related to the set $\mathcal{N}_t \subset M$ surviving for time t, by their ψ sections:

$$\mathfrak{N}_N|_{\psi=\psi_0} = \mathcal{N}_{2N\cos\psi_0}|_{\arcsin L = \psi_0} \tag{30}$$

Survival probability is the measure of orbits that do not escape until a fixed time t or a number of collisions equal N. We obtain different results depending on whether the measure is μ_M or $\mu_{\mathcal{M}}$ and whether the non-escaping orbits are $\mathcal{N}_t$ or $\mathfrak{N}_N$.

3 Structure of the set of orbits not escaping for large N or t

In this section we prove Thm. 1, that is, for $N > \frac{4\pi}{\Delta}$, the set $\mathfrak{N}_N \subset M$ of initial conditions surviving for at least N collisions consists of connected components. Each such component either (a) contains exactly one segment consisting of never escaping periodic orbits of the same period, or (b) contains only creeping orbits. Moreover, there are no other periodic orbits in the set $\mathfrak{N}_N$. After giving in detail the argument for the billiard map for one hole, we explain the few changes needed for continuous time (for the set $\mathcal{N}_t$) and for two holes, the most important of which is the treatment of creeping orbits ($\psi \approx \pm\frac{\pi}{2}$). Finally, we prove Lemma 1, by enumerating the creeping orbits and showing that they are negligible for survival probability.

We will start with the billiard map T, see Eq. (25), which sends unit vectors with the footpoints on the boundary of a billiard table and pointing to the interior of the billiard table into unit vectors that arise just after a moment of the next reflection off the boundary. Clearly, the only orbits that never escape are those periodic orbits with periods $n < \frac{2\pi}{\Delta}$ that never hit the holes $H_1 \cup H_2$. Again, for clarity and simplicity, we will study in detail only the case with a single hole H of size Δ on the boundary of a billiard table.

The phase space of the billiard map is a cylinder M, consisting of all points (β, ψ), where β is a boundary coordinate varying between $-\pi$ and π, and ψ is an angle coordinate varying from $-\pi/2$ to $\pi/2$.

Consider in the cylinder M all points with $\psi = \psi_{m,n} = \frac{\pi}{2} - \frac{\pi m}{n}$, where $m < n$ are co-prime integers. Clearly, all such points form a (horizontal) circle C on the cylinder M, where the (vertical) coordinate ψ is equal to $\psi_{m,n}$. For these points, the billiard map T corresponds to a rotation in the first coordinate by an angle $\frac{2\pi m}{n}$.

The set of orbits, which do not escape until time (iterate) N, consists of connected components which are adjacent to (horizontal) circles in the phase space (cylinder) M, which consist of periodic points of the billiard map with periods n not exceeding N. In each circle, we have the intersection of the hole with the images (or preimages) of the cylinder M and $n-1$ of this intersection. The complement to these n segments in the circle in M also consists of n equal segments with length $(2\pi/n - \Delta)$. We will show that the set of non-escaping until time N orbits, which is adjacent to this circle of periodic points (with period n), consists of $2n$ right triangles, each of which is adjacent to one of the n segments in the complement described above. Clearly, all points of the n segments, which correspond to the hole and its $n-1$ images (preimages), do not belong to this set.

Due to symmetry, it is enough to consider only non-negative angles ψ. In fact, for $-\psi$, both the consideration and formulas are the same. Small values of ψ are connected to the periodic orbit with $\psi = 0$, that is, $n = 2$. Consider now all preimages of the hole, from the first to the $(n-1)th$ preimage. Observe that the nth preimage of each hole coincides with this hole. In the horizontal circle in question, the set consisting of the hole H and its preimages under the billiard map contains n segments.

Observe that in the circle C the complement to the shifts of the hole also consists of n segments with ends sharing with end points of the shifts of the hole H. Denote this set by C'. Clearly, these n complementary segments will never escape because the billiard map sends each of them exactly to another. Therefore, the iterations of all points of these segments are forever kept away from the holes.

We will show that exactly such segments, consisting of periodic points (of period n in this case), are the core subsets of connected components of the set of not escaping until time N points (i.e., the Nth iterate of the billiard map), where $N > \frac{4\pi}{\Delta}$. Denote this set by Ne.

Clearly, there are points in Ne with the same (horizontal) coordinate x as some point y in a complement to the shifts of holes C' in C, and with angles close to $\psi_{m,n}$. Such points belong to the same vertical segment in the cylinder M as the points y in C. Indeed, let such a point z correspond to a larger rotation angle than the points in our circle C, (i.e., its vertical coordinate on the phase cylinder M is larger than the coordinates of points on C). Then, to avoid escape until time N, the horizontal coordinate of z must be close enough to the left end of a segment of C which belongs to the complement of the hole and its $n-1$ images (preimages) on C. In fact, N images of that must not fall into the hole. Therefore, the closer the horizontal coordinate of z is to the left end of a segment in the complement of the hole and its iterations, then the larger its vertical coordinate is allowed to be. Analogously, if the horizontal coordinate of z belongs to the hole or its images on C, then, the closer its horizontal coordinate on C is to the right end of hole, the larger its vertical coordinate is allowed to be. Analogous consideration is applied to the set of points that do not escape until time N, which have smaller vertical coordinates than the points on the circle C, where all right end points must be changed to left end points and vice versa.

Let the coordinates of such a point z be $(y, \pi m/n+\alpha)$. Here $\alpha > 0$, but small enough that during N iterations of the billiard map it never hits holes. Due to the extra rotation on α in each iterate of the billiard map, the point y must be sufficiently far from the right end of the segment consisting of never-escaping points to which this point belongs. The length of any segment of never-escaping points is at most $\frac{2\pi}{n} - \Delta$, in a gap formed by the n images of a single hole. In fact, let the distance of the projection of the point z to C to the left end of a segment in the complement of the hole and its images be equal to u. Then the height of the vertical segment containing z, which consists of not escaping until time N points, equals $2\pi/N - u/N$, where u varies between 0 and 2π. Therefore, all such points form a right triangle adjacent (from above) to segments in the complemental set to the union of the hole and its $n-1$ images on our circle C in M. If a point z is above our circle and its projection to the circle is in the hole or its images, then, the left end becomes the right end and vice versa. Moreover, in the above expression we have Δ instead of $2\pi/N - \Delta$.

Analogously, we can consider the case of negative α. Then the corresponding points are below y on the cylinder M. Therefore, y must be sufficiently far from the left end of a segment in the complementary set to the hole and its $n-1$ images, to not escape (hit a hole) during N iterations of the billiard map.

We will now estimate the "height" of the vertical segment on the point y from C which consists of not escaping in N iterating points.

Let the distance of the point y from the right end of the segment of never escaping points containing y equals l. Since the billiard map, Eq. (25), involves 2ψ, a perturbation of ψ leads to twice the perturbation of the position around the circle. Thus the height of the vertical segment over y cannot be larger than

$\frac{1}{2N}(\frac{2\pi}{n} - \Delta - l)$.

Therefore, the set Ne of not escaping till time N (Nth iterate of the billiard map) consists of two equal right triangles adjacent to a segment consisting of never escaping points; see Fig. 3. The orthogonal to horizontal segment side of upper triangle, that is, its height, equals

$$\frac{1}{2N}(\frac{2\pi}{n} - \Delta) \tag{31}$$

at the left end of the segment and zero at its right end. The lower triangle is symmetric to the upper one and thus has zero length of a vertical segment at the left end of the segment consisting of never escaping points, and the same height as above but now at the right end of the segment.

The period n of the non-escaping orbits cannot exceed $[\frac{2\pi}{\Delta}]$. Therefore, $n\Delta \leq 2\pi$, otherwise, the images of the hole will eventually cover the entire circle because the size Δ of the hole is finite. We will show now that such sets consisting of segments of never-escaping points and two adjacent triangles do not intersect. In order to do that, we must consider the closest one to our horizontal circle on the cylinder M, which also consists of periodic points on the billiard map. Consider the fractions with coprime numerators and denominators, where all denominators do not exceed a given fixed integer $\mathbf{N}$ (here equal to $[\frac{2\pi}{\Delta}]$).They form the so called Farey numbers $F_{\mathbf{N}}$. It is well known that the fraction closest to m/n in the set $F_{\mathbf{N}}$ is such fraction p/q that $|m/n - p/q| = 1/qn$.

Let us show that the neighborhoods of the corresponding two circles, which consist of right triangles (see above), do not intersect. To do it, consider the maximal lengths of the vertical sides of these triangles and compare their sum with the distance between these two horizontal circles on the cylinder M.

Assume that the number of collisions $N > \frac{4\pi}{\Delta} \geq n + q$ since $n, q \leq \frac{2\pi}{\Delta}$. The (vertical) distance between the horizontal circles (m, n) and (p, q) is $|\psi_{m,n} - \psi_{p,q}| = \pi \left|\frac{m}{n} - \frac{p}{q}\right| = \frac{\pi}{nq}$.

Our goal now is to show that the sum of the maximal lengths of the sides of right triangles in the sets of nonescaping orbits adjacent to these two circles (consisting of periodic orbits of the billiard map) is exceeded by the vertical distance between two corresponding circles on the phase space cylinder M. The sum of the maximal heights of the right triangles on these intervals equals

$$\frac{1}{2N}\left[(\frac{2\pi}{n} - \Delta) + (\frac{2\pi}{q} - \Delta)\right] = \frac{\pi}{N}\frac{n+q}{nq} - \frac{\Delta}{N} < \frac{\pi}{nq}$$

noting that $n + q < N$ as above.

We now exclude the possibility of orbits that survive for N collisions but are not connected to never escaping orbits of period at most $2\pi/\Delta$. Since in an orbit of at least $2\pi/\Delta$ collisions the orbit must return to within an angle $\leq \Delta$ of its starting point, we define the minimum number of collisions required to do so as the period $n \leq 2\pi/\Delta$. Thus, $N > 4\pi/\Delta > 2n$. A general orbit avoiding the hole(s) may be denoted as

$$\beta_j = \beta_0 + \left(\frac{2\pi m}{n} - 2\eta\right) j, \qquad 0 \leq j \leq 2n \tag{32}$$

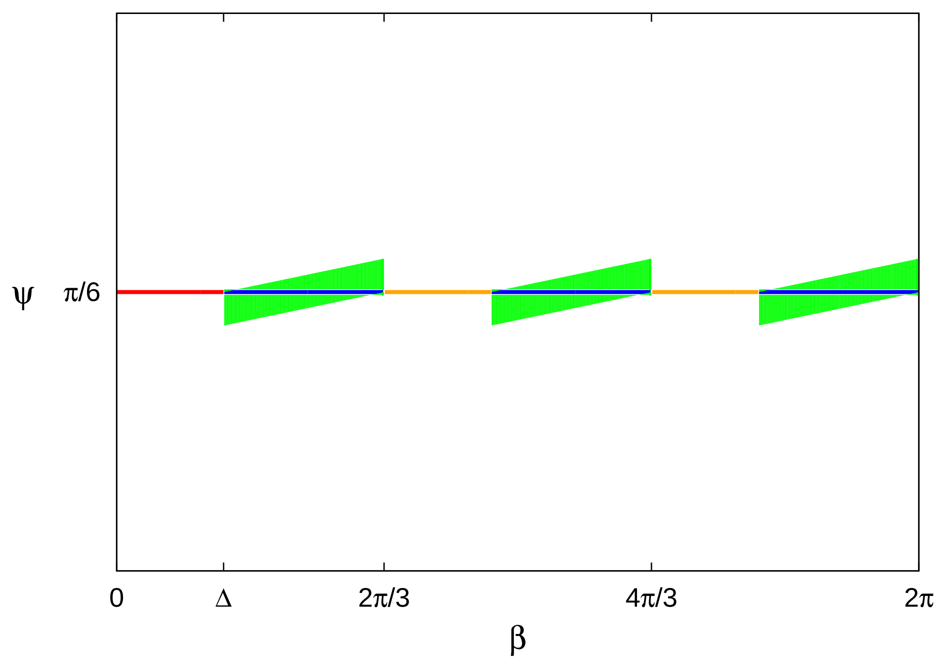

Figure 3: For a circular billiard with a single hole (ie $\theta = 0$) of size $\Delta < \frac{2\pi}{3}$, we illustrate the part of phase space near the $(m, n) = (1, 3)$ orbit. The set of points that never escape is shown in blue, together with the rest of the relevant circle ($\psi = \pi/6$; see Eq. 29), of which the red part corresponds to the hole and the orange parts its preimages. The set surviving for a large number N of collisions is shown in green, and consists of right triangles adjoining the points that never escape. Observe that for $\psi > \pi/6$, the dynamics, Eq. (25), gives a rotation around the circle of slightly less than $2\pi/3$. Thus the effective rotation is to the left in each of the upper green sets, so that the longest surviving orbits are those that start towards the right of these intervals.

where $\eta = \psi - \psi_{m,n}$ and we use Eqs. (25,29). Then, the periodic orbit

$$\bar{\beta}_j = \beta_n + \frac{2\pi m}{n} j, \qquad 0 \le j < n \tag{33}$$

is enclosed by the original orbit

$$\beta_{n+j} \le \bar{\beta}_j \le \beta_j, \qquad 0 \le j < n \tag{34}$$

for $\eta > 0$ and the reverse inequalities for $\eta < 0$. Since the original orbit returns to a distance less than Δ after n collisions, then it cannot enclose a hole. Thus, the periodic orbit also avoids the hole(s). The original arbitrary orbit may be continuously deformed to reach the periodic orbit. Hence, they are in the same connected component of orbits surviving for N collisions.

Therefore, the set of points Ne, which do not escape through the holes in N iterates of the billiard map consists of connected components. Each connected component of Ne is a neighborhood of a circle C on the cylinder M. Each such circle consists of periodic points of the billiard map with periods not exceeding $[2\pi/\Delta]$. Certainly, the holes and all their images (or preimages) under the first $(n-1)$ iterates of the billiard map do not belong to Ne.

For the case of two holes, the above analysis holds; depending on Δ and θ, the preimages of the holes may be completely disjoint, giving $2n$ non-escaping intervals, intersect, giving n non-escaping intervals, or cover the circle C entirely, giving no non-escaping intervals. The non-escaping intervals are at most the size of the one hole case, so the above argument with Farey fractions remains valid.

The case $n = 1$, corresponding to $\psi = \pm\frac{\pi}{2}$ is not a true periodic orbit, however its neighborhood is a right triangle as described above. These are the creeping orbits, those that move along an arc of the circle without crossing either of the holes. Thus for $N > \frac{4\pi}{\Delta}$ each connected component of $\mathfrak{N}_N$ contains a unique segment of periodic orbits, or contains only creeping orbits. This concludes the proof of Theorem 1.

For continuous time (a billiard flow) consideration is absolutely analogous to the one for discrete time (billiard map). Indeed, only periodic points may never escape. If we fix a large (continuous) time $t > \frac{8\pi}{\Delta}$ and note $\tau = 2\cos\psi \le 2$ so the number of collisions $N > \frac{4\pi}{\Delta}$ as assumed above. On each circle $\psi = \psi_{m,n}$ the same $2n$ (or n) segments of equal length will belong to the set of never escaping (and particularly till time t) points. Adjacent to each of these circles are as before the right triangles corresponding to $N = \lfloor \frac{t}{2\cos\psi} \rfloor$. Since N now varies with ψ, it is possible for most of the components corresponding to $\psi_{m,n}$ to correspond to a single value of N, while the tip of a right triangle corresponds to $N-1$ and can disconnect from the main component. The relative sizes of the right triangles also differ between periodic orbits at different $\psi_{m,n}$ compared to the discrete-time case. But the enumeration of the set $\mathcal{N}_t$ in terms of contributions from periodic orbits remains as the same in the case of discrete time.

Finally, we discuss the creeping orbits, which undergo N collisions with an arc of the circle without crossing either hole, and prove the claims of Lemma 1.

First, we deal with discrete time N. The creeping orbits have $\psi = \pm(\frac{\pi}{2} - \epsilon)$ with $0 < \epsilon < \frac{\pi}{N-1}$. Thus, their contribution to the survival probability is bounded above by

$$2\int_0^{2\pi} d\beta \int_{\frac{\pi}{2}-\frac{\pi}{N-1}}^{\frac{\pi}{2}} \frac{\cos\psi}{4\pi} d\psi = 1 - \cos\frac{\pi}{N-1} = O(N^{-2})$$

using Eq. (27). Using the same argument for Eq. (28) where initial conditions are in the interior, we find that the survival probability is $O(N^{-3})$. Thus in both cases this contribution is negligible compared with the periodic orbits, and their contribution is of order N^{-1} (see the next section).

For continuous time, we show that all creeping orbits escaped for $t > \min(5\sqrt{3}, \frac{8\pi}{\Delta})$. Let N be the number of collisions the creeping orbit makes with the boundary before entering or crossing one of the holes, and $t_m(N, \Delta)$ the supremum of the time the creeping orbit exists. Clearly $t_m(N, \Delta)$ is non-increasing in Δ. We have the following cases

- $N = 1$: Long creeping orbits start in the interior but arbitrarily close to the boundary, follow the diameter of the disk, collide, and then return along the diameter, giving $t_m(1, \Delta) = 4$.

- $N = 2$, $\Delta < \pi$: Long creeping orbits follow the diameter of the disk, make collisions with two opposite points, and then return along the diameter to give $t_m(2, \Delta) = 6$.

- $N = 2$, $\pi \leq \Delta < 2\pi$: Long creeping orbits have again escaping time just under 3τ, but now the maximum $\tau = 2\cos\psi = 2\sin\frac{2\pi-\Delta}{2}$. Thus, $t_m(2, \Delta) = 6\sin\frac{2\pi-\Delta}{2}$.

- $N \geq 3$: Long creeping orbits have escaping time just under $(N+1)\tau$, with the maximum $\tau = 2\cos\psi = 2\sin\frac{2\pi-\Delta}{2(N-1)}$ for the case of one hole, so that the largest hole-free arc of the boundary is of size $2\pi - \Delta$. Thus, $t_m(N, \Delta) \leq 2(N+1)\sin\frac{2\pi-\Delta}{2(N-1)}$

The above is for one hole; for two holes, $t_m(N, \Delta)$ may be less than this.

Now, we show that in each of these cases $t_m(N, \Delta) < \min(5\sqrt{3}, \frac{8\pi}{\Delta})$. This follows directly in the first two cases, since $\Delta < 2\pi$ in the first case, and $\Delta < \pi$ in the second case.

For the other cases we first show the inequality $t_m(N, \Delta) < \frac{8\pi}{\Delta}$. Note that the sine function is concave when its argument is in $[0, \pi]$ and that the graph of a concave function lies below any tangent to that graph. Then, the intersection of this tangent and the convex curve $t = \frac{8\pi}{\Delta}$ reduces to a quadratic equation with no real solutions.

For $N = 2$ and $\pi \leq \Delta < 2\pi$,

$$\begin{aligned} t_m(2,\Delta) &= 6 \sin \frac{2\pi - \Delta}{2} \\ &\leq 3\sqrt{3} - \frac{3}{2}(\Delta - \frac{4\pi}{3}) \\ &< \frac{8\pi}{\Delta} \end{aligned} \tag{35}$$

where the tangent to the graph is at $\Delta = \frac{4\pi}{3}$.

Similarly, for $N \geq 3$

$$\begin{aligned} t_m(N,\Delta) &\leq 2(N+1) \sin \frac{2\pi - \Delta}{2(N-1)} \\ &\leq \frac{N+1}{N-1}(2\pi - \Delta) \\ &< \frac{8\pi}{\Delta} \end{aligned} \tag{36}$$

where the tangent to the graph is at $\Delta = 2\pi$. The discriminant of the quadratic equation for the final inequality is $4\pi\frac{N+1}{N-1}\left(\pi\frac{N+1}{N-1} - 8\right)$ which is negative for $N > 2.3$. Thus, we have that $t_m(N,\Delta) < \frac{8\pi}{\Delta}$

Finally, we show the fixed bound $t_m(N,\Delta) \leq 5\sqrt{3}$. The fact that $t_m(N,\Delta)$ is non-increasing means that

$$t_m(N,\Delta) \leq t_m(N,0) \tag{37}$$

The concavity argument with the tangent at $\Delta = 2\pi$ gives again

$$t_m(N,0) \leq \frac{N+1}{N-1} 2\pi < 5\sqrt{3} \tag{38}$$

for $N > \frac{5\sqrt{3}+2\pi}{5\sqrt{3}-2\pi} \approx 6.2865$. Then, we can check the remaining cases $N \in \{3,4,5,6\}$ to show that $t_m(N,0) = 2(N+1)\sin\frac{\pi}{N-1} \leq 5\sqrt{3}$ which gives equality for $N = 4$. Thus, we have $t_m(N,\Delta) \leq 5\sqrt{3}$. In summary, all the creeping orbits escape by time $5\sqrt{3}$, or $\frac{8\pi}{\Delta}$ if this is earlier, so they give zero contribution to the survival probability if time is continuous. This concludes the proof of Lemma 1.

4 Limiting survival probabilities

In this section we prove Theorem 2, pertaining to the survival probability, that is, the measure of orbits that do not escape until a certain number of collisions N or time t. We can consider as initial conditions the measure with respect to the map μ_M or the flow $\mu_{\mathcal{M}}$, giving four expressions in total. As before, we consider in detail the simplest case, of discrete time in both the initial conditions and the dynamics, and the case of a single hole, and then discuss the straightforward modifications that need to be made for the other cases.

From Theorem 1 and Lemma 1 the survival probability for large N can be written in terms of contributions from connected components of periodic orbits:

$$\lim_{N\to\infty} N\mu_M(\mathfrak{N}_N) = \sum_{m,n,j} \lim_{N\to\infty} N\mu_M(\mathfrak{N}_{N,m,n,j}) \tag{39}$$

assuming that the limits exist. Here, $\{m, n, j\}$ label the non-escaping periodic orbits as before, with $1 \leq m < n < \frac{2\pi}{\Delta}$ and $(m, n) = 1$. Within each component, we write, following Eq. (29)

$$\psi = \psi_{m,n} + \eta \tag{40}$$

The label j enumerates the connected arcs of non-escaping periodic orbits; $1 \leq j \leq n$ in the one hole case. Eq. (39) is a finite sum, and the number of terms diverges as $\Delta \to 0$.

Each of the non-escaping periodic orbits is

$$\{(\beta, \psi) : \beta \in [\beta_j, \beta_j + \lambda_j], \psi = \psi_{m,n}\} \tag{41}$$

where λ_j is the length of the relevant arc and the dependence of β_j and λ_j on $\{m, n\}$ is omitted to simplify the notation.

Throughout the component of this periodic orbit, the billiard map, Eq. (25) is

$$T(\beta, \psi_{m,n} + \eta) = (\beta + \frac{2\pi m}{n} - 2\eta, \psi_{m,n} + \eta) \tag{42}$$

thus, iterated N times it becomes

$$T^N(\beta, \psi_{m,n} + \eta) = (\beta + \frac{2\pi m}{n}N - 2\eta N, \psi_{m,n} + \eta) \tag{43}$$

For each η, there is a set $\beta \in [\beta_-(\eta), \beta_+(\eta)]$ which survives after N collisions, defining the triangles depicted in Fig. 3. The value of β may extend beyond the periodic orbit, Eq. (41) near the start or end of the N collisions if it is in a pre-image of the hole, but not more than n collisions, the length of the periodic orbit. For $\eta > 0$ we find

$$\beta_j + 2\eta(N - n) \leq \beta_-(\eta) \leq \beta_j + 2\eta N \tag{44}$$

$$\beta_j + \lambda_j \leq \beta_+(\eta) \leq \beta_j + \lambda_j + 2\eta n \tag{45}$$

and similar expressions for $\eta < 0$. Thus, the length of the surviving set at fixed η, denoted $l(\eta) = \beta_+(\eta) - \beta_-(\eta)$ is

$$\max(\lambda_j - 2|\eta|N, 0) \leq l(\eta) \leq \lambda_j - 2|\eta|N + 4|\eta|n \tag{46}$$

This gives the maximum possible η for this component as

$$\eta_+ = \frac{\lambda_j}{2N - 4n} \tag{47}$$

with the minimum $\eta_- = -\eta_+$. We can now express the measure of the relevant component as an integral, using Eq. (27):

$$\mu_M(\mathfrak{N}_{N,m,n,j}) = \frac{1}{4\pi}\int_{-\eta_+}^{\eta_+} l(\eta)\cos(\psi_{m,n}+\eta)d\eta \tag{48}$$

We have

$$\cos(\psi_{m,n}+\eta) = \cos\psi_{m,n} + O(N^{-1}) \tag{49}$$

from Eq. (47). Substituting Eqs. (46,49) into Eq, (48) we obtain

$$\mu_M(\mathfrak{N}_{N,m,n,j}) = \frac{\lambda_j^2}{8\pi N}\cos\psi_{m,n} + O(N^{-2}) \tag{50}$$

Noting that the limits in Eq. (39) do indeed exist, we combine this with Eqs. (29,50) and find

$$\lim_{N\to\infty} N\mu_M(\mathfrak{N}_N) = \sum_{m,n,j}\frac{\lambda_j^2}{8\pi}\sin\frac{\pi m}{n} \tag{51}$$

As above, the sum is over $1 \le m < n < \frac{2\pi}{\Delta}$ with $(m,n) = 1$. In the one hole case there are n arcs of size $\lambda_j = \frac{2\pi}{n} - \Delta$ so we have

$$\lim_{N\to\infty} N\mu_M(\mathfrak{N}_N) = \frac{1}{8\pi}\sum_{m,n} n\left(\frac{2\pi}{n}-\Delta\right)^2\sin\frac{\pi m}{n} \tag{52}$$

This completes the calculation of the survival probability in the one hole case, for discrete time (both initial measure and escape time). For the remainder of this section, we show how to modify this result for two holes and continuous time (initial measure and/or escape time).

For the case of two holes, we note that the union of n preimages of the holes is periodic a period $\frac{2\pi}{n}$. Let us write $\theta' = \theta \pmod{\frac{2\pi}{n}}$.

Within the first unit cell, $\beta \in [0, \frac{2\pi}{n})$ there is a hole $H_1 = \{\beta : 0 \le \beta \le \Delta\}$ (noting that $\Delta < \frac{2\pi}{n}$). The other hole or one of its preimages is $H_2' = \{\beta : \theta' \le \beta \le \theta' + \Delta\}$. This leads to (potentially) two arcs of non-escaping periodic orbits in the unit cell, for $\Delta < \beta < \theta'$ and for $\theta' + \Delta < \beta < \frac{2\pi}{n}$. The lengths of these arcs are, respectively, $(\theta'-\Delta)_+$ and $\left(\frac{2\pi}{n}-\theta'-\Delta\right)_+$ where we use the notation $x_+ = \max(x,0)$ for some $x \in \mathbb{R}$ to account for cases where one or both of these arcs do not exist.

There are n copies of this unit cell, and these copies are periodic. Thus, there are n arcs of length $(\theta'-\Delta)_+$ and n arcs of length $\left(\frac{2\pi}{n}-\theta'-\Delta\right)_+$ of non-escaping periodic orbits. Thus, there are 0, n, or $2n$ arcs, depending on which of these quantities is positive. Eq. (51) becomes

$$\lim_{N\to\infty} N\mu_M(\mathfrak{N}_N) = \frac{1}{8\pi}\sum_{m,n} n\left[\left(\frac{2\pi}{n}-\theta'-\Delta\right)_+^2 + (\theta'-\Delta)_+^2\right]\sin\frac{\pi m}{n} \tag{53}$$

The one hole case, Eq. (52) is a special form of this equation, when $\theta = 0$. Note that we need not explicitly impose the condition $n < \frac{2\pi}{\Delta}$ as follows from the x_+ notation. We still require $(m,n) = 1$.

For escape in continuous time t, we note Eq. (30) and surrounding discussion. For each ψ we have $N = \lceil t/(2\cos\psi) \rceil$, and for each periodic orbit $\{m, n\}$ we have (see Eqs. (40,47)) that the values of ψ lie in

$$\psi_{m,n} - \eta_+ < \psi < \psi_{m,n} + \eta_+ \tag{54}$$

where $\eta_+ = O(N^{-1})$. This means that

$$N = \lceil t/(2\cos\psi) \rceil + O(1) \tag{55}$$

for any contribution at fixed t related to this periodic orbit. The $O(1)$ term contributes to the $O(N^{-2})$ term in Eq. (50) and so may be neglected. Thus Eq. (53) remains valid with an extra $2\cos\psi$ weighting on each orbit, leading to

$$\lim_{t\to\infty} t\mu_M(\mathcal{N}_t) = \frac{1}{4\pi}\sum_{m,n} n\left[\left(\frac{2\pi}{n} - \theta' - \Delta\right)_+^2 + (\theta' - \Delta)_+^2\right]\sin^2\frac{\pi m}{n} \tag{56}$$

In case of a flow, we place initial conditions in the interior of the billiard according to $\mu_{\mathcal{M}}$. Now, project them to M according to the projection proj. The measure of a set $A \subseteq M$ under the map and flow invariant measures are related by

$$\int_A d\mu_M = \frac{\pi}{2}\int_{\mathrm{proj}^{-1}A}\frac{d\mu_{\mathcal{M}}}{\tau\circ\mathrm{proj}} \tag{57}$$

That is, each part of the set A expands by a factor τ under proj^{-1}, so this has to be divided when integrating over $\mathcal{M}$. The prefactor $\frac{\pi}{2}$ arises as the ratio of the normalization constants in Eqs. (27,28) and it is easy to check that both sides of the equation are unity when $A = M$.

Thus, when integrating over the neighborhoods of periodic orbits of the flow measure, the results are for the map measure, weighted by $\frac{2\tau}{\pi}$. From Eqs. (53,56) we find

$$\lim_{N\to\infty} N\mu_{\mathcal{M}}(\mathrm{proj}^{-1}\mathfrak{N}_N) = \frac{1}{2\pi^2}\sum_{m,n} n\left[\left(\frac{2\pi}{n} - \theta' - \Delta\right)_+^2 + (\theta' - \Delta)_+^2\right]\sin^2\frac{\pi m}{n} \tag{58}$$

$$\lim_{t\to\infty} t\mu_{\mathcal{M}}(\mathrm{proj}^{-1}\mathcal{N}_t) = \frac{1}{\pi^2}\sum_{m,n} n\left[\left(\frac{2\pi}{n} - \theta' - \Delta\right)_+^2 + (\theta' - \Delta)_+^2\right]\sin^3\frac{\pi m}{n} \tag{59}$$

This completes the proof of Thm. 2.

Note that in the four expressions for the survival probability, Eqs. (53,56,58,59), differ only by an overall constant and the power of the sine function. This motivates the following simpler notation $\mathcal{P}_{j,\theta}(\Delta)$ given in Tab. 1 and in the statement of Thm. 2. The cases with $\sin^2\frac{\pi m}{n}$ are the same calculation as in Ref. [Bun05]. Here we also consider the remaining cases, Eqs. (53,59).

5 Möbius and Mellin transforms

In this section we prove the first part of Theorem 3. The Möbius inversion formulas (Ref. [DLMF, (27.5.3)] replacing d by n/d in the second equation) are

$$g(n) = \sum_{d|n} f(d) \qquad f(n) = \sum_{d|n} \mu(d) g(n/d) \tag{60}$$

If

$$f(n) = \sum_{\substack{m=1 \\ (m,n)=1}}^{n} h(m/n) \tag{61}$$

for some function h, then

$$\begin{aligned} g(n) &= \sum_{d|n} f(d) \\ &= \sum_{d|n} \sum_{\substack{k=1 \\ (k,d)=1}}^{d} h(k/d) \\ &= \sum_{m=1}^{n} h(m/n) \end{aligned} \tag{62}$$

where the second line enumerates the reduced fractions k/d of m/n in the third line.

Thus the Möbius inversion formula gives

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} h(m/n) = \sum_{d|n} \mu(d) \sum_{m=1}^{n/d} h(dm/n) \tag{63}$$

To apply this to Eq, (1) we write

$$\sin^2 \frac{\pi m}{n} = \frac{1}{2}\left(1 - \cos\frac{2\pi m}{n}\right) \tag{64}$$

$$\sin^3 \frac{\pi m}{n} = \frac{3}{4}\sin\frac{\pi m}{n} - \frac{1}{4}\sin\frac{3\pi m}{n} \tag{65}$$

So we need $h(x) = \sin(\pi x)$, $h(x) = \cos(2\pi x)$ or $h(x) = \sin(3\pi x)$. Writing the trigonometric functions as exponentials and summing geometric series, we find

$$\sum_{m=1}^{n} \sin\frac{\pi m}{n} = \cot\frac{\pi}{2n} \tag{66}$$

$$\sum_{m=1}^{n} \cos\frac{2\pi m}{n} = \delta_{n,1} \tag{67}$$

$$\sum_{m=1}^{n} \sin\frac{3\pi m}{n} = \cot\frac{3\pi}{2n} \tag{68}$$

where $\delta_{n,1}$ is one if $n = 1$ and zero otherwise.

Thus we find

$$\sum_{m=1}^{n} \sin \frac{m\pi}{n} = \cot \frac{\pi}{2n} \tag{69}$$

$$\sum_{m=1}^{n} \sin^2 \frac{m\pi}{n} = \frac{1}{2}(n - \delta_{n,1}) \tag{70}$$

$$\sum_{m=1}^{n} \sin^3 \frac{m\pi}{n} = \frac{3}{4} \cot \frac{\pi}{2n} - \frac{1}{4} \cot \frac{3\pi}{2n} \tag{71}$$

Now we apply Möbius inversion, Eq. (63) to get

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} \sin \frac{\pi m}{n} = \sum_{d|n} \mu(d) \cot \frac{\pi d}{2n} \tag{72}$$

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} \sin^2 \frac{\pi m}{n} = \sum_{d|n} \mu(d) \frac{1}{2} \left[\frac{n}{d} - \delta_{n/d,1} \right] = \frac{1}{2} [\phi(n) - \mu(n)] \tag{73}$$

$$\sum_{\substack{m=1 \\ (m,n)=1}}^{n} \sin^3 \frac{\pi m}{n} = \sum_{d|n} \mu(d) \left[\frac{3}{4} \cot \frac{\pi d}{2n} - \frac{1}{4} \cot \frac{3\pi d}{2n} \right] \tag{74}$$

In Eq. (73), we can explicitly sum over d as shown, where $\phi(n)$ is the Euler totient function (see Ref. [Bun05] for details). However, for consistency with the other cases it is simpler not to do so.

Then Eq. (1) becomes

$$\mathcal{P}_{j,\theta}(\Delta) = \frac{a_j}{8\pi^2} \sum_{n=1}^{\infty} n \left[\left(\frac{2\pi}{n} - \theta' - \Delta \right)_+^2 + (\theta' - \Delta)_+^2 \right] \sum_{d|n} \mu(d) C_j \left(\frac{n}{d} \right), \tag{75}$$

where a_j and $C_j(x)$ are defined in Tab. 1. The sum over n can be written in this unrestricted form because $C_j(1) = 0$ for all j. Hence, the $n = 1$ term vanishes. Also, both terms in the square brackets vanish when $n \geq \frac{2\pi}{\Delta}$.

Now, apply a Mellin transform in Δ. See Ref. [DLMF, §1.14(iv)]. Thus

$$\begin{aligned} \widetilde{\mathcal{P}}_{j,\theta}(s) &= \int_0^{\infty} \Delta^{s-1} \mathcal{P}_{j,\theta}(\Delta) d\Delta \\ &= P(\Delta, s) F_j(\theta, s) \end{aligned} \tag{76}$$

$$\mathcal{P}_{j,\theta}(s) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \Delta^{-s} \widetilde{\mathcal{P}}_{j,\theta}(s) ds \tag{77}$$

where $P(\Delta, s)$ and $F_j(\theta, s)$ are given in the statement of Thm 3. The Mellin transformation theorem and the interchange of sum and integral can both be justified when the integral is absolutely convergent. In fact, it holds to the right of all poles in the integrand, namely for $C > 1$. The result is Eq.(6), i.e., the first part of Thm. 3 holds.

6 Rational hole spacing

In this section, we prove the second part of Theorem 3. Let $\theta = 2\pi r/q$. Then $\Theta(n, \theta, s)$ is periodic in n with period q. Writing $n = cd$ and separating the sum according to the values of c and d modulo q (denoted $\bar{c}$ and $\bar{d}$ respectively), we obtain Eq. (10) with the sums defining $\tilde{C}(q, \bar{c}, s)$ and $\tilde{D}(q, \bar{d}, s)$ still to be evaluated.

Let us consider now $\tilde{D}(q, \bar{d}, s)$. Transform Eq. (12) by dividing all terms by the greatest common divisor $b = (\bar{d}, q)$. Then

$$\tilde{D}(q, \bar{d}, s) = \sum_{d' \equiv \bar{d}' (\bmod\ q')} \frac{\mu(bd')}{(bd')^{s+1}}, \tag{78}$$

where $d' = d/b$, $\bar{d}' = \bar{d}/b$ and $q' = q/b$. Here $(\bar{d}', q') = 1$.

To make the paper self-contained, we recall now some facts about the Dirichlet characters (see, e.g., [Dav13] for more details). The Dirichlet's characters to the modulus q are multiplicative functions $\chi(n)$ of an integer variable n which are periodic with period q. The conjugacy classes modulo q, which are coprime with q, form an abelian group under multiplication.

It is easy to see that the order of this group is equal to the Euler totient function $\phi(q)$. Moreover, it is a finite abelian group. Thus, it has $\phi(q)$ irreducible representations $\chi(n)$, where $(n, q) = 1$. The characters $\chi(n)$ are in this case the complex roots of unity, i.e., $\chi(m)\chi(n) = \chi(mn)$. This definition is extended by setting $\chi(n) = 0$, if $(n, q) > 1$.

By the orthogonality relation [Dav13]

$$\frac{1}{\phi(q)} \sum_{\chi} \bar{\chi}(a)\chi(n) = \delta_{a,n} \tag{79}$$

where $\delta_{a,n} = 1$, if $a \equiv n (\bmod\ q)$, zero otherwise, and $\bar{x}$ denotes a complex conjugate to a number x.

By inserting Eq. (79) into Eq. (78) we get

$$\tilde{D}(q, d, s) = \frac{1}{\phi(q')} \sum_{\chi} \bar{\chi}(\bar{d}') \sum_{d'=1}^{\infty} \chi(d') \frac{\mu(bd')}{(bd')^{s+1}} \tag{80}$$

Let $d' = \prod_p p^{\alpha_p}$ be the decomposition of d' into prime factors. Then $\chi(d') = \prod_p \chi(p)^{\alpha_p}$. Furthermore,

$$\mu(bd') = \begin{cases} \mu(b) \prod_p (-1)^{\alpha_p} & \text{if } bd' \text{ is square free} \\ 0 & \text{otherwise,} \end{cases} \tag{81}$$

Farther

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{s+1}} = \prod_p (1 - p^{-s-1}) = \zeta(s+1)^{-1}, \tag{82}$$

where $\zeta(s)$ is the Riemann zeta function. Analogously

$$\sum_n \frac{\chi(n)\mu(n)}{n^{s+1}} = L(s+1,\chi)^{-1}, \tag{83}$$

where

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1} \tag{84}$$

is the Dirichlet L function.

Now the Möbius μ function is not completely multiplicative. From Eq. (81), $\mu(bd') = \mu(b)\mu(d')$ when $(b,d') = 1$, otherwise it is zero. Thus, we may take $\mu(b)$ and b^{s+1} from the sum in Eq. (80) at a cost of removing all d' that have a common factor with b. In particular

$$\begin{aligned}\sum_{\substack{d'=1\\(d',b)=1}}^{\infty} \frac{\chi(d')}{d'^{s+1}} =& \prod_{p\nmid b} \left(1 - \frac{\chi(p)}{p^{s+1}}\right)^{-1} \\ =& L(s+1,\chi) \prod_{p|b} \left(1 - \frac{\chi(p)}{p^{s+1}}\right)\end{aligned} \tag{85}$$

Finally we arrive at Eq. (12).

If $q' = 1$. then $L(s,\chi)$ reduces to the Riemann zeta function $\zeta(s)$. For each q' there is a trivial character $\chi_1(\bar{d}')$ that assumes value 1 for all $\bar{d}'$ coprime to q'. Therefore

$$L(s,\chi_1) = \zeta(s) \prod_{p|q'} (1 - p^{-s}). \tag{86}$$

Consider now $\tilde{C}(q,\bar{c},s)$ in Eq. (11). We expand the cot terms using their power series [DLMF, (4.19.6)] to obtain

$$C_j(c) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} \frac{K_j(k)}{c^{2k-1}}, \tag{87}$$

where B_{2k} are the Bernoulli numbers, and $K_j(x)$ is defined in Tab. 1. Eq. (87) does not hold for the case $j \in \{2', 2''\}$ and $c = 1$, in which case $C_j(c) = 0$ directly from Tab. 1.

Explicitly, the cot expansions come to

$$\cot\frac{\pi}{2c} = \frac{2c}{\pi} - \frac{\pi}{6c} - \frac{\pi^3}{360c^3} - \frac{\pi^5}{15120c^5} + \dots \tag{88}$$

$$3\cot\frac{\pi}{2c} - \cot\frac{3\pi}{2c} = \frac{16c}{3\pi} + \frac{\pi^3}{15c^3} + \frac{\pi^5}{63c^5} + \dots \tag{89}$$

Note that the second expression does not have a term proportional to c^{-1}, since $K_3(1) = 0$.

The $C_1(c)$ series converges for $c > 1/2$, whilst the $C_3(c)$ series converges for $c > 3/2$. There is also an exception noted above for $c = 1$ when $j \in \{2', 2''\}$. In all cases, $C_j(1) = 0$, so we perform the sum for $c > 1$ in what follows.

Using the orthogonality of Dirichlet characters as before, we get

$$\begin{aligned}\sum_{c=\bar{c} \bmod q} \frac{1}{c^{s+2k}} &= \sum_{c'=\bar{c}' \bmod q''} \frac{1}{(b'c')^{s+2k}} \\ &= \frac{1}{\phi(q'')} \sum_{\chi} \bar{\chi}(\bar{c}') \sum_{c'=1}^{\infty} \frac{\chi(c')}{(b'c')^{s+2k}} \\ &= \frac{\sum_{\chi} \bar{\chi}(\bar{c}') L(s+2k, \chi)}{b'^{s+2k} \phi(q'')}\end{aligned} \tag{90}$$

where we denote $b' = (\bar{c}, q)$, $q'' = q/b'$, $\bar{c}' = \bar{c}/b'$ and Dirichlet characters are of modulus q''. Finally, we require $c > 1$:

$$\sum_{\substack{c = \bar{c} \bmod q \\ c > 1}} \frac{1}{c^{s+2k}} = \frac{\sum_{\chi} \bar{\chi}(\bar{c}') L(s+2k, \chi)}{b'^{s+2k} \phi(q'')} - \delta_{\bar{c},1}, \tag{91}$$

This expression equals $O(2^{-s-2k})$, as seen from the left hand side.

Combining Eqs. (87,91) and the first equality in Eq. (11) we arrive at the second equality in Eq. (11). The series in k converges exponentially, except exactly at a pole of the zeta function, that is, $s = 1 - 2k$, since $\frac{\pi^{2k} B_{2k}}{(2k)!} \sim 2^{1-2k}$ [DLMF, (24.11.1)], $K_j(k) = O(3^{2k})$ and the square bracket in Eq. (11), namely Eq. (91), is $O(2^{-2k})$. This completes the proof of Theorem 3.

7 Poles and residues

In this section we study the poles of the integrand $P(\Delta, s) F_j(\theta, s)$ of the Mellin transform, Eq. (6), for rational hole spacing $\theta = 2\pi r/q$, in decreasing order of $\Re(s)$. The results are summarized in Sec. 7.7.

7.1 $s = 1$

The residue of the pole at $s = 1$ arises only from the $k = 0$ term of the c series. We have

$$\mathrm{Res}_{s=1}\ \tilde{C}_j(q, \bar{c}, s) = \frac{2K_j(0)}{\pi q} \tag{92}$$

independent of $\bar{c}$. The full residue is thus

$$\mathrm{Res}_{s=1}\ P(\Delta, s) F_j(\theta, s) = \frac{\alpha_j}{q\Delta} \sum_{\bar{c}=0}^{q-1} \sum_{\bar{d}=0}^{q-1} \Theta(\bar{c}\bar{d}, \frac{2\pi r}{q}, 1) \tilde{D}(q, \bar{d}, 1) \tag{93}$$

with $\alpha_j = \frac{2}{3} K_j(0) a_j$ given in Tab. 1.

Assuming $(q,r)=1$ without loss of generality and that $\Theta(\bar{c}\bar{d}, 2\pi r/q, 1)$ has period q in its first argument, we note that any $r \neq 1$ just permutes the sum over $\bar{c}$, so we may take $r=1$. The sum over $\bar{c}$ then gives

$$\begin{aligned} \frac{1}{q}\sum_{\bar{c}=0}^{q-1}\Theta\left(\bar{c}\bar{d}, \frac{2\pi r}{q}, 1\right) &= \frac{1}{q'}\sum_{j=0}^{q'-1}\Theta\left(j, \frac{2\pi}{q'}, 1\right) \\ &= \frac{1}{q'^4}\left(q'^3 + 2\sum_{j=1}^{q'-1} j^3\right) \\ &= \frac{q'^2+1}{2q'^2} \end{aligned} \tag{94}$$

where, as before, $q' = q/b$, $b = (q,\bar{d})$. In the first line we note that $\bar{c}\bar{d}/q$ simplifies by dividing through by the common factor b and that $\bar{d}/b$ is coprime with q' and so permutes the sum as above. The second line uses the definition of Θ, and separates the terms with $j=0$ and $j>0$. The third line uses the well-known expression for sums of cubes

Summing over $\bar{d}$ within the classes defined by constant b all non-principal characters cancel to give

$$\sum_{\bar{d}:(q,\bar{d})=b} \tilde{D}(q,\bar{d},1) = \frac{6}{\pi^2}\frac{\mu(b)}{b^2}\frac{1}{\prod_{p|q}(1-p^{-2})}$$

where we have substituted $\zeta(2) = \pi^2/6$. Finally, Eq. (93) becomes

$$\begin{aligned} \mathrm{Res}_{s=1}\ P(\Delta,s)F_j(\theta,s) &= \frac{\alpha_j}{\Delta}\frac{6}{\pi^2}\sum_{b|q}\frac{(q/b)^2+1}{2(q/b)^2}\frac{\mu(b)}{b^2}\frac{1}{\prod_{p|q}(1-p^{-2})} \\ &= \frac{\alpha_j}{\Delta}\frac{3}{\pi^2}\frac{q^2\sum_{b|q}\mu(b)/b^2 + \sum_{b|q}\mu(b)}{q^2\prod_{p|q}(1-p^{-2})} \\ &= \frac{\alpha_j}{\Delta}\frac{3}{\pi^2}(1+\delta_{q,1}) \end{aligned} \tag{95}$$

Thus the residue has a simple closed form and is independent of θ except for $q=1$ (one hole case); Eq. (21) claims that this independence holds even when θ is not a rational multiple of π.

7.2 $s=0$

For $s=0$ there is a pole arising from $P(\Delta,s)$. However for $q \in \{1,2,3,4,6\}$ there are no non-principal even characters, so this is cancelled by the zeta function in the denominator of $\tilde{D}(q,\bar{d},s)$. For other values of q, the Dirichlet L-functions at argument 1 can be evaluated exactly using the discrete Fourier transform [MSE]

$$L(1,\chi) = -\frac{1}{q}\sum_{k=1}^{q-1}\ln(1-e^{2\pi ik/q})\sum_{n=1}^{q}\chi(n)e^{-2\pi ink/q} \tag{96}$$

See for example Tab. 2 below, $q = 5$ at $s = 0$. However there do not appear to be any significant simplifications to the expression for the residue.

7.3 $\Re(s) = -\frac{1}{2}$

These arise due to zeros of zeta and L-functions in the denominator of $\tilde{D}(q, \bar{d}, s)$. If the Generalized Riemann Hypothesis is false, then other poles exist in the critical strip with real part $[-1, 0]$. The density of poles increases with q as more and more L-functions contribute; see Fig. 2. This suggests that for θ not a rational multiple of π, it may not be possible to analytically continue the Mellin transform to smaller real part.

7.4 $s = -1$

For $s = -1$ there is a pole arising from $P(\Delta, s)$ and for $j = 1$ also one from $\tilde{C}_j(q, \bar{c}, s)$.

It is tempting to note

$$\Theta(n, \theta, -1) = 1 \tag{97}$$

for all θ, not just rational multiples of π. For rational cases, the sums over $\bar{c}$ and $\bar{d}$ decouple, and we can perform them:

$$\sum_{\bar{c}=0}^{q-1} \tilde{C}_j(q, \bar{c}, s) = \tilde{C}_j(1, 0, s) \tag{98}$$

$$\sum_{\bar{d}=0}^{q-1} \tilde{D}(q, \bar{d}, s) = \tilde{D}(1, 0, s) \tag{99}$$

with the relevant expressions in Sec. 8.1 below. Substituting $s = -1$ we have $\tilde{D}(1, 0, -1) = -2$, that is, finite.

However, this is too naive. For $q > 1$ the limit $s \to -1$ is singular, due to the factor $\prod_{p|b}(1 - \chi(p)p^{-s-1})$ in the denominator of $\tilde{D}(q, \bar{d}, s)$ which gives a pole of order the number of prime factors of q. These poles cancel in the sum over $\bar{d}$ exactly at $s = -1$ but still lead to divergence in the limit $s \to -1$ when there is more than one prime factor (for example $q = 6$), leading to an overall increase in the order of the pole at $s = -1$. Even when there is a single prime factor, the singular limit needs to be addressed correctly. We do not have a general expression, but the residues for $q \leq 6$ are given in Tab. 2 below.

7.5 $\Re(s) = -1$, $\Im(s) \neq 0$

The factor $\prod_{p|b}(1 - \chi(p)p^{-s-1})$ in the denominator of $\tilde{D}(q, \bar{d}, s)$ can also lead to further poles with imaginary part equal to -1. It is zero when

$$s = -1 + \frac{i(\arg \chi(p) + 2\pi l)}{\ln p}, \qquad l \in \mathbb{Z} \tag{100}$$

However, some of these poles may be canceled when the sums are organized to construct $F_j(2\pi/q, s)$, and indeed they do not appear for $q \in \{1, 2\}$. See Sec. 8.5 for details.

7.6 $s = -2$

For $j \in \{2', 2''\}$ there is another case for which the residue can be found exactly, namely $s = -2$. Unlike the case $s = -1$ in the previous section, both $\tilde{C}$ and $\tilde{D}$ are well behaved at this point, so there is no singular limit to consider.

Because $j \in \{2', 2''\}$ we need consider only $k = 0$, and noting that odd L-functions cancel and even L-functions have a zero at $s = -2$, we find

$$\tilde{C}_{2'}(q, \bar{c}, -2) = -\delta_{\bar{c},1} \tag{101}$$

Moreover, at $s = -2$ the exponents in $\Theta(\bar{c}\bar{d}, \theta, s)$ are zero. Analytically continuing from $s > -2$ we note that if $\bar{c}\bar{d} = 0 \bmod q$ there is only a single term, otherwise two terms. So,

$$\Theta\left(\bar{c}\bar{d}, \frac{2\pi r}{q}, -2\right) = 2 - \delta_{\bar{c}\bar{d},0} \tag{102}$$

Summing over all $\bar{d}$ give the zeta function

$$\sum_{\bar{d}=0}^{q-1} \tilde{D}(q, \bar{d}, s) = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^{s+1}} = \frac{1}{\zeta(s+1)} \tag{103}$$

Thus by analytic continuation

$$\sum_{\bar{d}=0}^{q-1} \tilde{D}(q, \bar{d}, -2) = \frac{1}{\zeta(-1)} = -12 \tag{104}$$

Then, for $\bar{d} = 0$ we have

$$\tilde{D}(q, 0, -2) = \frac{q\mu(q)}{\zeta(-1)\prod_{p|q}(1-p)} = -12\frac{q\mu(q)^2}{\phi(q)} \tag{105}$$

Putting it together and noting that $j = 2''$ differs only by a constant, we find

$$\begin{aligned}\frac{\pi}{2}\mathrm{Res}_{s=-2}\ P(\Delta, s)F_{2''}\left(\frac{2\pi r}{q}, s\right) &= \mathrm{Res}_{s=-2}\ P(\Delta, s)F_{2'}\left(\frac{2\pi r}{q}, s\right) \\ &= \frac{3\Delta^2}{\pi}\left(1 - \frac{q\mu(q)^2}{2\phi(q)}\right)\end{aligned} \tag{106}$$

for coprime (r, q). Note that this formula applies also to $q = 1$, which is not obvious from the above derivation but can be checked separately. Also, for $q = 2$ (and only $q = 2$) the residue is zero - the pole is cancelled.

7.7 Overall pole structure

To summarise, the poles (covering all values of j) are given as follows. See also Fig. 2. Unless otherwise stated, the residues for $j \in \{1, 3\}$ are written as series over k and for $j \in \{2', 2''\}$ they are written in closed form, involving values of L-functions.

$s = 1$ There is a simple pole with residue given exactly in Eq. (95).

$s = 0$ There is a pole in $P(\Delta, s)$, which is canceled by the zeta function in the denominator of $\tilde{D}(q, \bar{d}, s)$ for $q \in \{1, 2, 3, 4, 6\}$. For other q and $j \in \{2', 2''\}$ the residue can be found in closed form, and for $j \in \{1, 3\}$ as a series over k.

$\Re(s) = -\frac{1}{2}$ These arise from zeros in the zeta and L functions in the denominator of $\tilde{D}(q, \bar{d}, s)$. If the Riemann hypothesis is false, then other poles exist with a real part in the critical strip $[-1, 0]$; this may also happen for some q if the Generalized Riemann Hypothesis is false, depending on what L-functions are present.

$s = -1$ There is a pole in $P(\Delta, s)$, and also $\tilde{C}_j(q, \bar{c}, s)$ if $j = 1$. In addition, the factor $\prod_{p|b}(1 - \chi(p)p^{-s-1})$ in the denominator of $\tilde{D}(q, \bar{d}, s)$ increases the order of the pole by one less than the number of distinct prime factors of q.

$\Re(s) = -1$ The factor $\prod_{p|b}(1-\chi(p)p^{-s-1})$ in the denominator of $\tilde{D}(q, \bar{d}, s)$ leads to one or more families of equally spaced poles for $q \geq 3$.

$s = -2$ There is a pole in $P(\Delta, s)$. For $j \in \{2', 2''\}$ the residue is given exactly in Eq. (106), vanishing and canceling the pole for $q = 2$.

$s \leq -3$**, odd** There is a pole in $\tilde{D}(q, \bar{d}, s)$, and also in $\tilde{C}_j(q, \bar{c}, s)$ if $j \in \{1, 3\}$.

$s \leq -4$**, even** There are no poles, since all odd L-functions in $\tilde{D}(q, \bar{d}, s)$ cancel after summing over $\bar{c}$ and $\bar{d}$ as noted after the statement of Thm. 3.

Note that the above description is conditional on the (as almost certainly true) statement that $\tilde{C}_j(q, \bar{c}, s)$ has no zeros for $s \in \{-2, 0\}$ and $j \in \{1, 3\}$. If it is false, then it is possible that these poles may be canceled.

8 Calculations for fixed q

In this section, we give explicit expressions for the integrand $P(\Delta, s)F_j(\theta, s)$ of the Mellin transform, Eq. (6) in the case where $\theta = 2\pi r/q$ and $q \leq 6$, that is, there is only a single hole ($\theta = 0$), or the angle between the two holes is a simple rational multiple of π, using the equations of Sec. 6.

8.1 $q = 1$: One hole

This is the case of one hole. We have $r = \bar{c} = \bar{d} = 0$.

$$\Theta(0, 0, s) = 1 \tag{107}$$

$$\tilde{C}_j(1, 0, s) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \left[\zeta(s + 2k) - 1\right] \tag{108}$$

$$\tilde{D}(1, 0, s) = \frac{1}{\zeta(s + 1)} \tag{109}$$

$$F_j(0, s) = a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{\zeta(s + 2k) - 1}{\zeta(s + 1)} \tag{110}$$

The residues are given in Tab. 2. The residue for $s = 1$ is exactly twice that of the other cases ($q > 1$).

8.2 $q = 2$: Two symmetric holes

The case $q = 2$ is that of two symmetric holes. We have $r = 1$, $0 \le \bar{c}, \bar{d} \le 1$.

$$\Theta(0, \pi, s) = 1 \tag{111}$$

$$\Theta(1, \pi, s) = \frac{1}{2^{s+1}} \tag{112}$$

$$\tilde{C}_j(2, 0, s) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \zeta(s + 2k) 2^{-s-2k} \tag{113}$$

$$\tilde{C}_j(2, 1, s) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \left[\zeta(s + 2k)(1 - 2^{-s-2k}) - 1\right] \tag{114}$$

$$\tilde{D}(2, 0, s) = \frac{1}{\zeta(s + 1)(1 - 2^{s+1})} \tag{115}$$

$$\tilde{D}(2, 1, s) = \frac{1}{\zeta(s + 1)(1 - 2^{-s-1})} \tag{116}$$

This gives

$$F_j(\pi, s) = a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{\zeta(s + 2k)}{\zeta(s + 1) 2^{s+2k}} \tag{117}$$

The residues are given in Tab. 2. For $j \in \{2', 2''\}$ the zeta function in the numerator cancels the pole at $s = -2$. For $q > 2$ symmetric holes, see Sec. 9 below.

8.3 $q = 3$

This is the case of two holes with angle $\frac{2\pi}{3}$ (equivalent to $\frac{4\pi}{3}$ by interchanging the holes). We have $r = 1$ (without loss of generality), $0 \le \bar{c}, \bar{d} \le 2$.

$$\Theta(0, \frac{2\pi}{3}, s) = 1 \tag{118}$$

$$\Theta(1, \frac{2\pi}{3}, s) = \Theta(2, \frac{2\pi}{3}, s) = \frac{1 + 2^{s+2}}{3^{s+2}} \tag{119}$$

$$\tilde{C}_j(3, 0, s) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \zeta(s+2k) 3^{-s-2k} \tag{120}$$

$$\tilde{C}_j(3, 1, s) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \times \left[\frac{\zeta(s+2k)(1 - 3^{-s-2k}) + L(s+2k, \chi_2)}{2} - 1 \right] \tag{121}$$

$$\tilde{C}_j(3, 2, s) = \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \times \left[\frac{\zeta(s+2k)(1 - 3^{-s-2k}) - L(s+2k, \chi_2)}{2} - 1 \right] \tag{122}$$

$$\tilde{D}(3, 0, s) = \frac{1}{\zeta(s+1)(1 - 3^{s+1})} \tag{123}$$

$$\tilde{D}(3, 1, s) = \frac{1}{2} \left[\frac{1}{\zeta(s+1)(1 - 3^{-s-1})} + \frac{1}{L(s+1, \chi_2)} \right] \tag{124}$$

$$\tilde{D}(3, 2, s) = \frac{1}{2} \left[\frac{1}{\zeta(s+1)(1 - 3^{-s-1})} - \frac{1}{L(s+1, \chi_2)} \right] \tag{125}$$

where χ_2 is the non-principal (and odd) character of modulus 3. Now the sum over $\bar{c}$ and $\bar{d}$ involves only $\tilde{C}_j(3, 0, s)$ and $\tilde{D}(3, 0, s)$ together with the combinations $\tilde{C}_j(3, 1, s) + \tilde{C}_j(3, 2, s)$ and $\tilde{D}(3, 1, s) + \tilde{D}(3, 2, s)$. Hence, all terms with the L-function of character χ_2 cancel. The result is

$$F_j\left(\frac{2\pi}{3}, s\right) = a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{1}{\zeta(s+1) 3 (3^{s+1} - 1)} \times \left[(2^{s+2} - 2)(\zeta(s+2k) - 1) + \frac{3^{s+2} - 2^{s+2} - 1}{3^{s+2k}} \zeta(s+2k) \right] \tag{126}$$

The residues are given in Tab. 2.

8.4 $4 \leq q \leq 6$

The following were found using mathematica.

$$\begin{aligned}
F_j\left(\frac{\pi}{2}, s\right) =& a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{1}{\zeta(s+1)(2^{s+1}-1)4^{s+2k+1}} && (127)\\
&\times \left[\zeta(s+2k)((2^{s+4k}-2^{2k})(3^{s+2}-3)+2^{s+4}-8)-2^{s+4k}(3^{s+2}-3)\right]\\
F_j\left(\frac{2\pi}{5}, s\right) =& a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{1}{\zeta(s+1)L(s+1,\chi_3)(5^{s+1}-1)5^{s+2k+2}}\\
&\times \big(5L(s+1,\chi_3)[\zeta(s+2k)(2\times 5^{s+2}-4^{s+2}-3^{s+2}-2^{s+2}-1)\\
&+5^{s+2k}(\zeta(s+2k)-1)(4^{s+2}+3^{s+2}+2^{s+2}-9)] && (128)\\
&+(L(s+2k,\chi_3)-1)\zeta(s+1)(5^{s+2k+1}-5^{2k})(4^{s+2}-3^{s+2}-2^{s+2}+1)\big)\\
F_j\left(\frac{4\pi}{5}, s\right) =& a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{1}{\zeta(s+1)L(s+1,\chi_3)(5^{s+1}-1)5^{s+2k+2}}\\
&\times \big(5L(s+1,\chi_3)[\zeta(s+2k)(2\times 5^{s+2}-4^{s+2}-3^{s+2}-2^{s+2}-1)\\
&+5^{s+2k}(\zeta(s+2k)-1)(4^{s+2}+3^{s+2}+2^{s+2}-9)] && (129)\\
&-(L(s+2k,\chi_3)-1)\zeta(s+1)(5^{s+2k+1}-5^{2k})(4^{s+2}-3^{s+2}-2^{s+2}+1)\big)\\
F_j\left(\frac{\pi}{3}, s\right) =& a_j \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k} B_{2k}}{(2k)!} K_j(k) \frac{1}{\zeta(s+1)(2^{s+1}-1)(3^{s+1}-1)6^{s+2k}}\\
&\times \big[\zeta(s+2k)(6^{s+2k}(1-5^{s+2})+5^{s+2}(3^{s+2k}+2^{s+2k}-1) && (130)\\
&+4^{s+2}(1-3^{s+2k})+3^{s+2k}(2^{2s+2k+3}+2^{s+3}-5)+2\times 3^{s+2}\\
&-2^{2s+2k+3}-2^{s+2k}-2^{s+2}-1)+6^{s+2k}(5^{s+2}-2^{s+3}-1)\big]
\end{aligned}$$

Here, χ_3 is the non-principal even character of modulus 5. All $q > 6$ have non-principal even characters, so we would expect the expressions to include the relevant non-principal L-functions.

The residues are given in Tab. 2.

For $q = 5$, the non-principal L function leads to a pole at $s = 0$.

For $q = 6$ an increase in the order of the pole is found for the first time at $s = -1$ due to $\tilde{D}(q, \bar{d}, s)$. That is, for $j = 1$, it is now a triple pole and for $j \in \{2', 2'', 3\}$ it is a double pole. See Sec. 7.4.

8.5 Table of residues

The residues of the functions found for $q \leq 6$ are given in Tab. 2. They were found using mathematica symbolic algebra, then numerical evaluation when involving a sum over k. The case $j = 3$ requires $k \leq 16$ for this precision. The table confirms many of the results shown in Sect. 7, namely, the residues for $s = 1$, those for $s = -2$ and $j \in \{2', 2''\}$, and the order of all poles, including the absence of poles for $s = 0$ and $q \in \{1, 2, 3, 4, 6\}$ and for $s = -2$, $j \in \{2', 2''\}$ and $q = 2$.

q	r	j	$s=1$	$s=0$	$s=-1$	$s=-2$
1	0	1	$\frac{4}{\pi}\Delta^{-1}$	0	$-0.486\Delta + 0.524\Delta\ln\Delta$	$-0.0437\Delta^2$
		$2'$	$2\Delta^{-1}$	0	$-\frac{13}{12}\Delta$	$\frac{3}{2\pi}\Delta^2$
		$2''$	$\frac{4}{\pi}\Delta^{-1}$	0	$-\frac{13}{6\pi}\Delta$	$\frac{3}{\pi^2}\Delta^2$
		3	$\frac{64}{3\pi^2}\Delta^{-1}$	0	-0.662Δ	$-0.136\Delta^2$
2	1	1	$\frac{2}{\pi}\Delta^{-1}$	0	$-0.0894\Delta + 0.262\Delta\ln\Delta$	$-0.107\Delta^2$
		$2'$	Δ^{-1}	0	$-\frac{1}{6}\Delta$	0
		$2''$	$\frac{2}{\pi}\Delta^{-1}$	0	$-\frac{1}{3\pi}\Delta$	0
		3	$\frac{32}{3\pi^2}\Delta^{-1}$	0	0.242Δ	$-0.478\Delta^2$
3	1	1	$\frac{2}{\pi}\Delta^{-1}$	0	$-0.217\Delta + 0.321\Delta\ln\Delta$	$-0.0805\Delta^2$
		$2'$	Δ^{-1}	0	$-\frac{20\ln 2+9\ln 3}{36\ln 3}\Delta$	$\frac{3}{4\pi}\Delta^2$
		$2''$	$\frac{2}{\pi}\Delta^{-1}$	0	$-\frac{20\ln 2+9\ln 3}{18\pi\ln 3}\Delta$	$\frac{3}{2\pi^2}\Delta^2$
		3	$\frac{32}{3\pi^2}\Delta^{-1}$	0	-0.391Δ	$-0.139\Delta^2$
4	1	1	$\frac{2}{\pi}\Delta^{-1}$	0	$-0.427\Delta + 0.442\Delta\ln\Delta$	$0.00671\Delta^2$
		$2'$	Δ^{-1}	0	$-\frac{16\ln 2+33\ln 3}{48\ln 3}\Delta$	$\frac{3}{\pi}\Delta^2$
		$2''$	$\frac{2}{\pi}\Delta^{-1}$	0	$-\frac{16\ln 2+33\ln 3}{24\pi\ln 3}\Delta$	$\frac{6}{\pi^2}\Delta^2$
		3	$\frac{32}{3\pi^2}\Delta^{-1}$	0	-1.407Δ	$0.611\Delta^2$
5	1	1	$\frac{2}{\pi}\Delta^{-1}$	-0.139	$-0.312\Delta + 0.371\Delta\ln\Delta$	$-0.0633\Delta^2$
		$2'$	Δ^{-1}	$\frac{-\pi}{10\sqrt{5}\ln\varphi}$	$-\frac{40\ln 2+12\ln 3+25\ln 5}{60\ln 5}\Delta$	$\frac{9}{8\pi}\Delta^2$
		$2''$	$\frac{2}{\pi}\Delta^{-1}$	$\frac{-2}{10\sqrt{5}\ln\varphi}$	$-\frac{40\ln 2+12\ln 3+25\ln 5}{30\pi\ln 5}\Delta$	$\frac{9}{4\pi^2}\Delta^2$
		3	$\frac{32}{3\pi^2}\Delta^{-1}$	-0.376	-0.580Δ	$-0.113\Delta^2$
5	2	1	$\frac{2}{\pi}\Delta^{-1}$	0.139	$-0.312\Delta + 0.371\Delta\ln\Delta$	$-0.0633\Delta^2$
		$2'$	Δ^{-1}	$\frac{\pi}{10\sqrt{5}\ln\varphi}$	$-\frac{40\ln 2+12\ln 3+25\ln 5}{60\ln 5}\Delta$	$\frac{9}{8\pi}\Delta^2$
		$2''$	$\frac{2}{\pi}\Delta^{-1}$	$\frac{2}{10\sqrt{5}\ln\varphi}$	$-\frac{40\ln 2+12\ln 3+25\ln 5}{30\pi\ln 5}\Delta$	$\frac{9}{4\pi^2}\Delta^2$
		3	$\frac{32}{3\pi^2}\Delta^{-1}$	0.376	-0.580Δ	$-0.113\Delta^2$
6	1	1	$\frac{2}{\pi}\Delta^{-1}$	0	$-0.374\Delta+0.812\Delta\ln\Delta-0.101\Delta(\ln\Delta)^2$	$-0.173\Delta^2$
		$2'$	Δ^{-1}	0	$[5\ln 5(2+10\ln 3-7\ln 5-24\ln A)$ $+\ln 2(-8-76\ln 3+55\ln 5+96\ln A)$ $+(70\ln 5-56\ln 2)\ln\Delta]\frac{\Delta}{72\ln 2\ln 3}$	$-\frac{3}{2\pi}\Delta^2$
		$2''$	$\frac{2}{\pi}\Delta^{-1}$	0	$[5\ln 5(2+10\ln 3-7\ln 5-24\ln A)$ $+\ln 2(-8-76\ln 3+55\ln 5+96\ln A)$ $+(70\ln 5-56\ln 2)\ln\Delta]\frac{\Delta}{36\pi\ln 2\ln 3}$	$-\frac{3}{\pi^2}\Delta^2$
		3	$\frac{32}{3\pi^2}\Delta^{-1}$	0	$-0.242\Delta + 1.435\Delta\ln\Delta$	$-0.706\Delta^2$

Table 2: Residues for $q \leq 6$ and $s \geq -2$ and real. $A = \exp[\frac{1}{12} - \zeta'(-1)]$ is the Glaisher-Kinkelin constant, and $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden ratio. Where the residue is zero, there is no pole. See Sec. 8.5.

The table presents details of poles on the real axis for $s \geq -2$. In addition, as noted in Sec. 7 and using the expressions for $F_j(2\pi/q, s)$ earlier in Sec. 8, there are the following additional families of poles off the real axis for $q \leq 6$: For $\Re(s) = -1/2$ at locations corresponding to the non-trivial zeros of $\zeta(s)$ for all q, and also $L(s, \chi_3)$ for $q = 5$. For $\Re(s) = -1$,

$$\begin{aligned} s &= -1 + \frac{2\pi l}{\ln 2} \qquad q \in \{4, 6\} \\ s &= -1 + \frac{2\pi l}{\ln 3} \qquad q \in \{3, 6\} \\ s &= -1 + \frac{2\pi l}{\ln 5} \qquad q = 5 \end{aligned} \tag{131}$$

where $l \in \mathbb{Z}$. Note that complex poles with $\Re(s) = -1$ do not appear for $q \in \{1, 2\}$.

9 Symmetrically placed holes

In this section we consider a different scenario, that of $q \geq 2$ symmetric holes, that is, $H = \cup_{j=0}^{q-1}[\frac{2\pi j}{q}, \frac{2\pi j}{q} + \Delta]$. The analysis is exactly the same to the derivation of the one hole survival probability, Eq. (52). Then, reducing the dynamics mod $\frac{2\pi}{n}$, we find $\tilde{q}$ arcs of length $\frac{2\pi}{n\tilde{q}}$ of non-escaping periodic orbits of length n, where $\tilde{q} = \frac{q}{(n,q)}$. Thus Eq. (1) becomes

$$\mathcal{P}_{j,q}^{\mathrm{sym}}(\Delta) = \frac{2^j a_j}{16\pi^2} \sum_{m,n} [n, q] \left(\frac{2\pi}{[n, q]} - \Delta \right)_+^2 \sin^j \frac{\pi m}{n} \tag{132}$$

where $[n, q] = n\tilde{q}$ is the least common multiple, and again the sum is over $1 \leq m < n$ with $(m, n) = 1$. Applying the Möbius and Mellin transforms as before, Eq, (6) becomes

$$\mathcal{P}_{j,q}^{\mathrm{sym}}(\Delta) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} P(\Delta, s) F_j^{\mathrm{sym}}(q, s) ds \tag{133}$$

$$F_j^{\mathrm{sym}}(q, s) = a_j \sum_n \frac{1}{[n, q]^{s+1}} \sum_{d|n} \mu(d) C_j(n/d) \tag{134}$$

We can substitute Eq. (87). In addition, the multiplicative structure in the sums imply the identity

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{[n,q]^{s+1}} \sum_{d|n} \mu(d) \left(\frac{d}{n}\right)^{2k-1} \\
&= \frac{1}{q^{s+1}} \sum_{n=1}^{\infty} \frac{(n,q)^{s+1}}{n^{s+2k}} \prod_{p|n} (1-p^{2k-1}) \\
&= \frac{1}{q^{s+1}} \prod_{p} \left[1 + (1-p^{2k-1}) \sum_{j=1}^{\infty} p^{-(s+1)\max(j-\beta,0)-(2k-1)j} \right] \qquad (135) \\
&= \frac{1}{q^{s+1}} \prod_{p} \left[1 + (1-p^{2k-1}) \left(\sum_{j=1}^{\beta} p^{-j(2k-1)} + p^{\beta(s+1)} \sum_{j=\beta+1}^{\infty} p^{-j(s+2k)} \right) \right] \\
&= \frac{1}{q^{s+1}} \prod_{p} \left[p^{-\beta(2k-1)} \frac{1-p^{-(s+1)}}{1-p^{-(s+2k)}} \right] \\
&= \frac{\zeta(s+2k)}{\zeta(s+1) q^{s+2k}}
\end{aligned}
$$

where $\beta = \mathrm{ord}_p(q)$ and $j = \mathrm{ord}_p(n)$. Combining this together, we obtain the statement of Theorem 4.

For poles where only $k = 0$ contributes, that is, $s = 1$ or $j \in \{2', 2''\}$, the residue is proportional to q^{-s}. For example, we have from Eq. (95)

$$
\mathrm{Res}_{s=1} P(\Delta, s) F^{\mathrm{sym}}_{j,q}(s) = \frac{6\alpha_j}{\pi^2 \Delta q} \qquad (136)
$$

10 Sum of real residues

Here we state and prove a lemma that demonstrates convergence of the sum over the residues on the real axis.

Lemma 2. *Let* $q \in \{1, 2, 3, 4, 6\}$*, then*

$$
\sum_{l=1}^{\infty} \mathrm{Res}_{s=-l} P(\Delta, s) F_j(2\pi/q, s) < C\Delta \left| \ln \frac{\Delta}{4\pi} \right|^2 \qquad (137)
$$

where $C > 0$ *is a constant.*

Note that to have non-overlapping holes, we must have $\Delta < \frac{2\pi}{q}$. The right-hand side is determined by the pole at $s = -1$, so the presence of the logarithm depends on the order of this pole; see Tab. 2. We write this as $\ln(\Delta/(4\pi))$ rather than $\ln \Delta$ as the latter is zero when $\Delta = 1$. Where there is no pole (even l except perhaps $l = 2$), we consider its residue to be zero.

Proof. Refer to Eq. (7) for $P(\Delta, s)$ and Sec. 8 for $F_j(2\pi/q, s)$. The sum over k converges (see the end of the proof of Theorem 3) and so each residue is finite and $O(\Delta^l|\ln(\Delta/(4\pi))|^2)$ as $\Delta \to 0$. Thus, we need only a bound on the residue for large l.

For the Riemann zeta function, we have the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{138}$$

Now $1 < \zeta(s) \leq \zeta(2)$ for $s \geq 2$. Thus, as $s \to -\infty$, $\zeta(s)$ oscillates with amplitude dominated by the $\Gamma(1-s)$ term. This appears in the denominator of $F_j(2\pi/q, s)$ as $\zeta(s+1)$, whilst in the numerator $\zeta(s+2k)$ has a similar amplitude for $k=0$ but decreases exponentially for $k>0$:

$$|\zeta(s+2k)| \approx |\zeta(s)| \left(\frac{2\pi}{|s|}\right)^{2k} \tag{139}$$

Thus, we need consider only the $k=0$ term. Apart from constants and exponentially small terms, this is

$$\mathrm{Res}_{s=-l} P(\Delta, s) \frac{\zeta(s)}{\zeta(s+1)q^s} = \begin{cases} -\left(\frac{q\Delta}{2\pi}\right)^l \frac{\zeta(l+1)}{(1-l)(2-l)\zeta(l)} & l \text{ odd} \\ 0 & l \text{ even} \end{cases} \tag{140}$$

thus the sum over l converges for $\Delta < \frac{2\pi}{q}$ as required. □

Remark 1. *For other values of q, it also makes sense to consider $\Delta > \frac{2\pi}{q}$. For example, when $q=5$, $r=2$, we have $\theta = 4\pi/5$ and it is valid to consider $\Delta > \frac{2\pi}{5}$. It seems the series does not converge in this case.*

11 Numerical results

The numerical results in this section are given in support of Conj. 1, that the sum of residues is absolutely convergent and gives the coefficient of the long time survival probability. The sum of the absolute value of the residues is plotted in Fig. 4, showing convergence.

The infinite time limit of the survival probability $\mathcal{P}_{j,\theta}(\Delta)$ was estimated by direct simulation, using a sample of 10^9 initial points distributed with respect to the relevant measure. Then, the leading term $\mathrm{Res}_{s=1} P(\Delta, s) F_j(\theta, s)$ was subtracted from this, for different values of Δ. The results are as in Fig. 5 (points), together with the sum of residues for $s = 0, -1, -2$ (curves); refer to Tab. 2.

If the survival probability is given by the sum of the residues as in Conjecture 1, the $\Delta \to 0$ behavior should then be determined by the residues of the remaining ($s \neq 1$) poles. It turns out that the poles on the critical axis (real part $-1/2$) have residues of magnitude about 10^{-3} and hence are not visible at this scale. For $q \in \{1, 2, 3\}$ the subleading real pole is at $s = -1$ whilst for

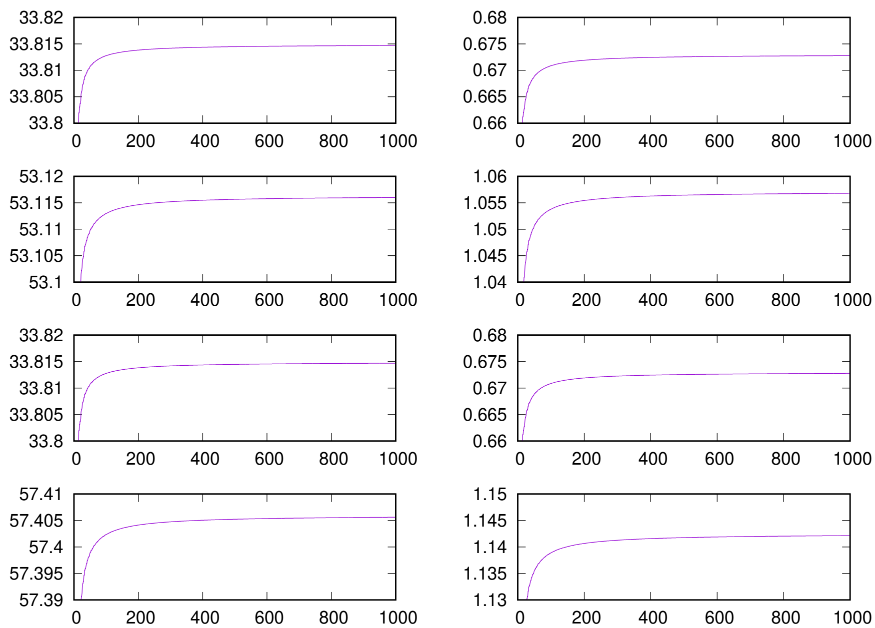

Figure 4: Sum of the absolute value of the residues in Conj. 1, plotted against the maximum absolute value of s^*, showing convergence. The left panels are for $q = 1$ and the right panels for $q = 2$. From top to bottom, they are for $j \in \{1, 2', 2'', 3\}$. The hole size $\Delta = 2\pi/q$, the supremum of possible values.

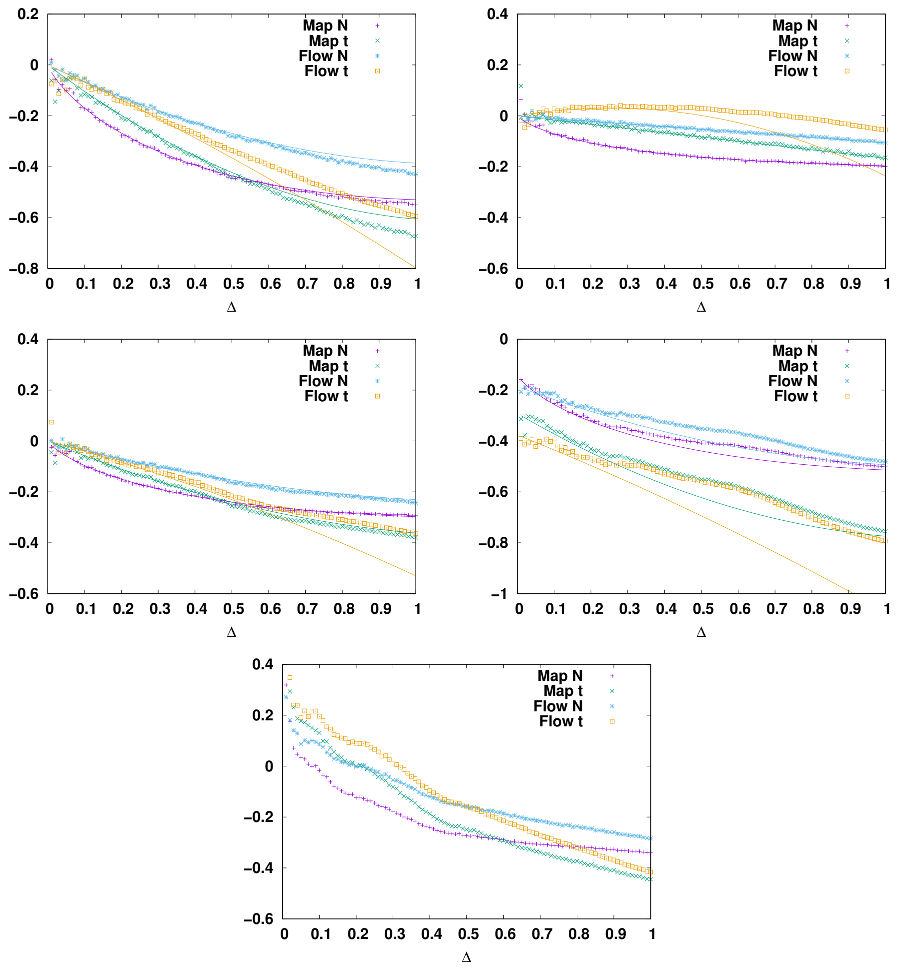

Figure 5: Numerically computed $\mathcal{P}_{j,\theta}(\Delta)$ as a function of Δ, after subtracting the leading ($\sim \Delta^{-1}$) term (points) compared with the sum of residues for $s = 0, -1, -2$ (curves). Top left: $q = 1$. Top right $q = 2$. Middle left: $q = 3$. Middle right $q = 5$, $r = 1$. Lower panel (points only): $\theta = (3 - \sqrt{5})\pi$.

$q = 5$ it is at $s = 0$. We found good agreement for small Δ. At large Δ, other poles (more negative s) are more relevant. For $q = 5$ the agreement is poorer for large Δ; here there are more omitted poles, with real parts $-1/2$ and -1; see Fig. 2. For $\theta = (3 - \sqrt{5})\pi$ we do not have an expression in terms of L-functions but subtract the same leading term. As with $q = 5$, the curves do not approach zero as $\Delta \to 0$, suggesting pole(s) with $0 \leq \Re s < 1$.

References

[Alt13] E. G. Altmann, J. S. Portela and T. Tél, "Leaking chaotic systems," Reviews of Modern Physics **85**, 869 (2013).

[Bun05] L. A. Bunimovich and C. P. Dettmann, "Open circular billiards and the Riemann hypothesis," Physical Review Letters **94**, 100201 (2005).

[Bun06] L. A. Bunimovich and C. P. Dettmann, "Escape from a circle and Riemann hypotheses," arXiv preprint math/0603373 (2006).

[Bun07] L. A. Bunimovich and C. P. Dettmann, "Peeping at chaos: Nondestructive monitoring of chaotic systems by measuring long-time escape rates," Europhysics Letters **80**, 40001 (2007).

[Bun23] L. A. Bunimovich and Y. Su, "Maximal large deviations and slow recurrences in weakly chaotic systems," Advances in Mathematics **432**, 109267 (2023).

[Bun24] L. A. Bunimovich and Y. Su, "Back to boundaries in billiards," Communications In Mathematical Physics **405**, 140 (2024).

[Cre15] C. E. Creffield and G. Sierra, "Finding zeros of the Riemann zeta function by periodic driving of cold atoms," Physical Review A **91**, 063608 (2015).

[Dav13] H. Davenport, *Multiplicative number theory*, volume 74 of *Graduate Texts in Mathematics* (Springer Science & Business Media, 2013).

[Det14] C. P. Dettmann and M. R. Rahman, "Survival probability for open spherical billiards," Chaos: An Interdisciplinary Journal of Nonlinear Science **24**, 043130 (2014).

[Det17] C. P. Dettmann, J. Marklof and A. Strömbergsson, "Universal hitting time statistics for integrable flows," Journal of Statistical Physics **166**, 714–749 (2017).

[Det21] C. P. Dettmann and M. R. Rahman, "Spherical billiards with almost complete escape," Chaos: An Interdisciplinary Journal of Nonlinear Science **31**, 123119 (2021).

[DLMF] "*NIST Digital Library of Mathematical Functions*," `https://dlmf.nist.gov/`, Release 1.1.10 of 2023-06-15, f. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.

[Fri01] N. Friedman, A. Kaplan, D. Carasso and N. Davidson, "Observation of chaotic and regular dynamics in atom-optics billiards," Physical review letters **86**, 1518 (2001).

[Hay20] N. Haydn and F. Yang, "Local escape rates for ϕ-mixing dynamical systems," Ergodic Theory and Dynamical Systems **40**, 2854–2880 (2020).

[He21] R. He, M.-Z. Ai, J.-M. Cui, Y.-F. Huang, Y.-J. Han, C.-F. Li, G.-C. Guo, G. Sierra and C. E. Creffield, "Riemann zeros from Floquet engineering a trapped-ion qubit," npj Quantum Information **7**, 109 (2021).

[Mil01] V. Milner, J. Hanssen, W. Campbell and M. Raizen, "Optical billiards for atoms," Physical Review Letters **86**, 1514 (2001).

[MSE] `https://math.stackexchange.com/questions/3074336`, [Online; accessed 3-October-2024].

[Pia79] G. Pianigiani and J. A. Yorke, "Expanding maps on sets which are almost invariant: Decay and chaos," Transactions of the American Mathematical Society **252**, 351–366 (1979).

[Su22] Y. Su and L. A. Bunimovich, "Poisson approximations and convergence rates for hyperbolic dynamical systems," Communications in Mathematical Physics **390**, 113–168 (2022).