General Relativity

Contents

1 Introduction **1**
1.1 Books 2
1.2 History 2
1.3 The equivalence principle 3
1.4 Classical effects 4
1.5 Curved space-time 6
1.6 Overview 7

2 Preliminaries **8**
2.1 Dimensions 8
2.2 Newtonian gravity 9
2.3 Galilean relativity and index notation 11
2.4 Variational mechanics 14
2.5 Nonrelativistic Kepler problem 17

3 Special relativity **18**
3.1 Minkowski space-time 18
3.2 Applications of the Lorentz transformations . 20
3.3 Scalars and vectors in special relativity 22
3.4 Vectors and covectors: general theory 25
3.5 Tensors: metric and Levi-Civita 27
3.6 Tensors: general theory 31
3.7 Continua in special relativity 33

4 The metric **36**
4.1 Curvilinear coordinates 36
4.2 Length, angle and volume 38
4.3 Differentiable manifolds 41

5 The connection **42**
5.1 Curvilinear coordinates 42
5.2 Curved manifolds 46
5.3 Parallel transport and geodesics 47

6 Curvature **51**
6.1 The curvature tensor 51
6.2 The Einstein field equations 53
6.3 Weakly curved spacetimes 55
6.4 Gravitational waves 57

7 Spherical symmetry **59**
7.1 Spherical symmetry in GR 59
7.2 Spherical stars 60
7.3 Orbits in Schwarzchild spacetime 62
7.4 The event horizon, singularity and extensions 66

8 Experimental tests of GR **68**

1 Introduction

Lecturer; outline; problems, classes and credit; exam

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Small text is supplementary non-examinable material, for example from previous years. It will improve your understanding of the rest of the course material.

1.1 Books

- "A first course in general relativity", Bernard F Schutz, Cambridge University Press, 1985, 400 pages. First recommendation. Good beginner's book with very little assumed background. Covers the course but not much more.
- "General Relativity", Robert M Wald, U. Chicago press, 1984, 500 pages. Not well suited to beginners, but strong students with a more mathematical background should consider it. Contains more recent theory, so useful if you plan to take this subject further.
- "A short course on general relativity" J Foster, Springer-Verlag, 1995 250 pages. Good beginner's book, concise and logical, but maybe too short ie. not as much explanation as Schutz above.
- "Relativity" H. Stephani, Cambridge, third ed. 2004, 400 pages. A little too advanced on its own, but good to extend a more elementary text. A more modern approach, and more physical than Wald.
- "A concise overview of the classical theory" P A M Dirac, 1996 Princeton University Press, 80 pages. Concise and logical, but no modern developments, figures or problems.
- "Gravitation" C W Misner, K S Thorne, J A Wheeler, Freeman, 1973, 1300 pages. Exhaustive classic, not for beginners, but good for detailed pictorial arguments on many topics. Physical approach.
- "Geometrical methods of mathematical physics" Bernard F Schutz, Cambridge University Press, 1980, 250 pages. Not a text on general relativity per se, but delves deeper into the mathematical structure, and so useful to extend one of the above texts. This book is used in the differentiable manifolds unit.

1.2 History

Following success in 1905 in his Special Theory of Relativity (abbreviated SR in these notes), incorporating the relativity principle and electromagnetism, Einstein tried unsuccessfully to incorporate gravity into the theory. In particular, a scalar theory (ie the gravitational potential is a scalar field) does not predict that light is bent by a gravitational field, which Einstein predicted and which was confirmed experimentally later (1919); a vector theory such as electromagnetism but with attraction of like "charges" leads to waves of negative energy; a tensor theory is inconsistent with regard to energy conservation. (more details: MTW chapter 7). In 1915 he succeeded by generalising the space-time of the special theory using the theory of curved spaces formulated by Georg Riemann in 1854. This is known as the general theory of relativity (abbreviated GR in these notes). More recent work (1950's and 60's) has shown that this tensor theory may be made consistent, in a form equivalent to the general theory! Since 1950 there have been a number of interesting developments, including singularity theorems and Hawking radiation from black holes (beyond the scope of the course). Almost all weak field predictions of the theory have agreed with experiment (exception: gravity waves not yet observed), but only indirect evidence for strong field predictions such as black holes.

We turn to the centrepiece of the new theory - the equivalence principle, and use it to derive some important effects of general relativity. We skip quickly over some aspects of gravity and relativity for this description, but both will be discussed more carefully later.

1.3 The equivalence principle

Why is gravity different to electromagnetism, and indeed any other force (nuclear, etc.)? The gravitational force on an object is proportional to the mass, as appears also in Newton's second law, while the other forces are not (eg. the electric force is proportional to the electric charge.) Simply stated,

$$m\mathbf{a} = m\mathbf{g}$$

and so the masses cancel - gravity can be thought of as an acceleration, rather than a force.

The only other forces which are proportional to the mass are the so called *inertial* forces, ie fictitious forces observed in accelerated or rotating frames of reference. For example, if

$$x' = x + bt^2$$
$$\frac{d^2x'}{dt^2} = \frac{d^2x}{dt^2} + 2b$$
$$f' = m\frac{d^2x'}{dt^2} = m\frac{d^2x}{dt^2} + 2mb = f + 2mb$$

so in the primed frame, the force is given as in the unprimed frame, together with an extra, inertial force $2mb$. In rotating frames, there are two inertial forces called the centrifugal and Coriolis forces.

Einstein understood this as a fundamental principle, called the equivalence principle (1911):

> Local laws of physics are the same in a gravitational field as in an accelerated frame of reference.

The "local" is because the gravitational field is clearly not constant; a different accelerating frame is needed near each point in space and time (*event*). The EP states that gravity is an inertial force.

In special relativity we had *inertial frames of reference*, defined so that Newton's first law held.

> SR: An inertial frame is a coordinate system in which a free particle moves with constant velocity.

Constant velocity of course means constant speed and direction. We can now modify this definition:

> GR: A local inertial frame is a local coordinate system in which a particle free from non-gravitational forces moves with constant velocity.

As in special relativity there are many local inertial frames, with constant relative velocities. They are constructed as before using standard rods, clocks and light beams. However they are now "local", defined separately in a small region around each space-time point. We can restate the EP as

> In local inertial frames, the usual laws of physics hold as described by Special Relativity.

At the surface of the Earth we observe these local inertial frames as freely falling, so these frames of reference are also called *freely falling frames*. Objects are held stationary with respect to the Earth by normal reaction forces from the ground, so in the local inertial frame, these are seen to be accelerating upwards with magnitude g. (Fig: two pictures).

The gravitational field itself is not physically observable, since it is not felt by a local inertial observer. For example - we accelerate around the centre of the galaxy, but this is hard to measure within the solar system since the sun and planets have (almost) the same acceleration as each other. Diagram. However, differences or gradients of gravitational fields are observable. For example, we can compare notes with people at the other side of the world, and discover that we have a relative acceleration of $2g$. The gradient of the gravitational field is responsible for *tidal forces*, ie the Avon rises in response to a stronger gravitational force from the moon or sun than the ground when these are above us, and to a weaker gravitational force when these are below us (Fig).

1.4 Classical effects

The earliest effect, known before Einstein, was the perihelion precession of Mercury (diagram). After subtracting known effects such as the gravitational perturbations of the other planets, 43" per century can be attributed to relativistic effects. This is in good agreement with theory (we will calculate a simplified version), but is not surprising in itself as almost any perturbation breaks the invariance of the Runge-Lenz vector associated with the closure of the elliptical orbits.

The next effect, verified in 1919 and the first success of the new theory, was the bending of light in a gravitational field (in particular starlight passing near the sun was observed during an eclipse in 1919, predicted deflection 1.75"). We predict this qualitatively using the equivalence principle: a light ray moving horizontally in a frame accelerating with g towards the Earth clearly looks curved with respect to a frame on the Earth's surface, in the same way as any horizontal projectile is deflected. Diagram. Recall that this effect ruled out the special relativistic scalar theory of gravity.

Finally, there is the gravitational time dilation.

Again we use the equivalence principle: Consider two people A and B, moving in one dimension as follows:

$$x_A = gt^2/2$$

$$x_B = gt^2/2 + d$$

we only consider times such that their speeds are much less than c (ie $t \ll c/g$), and also assume that $gd \ll c^2$ so that they have time to exchange many light beams. If A emits a beam towards B at time t_A, its worldline is

$$x_L = c(t - t_A) + gt_A^2/2$$

so as to reach B when

$$c(t_B - t_A) + gt_A^2/2 = gt_B^2/2 + d$$

If A sends two beams with interval dt_A, they will be received at B with interval $dt_B = dt_A \times (dt_B/dt_A)$. Differentiating,

$$c(dt_B/dt_A - 1) + gt_A = gt_B(dt_B/dt_A)$$

$$dt_B/dt_A = \frac{c - gt_A}{c - gt_B} \approx 1 + g(t_B - t_A)/c \approx 1 + gd/c^2$$

thus B receives a lower rate than A sends. If B sends to A the argument is reversed, d changes sign, and we still find that A receives beams at a higher rate than B sends. In other words, A's clock seems to be moving slower than B's in this accelerating frame. In the gravitational context it means that a clock B a distance h above a clock A runs faster by a factor approximately $1 + gd/c^2$. We will find a precise prediction later.

Doppler shift argument: As above, but in the time d/c taken by the light beam, B's velocity has increased by gd/c. The result follows using the Doppler red shift formula ($f/f' = 1 + u\cos\theta/c$, NR, f is frequency, $\theta = 0$, f' is moving receiver, f=stationary source).

Energy conservation argument: Use quantum mechanics, in which the energy of a photon (light particle) is hf where $h = 6 \times 10^{-34} Js$ is Planck's constant and f is frequency. A mass m at position B (height d above A) is dropped to A, giving it mgd kinetic energy. It is then converted into a photon of energy $mgd+mc^2$ of frequency $m(gd+c^2)/h$ and sent back to B, where it is converted back to mass again. We need to end up with the same amount of mass, hence the frequency at A should be mc^2/h. Thus there is a gravitational redshift: $f_A/f_B = 1 + gd/c^2 = 1 + \Delta\Phi/c^2$ in general (Φ is graviational potential). Side remark - energy/momentum conservation actually prohibits conversion of mass to a single photon, but not conversion to two or more photons. The argument follows in the same manner.

Example: The GPS satellites have orbital periods of 12 hours. Compute their distance from the Earth's centre and velocity, and hence the SR and GR time dilation effects. The atomic clocks are precise to one part in 10^{13}; are relativistic effects important?

The orbital parameters are calculated using Newtonian gravity:

$$R_S = \left(\frac{GM_E T_S^2}{4\pi^2}\right)^{1/3} = 2.66 \times 10^7 m$$

$$u_S = 2\pi R_S/T_S = 3.87 \times 10^3 ms^{-1}$$

Here $T_S = 43200$, $G = 6.67 \times 10^{-11}$, $M_E = 5.98 \times 10^{24}$ in SI units.

SR effect (discussed later, for now just note formula):

$$\gamma = (1 - u_S^2/c^2)^{-1/2} \approx 1 + u_S^2/2c^2 = 1 + 8.33 \times 10^{-11}$$

where $c = 3 \times 10^8$ is the speed of light. The satellite's time appears to moves more slowly due to this effect.

The GR effect is $1 + \Delta\Phi/c^2$.

$$1+\Delta\Phi/c^2 = 1+(GM/R_S c^2)-(GM/R_E c^2) = 1-5.28\times10^{-10}$$

where $R_E = 6.38 \times 10^6$ is the radius of the Earth. The satellite's time appears to move more quickly due to this effect, which dominates. Both are orders of magnitude larger than the precision of the clock, and so must be taken into account.

In addition to these *classical effects*, there are some observational effects that have been predicted since the theory was established. These include the black holes and gravitational waves, to be discussed later in the course. In both of these cases, we have indirect evidence of their existence, but this evidence is astronomical rather than solar system measurements.

1.5 Curved space-time

The time dilation effect is an example of the breakdown of usual geometrical principles in general relativity. In a space-time diagram of A sending light rays to B, the parallelogram defined by the equal angles has opposite sides of different length. Even if light does not move in straight lines in the space-time diagram, the parallel argument holds.

What does a curved space-time (for that matter a curved space) look like? Example: ball. Locally it looks flat - this corresponds to the local inertial frames discussed previously. However these flat local regions are stitched together in a nontrivial manner.

Straight lines may be generalised in at least two possible ways - either they are defined as to look as straight as possible ("parallel transport of tangent vector"), or to satisfy a variational condition such as being the shortest distance between two points. Actually we will discover that these two correspond to the same object, called a "geodesic". Free particles (ie with no non-gravitational forces) will move along these geodesics.

Example: If I throw a ball up in a constant gravitational field so that it returns in time T, what path does it take in space-time? Brief answer: it takes the longest possible proper time by moving further away from the Earth for a while (increasing the time due to graviational time dilation) but going too far is counterproductive as the SR time dilation effect takes over as it moves faster. The (nearly) parabolic path in space-time gives the optimal balance between these effects.

Detailed answer:

Newtonian gravity: $z = u_0 t - gt^2/2$ which is zero at $t = T$. Thus $u_0 = gT/2$ and the path taken is $z = (T - t)gt/2$.

Geodesic approach: We maximise the proper time (registered by a clock moving with the ball), using time dilation associated with velocity (special relativity) and height (general relativity), assuming constant acceleration a for simplicity. If $z = (T - t)at/2$, and proper time (ignoring higher order effects) is assumed to be

$$d\tau = 1 + gz/c^2 - u^2/2c^2 + O(1/c^4)$$

$$\tau = \int_0^T d\tau$$

$$\tau = \int_0^T dt(1 + ag(T - t)t/(2c^2) - a^2(T^2/4 - Tt + t^2)/(2c^2))$$

$$\tau = T + aT^3(g/12 - a/24)/c^2$$

$$\frac{d\tau}{da} = T^3(g/12 - a/12)/c^2 = 0$$

$$a = g$$

This also gives the time recorded:

$$\tau = T + g^2T^3/(24c^2)$$

In other words, the ball follows a path in which to take advantage of the faster clock rate at a higher altitude, while keeping a moderate speed and hence minimising the special relativistic time dilation effect. In the freely falling frame, everything looks simple: the ball remains fixed, so it is clearly the "shortest" path in space-time. We make this calculation more general and precise later.

Still further discussion: We are given the metric (assuming slow moving weak sources of gravitational field; we derive this later)

$$ds^2 = (1 + 2\Phi)dt^2 - (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

where $c = 1$, $\Phi \ll 1$ is the Newtonian gravitational potential, for example $-GM/r$ outside a spherical mass. Assuming a particle moves slowly $\mathbf{u} \ll 1$ the rate of a clock is given by

$$d\tau = ds = dt\sqrt{1 + 2\Phi - (1 - 2\Phi)\mathbf{u}^2} \approx dt(1 + \Phi - \mathbf{u}^2/2)$$

exhibiting both GR and SR time dilation effects. Now the Euler-Lagrange equations (also derived later)

$$\frac{\partial L}{\partial x^i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i}$$

compute the stationary value of $\int L dt$ where $L(x^i, \dot{x}^i)$ is the Lagrangian function, with the initial and final points of the path fixed. The index $i = 1, 2, 3$ corresponds to x, y, z coordinates. Thus a geodesic (stationary proper time) is given by the E-L equations for $L = d\tau/dt$, namely

$$\frac{\partial \Phi}{\partial x^i} = \frac{d}{dt}(-\frac{dx^i}{dt})$$

which is just the Newtonian equations of motion.

We can construct more complicated geometrical figures from these geodesics, but note that, for example, the angles in a triangle of geodesics need not add to 180 degrees; we have already noted that a parallelogram may have opposite sides of different length.

We may attempt to compare vectors at different points by transporting them along curves ("parallel transport"), but notice that parallel transporting a vector back to its original point may lead to a vector pointing in a different direction! This means that vectors are only defined with respect to a point in space-time, and care must be taken in differentiating them.

Before concluding this section on curvature, we need to make an important distinction, between intrinsic and extrinsic curvature. A two dimensional intrinsically flat surface (ie with all the usual geometrical properties) can be embedded into three dimensions as a curved surface, eg fold or roll a piece of paper. The latter is called extrinsic curvature.

Example: the curved surface of a cylinder is intrinsically flat, as is all but the apex of a cone (demonstration). A sphere is, however, intrinsically curved, as anyone who has tried to wrap a spherical present will know.

General relativity is a theory in which space-time is a four dimensional *intrinsically* curved surface.

1.6 Overview

We are now in a position to describe the course and its applications:

We will need to cover, perhaps in a slightly new language, topics you may have seen before: Dimensional analysis, Newtonian gravity, variational mechanics (Lagrangians and Hamiltonians), change of coordinates and special relativity, including tensors.

The mathematics of curvature comes in three parts - first the curved space itself and its scalar vector and tensor fields, including the metric, which gives lengths, angles and volumes, and corresponds to the gravitational potential as we have seen from the gravitational time dilation effect.

Then the derivative of the metric gives the "connection", from which comes the geodesic equation, parallel transport, and differentiation of vectors and tensors. It corresponds to the gravitational field, and, being unobservable can be set to zero locally by a judicious choice of coordinates. At this point the equations of physics in a gravitational field can be constructed.

The second derivative of the metric gives the "curvature" itself, corresponding to the tidal forces (derivative of the gravitational field). At this point we discuss the other part of the theory, how a mass distribution (actually an object called the stress-energy tensor) determines the curvature of space-time, the Einstein field equations.

Finally we discuss simple solutions to the Einstein field equations, in particular the Schwarzchild metric corresponding to the outside of a spherical mass distribution. We calculate the perihelion precession, deviation of light and gravitational redshift, and then turn to more exotic predictions of the solution, in particular black holes.

Other solutions we will discuss briefly are cosmological (big bang) models, and gravitational waves (which have yet to be found experimentally).

The mathematics of curved spaces has many other applications, for example spherical geometry and constrained problems in mechanics. The techniques of change of variables are useful for solving partial differential equations with different shaped boundaries. Last but not least, many study the theory for its renowned power and elegance.

2 Preliminaries

[unfortunately somewhat fragmented, but necessary to avoid distractions later on]

2.1 Dimensions

[The main concepts are used in many branches of mathematics and physics; you may have seen this elsewhere. This is used throughout the unit.]

There are two fundamental constants appearing in general relativity: $G = 6.67\times10^{-11}Nm^2kg^{-2}$ is Newton's gravitational constant. $c = 2.997792458\times10^8 ms^{-1}$ is the speed of light in a vacuum. These relate a length to a time: $l = ct$, or a length to a mass: $l = Gm/c^2$. Many people use "geometricised units" in which $G = c = 1$. This is accomplished by making times and masses have the units of length. In general, this simplifies the algebra, but we will reintroduce these constants when we do explicit calculations, or use, say, $1/c$ as an expansion parameter for the nonrelativistic limit.

Example: Find the lengths corresponding to (i) a nanosecond, (ii) the mass of the Earth. A nanosecond $10^{-9}s$ corresponds to $0.3m$ or about 1 foot. The mass of the Earth $5.98\times10^{24}kg$ corresponds to $0.0044m$ or 3/16 inch.

Example: Show that Gm/c^2 has dimensions of length.

$$[m] = M$$

$$[c] = LT^{-1}$$

$$[G] = MLT^{-2}L^2M^{-2} = M^{-1}L^3T^{-2}$$
$$[Gm/c^2] = M^{-1}L^3T^{-2}ML^{-2}T^2 = L$$

Example: The event horizon of a black hole (beyond which no light can escape) is given as $r = 2m$. In ordinary units this would be $r = 2Gm/c^2$. Incidentally, this is also the Newtonian prediction obtained by setting the escape velocity $\sqrt{2Gm/r}$ equal to c. Warning: m and r have not yet been defined in the context of general relativity. m will be the mass calculated from gravitational forces at infinity not the amount of mass obtained by adding the protons and neutrons, r will be the proper circumference divided by 2π but will not measure radial distance.

Example: Obtain the gravitational redshift $dt'/dt = 1 + z$ in terms of g, d, G and c using dimensional analysis. The gravitational redshift is a dimensionless ratio of two times or frequencies. The dimensions of the quantities are:

$$[g] = LT^{-2}$$
$$[d] = L$$
$$[c] = LT^{-1}$$
$$[G] = MLT^{-2}L^2M^{-2} = M^{-1}L^3T^{-2}$$

Suppose $g^\alpha d^\beta c^\gamma G^\delta$ is dimensionless, then

$$(LT^{-2})^\alpha L^\beta (LT^{-1})^\gamma (M^{-1}L^3T^{-2})^\delta = 1$$
$$\delta = 0$$
$$\alpha + \beta + \gamma + 3\delta = 0$$
$$-2\alpha - \gamma - 2\delta = 0$$

thus

$$\gamma = -2\alpha$$
$$\beta = \alpha$$

$(gd/c^2)^\alpha$ is the only dimensionless combination.

If we consider three observers, we conclude that $1 + z(d_1 + d_2) = (1 + z(d_1))(1 + z(d_2) \approx 1 + z(d_1) + z(d_2)$ thus $z(d)$ is linear in d, and so we have $z(d) = const \times gd/c^2$.

2.2 Newtonian gravity

[Prior experience of electromagnetism helpful here; force and potential energy are familiar from level 1 mechanics. This is required for the rest of section 2, and for section 6.2 and following]

We cannot hope to describe a relativistic theory of gravity without some understanding of the nonrelativistic theory.

Newton's original theory of gravity consists of pairwise inverse square forces.

$$m_i \frac{d^2\mathbf{x}_i}{dt^2} = -\sum_{j \neq i} \frac{Gm_im_j}{r_{ij}^2}\hat{\mathbf{r}}_{ij}$$

where $r_{ij} = \mathbf{x}_j - \mathbf{x}_i$. The mass on the LHS is the "inertial mass" while the masses on the RHS are "gravitational masses". Newton assumed that they were the same, as does the equivalence principle, allowing us to cancel the m_i on both sides.

Newton's theory is "action at a distance", which we must avoid in relativity; signals can travel only at the speed of light. A local formulation is as follows:

$$\mathbf{f}_i = m_i\mathbf{g}(\mathbf{x}_i)$$

where $\mathbf{f}_i$ is the gravitational force on a mass m_i due to a gravitational field $\mathbf{g}$ at position $\mathbf{x}_i$. Aside: in electromagnetism the force is $q_i\mathbf{E}(\mathbf{x}_i)$ where the charge q_i is analogous to the mass, and the electric field $\mathbf{E}$ is analogous to the gravitational field.

We also need an equation relating the gravitational field at different points in space, and in particular to the mass distribution that generated it:

$$\nabla \cdot \mathbf{g} = -4\pi G\rho$$

Here G is Newton's constant and ρ is the mass per unit volume. Aside: the analogous equation is $\nabla \cdot \mathbf{E} = \rho_e/\epsilon_0$.

If we apply the *divergence theorem* in a domain D

$$\int_{\partial D} \mathbf{V} \cdot d\mathbf{S} = \int_D \nabla \cdot \mathbf{V} dV$$

we find

$$\int_{\partial D} \mathbf{g} \cdot d\mathbf{S} = -4\pi GM$$

where M is the total mass inside the region. If the region is a sphere of radius r, and the mass distribution is spherically symmetric so that $\mathbf{g}$ can be assumed to be radial, we find

$$4\pi r^2|\mathbf{g}| = -4\pi GM$$

$$\mathbf{g} = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

Note that the gravitational field depends only on the amount of mass M within radius r, not its distribution. This was first proved by Gauss, and a modified version also holds in general relativity.

Thus we return to Newton's formulation:

$$\mathbf{f} = -\frac{GmM}{r^2}\hat{\mathbf{r}}$$

There is another equation for the gravitational field (implicit in the assumption that $\mathbf{g}$ is spherically symmetric, above; obviously required because the first equation is only a single equation for the three components of $\mathbf{g}$),

$$\nabla \times \mathbf{g} = 0$$

from which we deduce the existence of the *gravitational potential* Φ,

$$\mathbf{g} = -\nabla\Phi$$

(cf $\mathbf{E} = -\nabla\mathbf{V}$). This follows since $\Phi = -\int \mathbf{g} \cdot d\mathbf{r}$ is uniquely defined: Apply Stokes' theorem, and assume a simply connected domain.

The potential *energy* is obtained by multiplying this by the mass,

$$U = m\Phi$$

(cf U=qV) so we can write

$$\mathbf{f} = m\mathbf{g} = -m\nabla\Phi = -\nabla U$$

$$U = -\frac{GmM}{r}$$

for a particle of mass m moving in the field of a mass M.

Thus we can write the equations of Newtonian gravity as

$$\mathbf{f} = -m\nabla\Phi$$

$$\nabla^2\Phi = 4\pi G\rho$$

The solution to the second equation comes from summing over the mass distribution:

$$\Phi(\mathbf{x}) = -\int_D \frac{G\rho dx'}{|\mathbf{x}-\mathbf{x}'|}$$

Example: the gravitational potential in a uniform gravitational field

$$\mathbf{g} = -g\mathbf{k}$$

is simply gz.

Example: the gravitational potential due to an infinite line of mass per unit length λ along the z axis. Radial by symmetry, so use a cylindrical domain,

$$2\pi rh|\mathbf{g}| = 4\pi G\lambda h$$

$$|\mathbf{g}| = \frac{2G\lambda}{r}$$

$$\Phi(r) = -\int \mathbf{g}\cdot d\mathbf{r} = 2G\lambda\ln r$$

Example: the gravitational potential from a cone of height h, radius R and density ρ measured along the axis. We simply add contributions from different parts in cylindrical coordinates:

$$\Phi(d) = \int_{z=0}^{h}\int_{r=0}^{Rz/h}\int_{\phi=0}^{2\pi}\frac{G\rho r dr dz d\phi}{\sqrt{r^2+(z-d)^2}}$$

2.3 Galilean relativity and index notation

[Familiar concepts; new notation. Index notation is essential for performing calculations in general relativity, and is used from section 3.3 onwards.]

As a consequence of the fact that Newtonian gravity determines the acceleration in terms of the positions of the masses (not their velocities), the equations also satisfy a symmetry of the equations, physically known as *Galilean Relativity*:

$$\begin{aligned}\mathbf{x}' &= \mathbf{x}-\mathbf{v}t\\ \mathbf{t}' &= t\end{aligned}$$

which has the following physical interpretation: An observer O' moves with velocity $\mathbf{v}$ as measured by observer O, and measurements are made with respect to the same spatial origin at $t=0$. This is called "Standard configuration" (also in special relativity). O and O' agree about (Newtonian absolute) time (unlike in special relativity).

Example: A point marked on a rotating wheel has acceleration $\omega^2 r$ in a frame of reference moving with its centre, so it also has this acceleration with respect to the ground. Differentiate the Galilean transformation twice.

In addition, the equations have translational and rotational symmetry. To denote the rotation, we need to develop some notation. Let the Latin letters $i, j = 1, 2, 3$ denote the three coordinates, x^i as the ith component of the vector $\mathbf{x}$, and Λ^1_2 or $\Lambda^1{}_2$ or Λ_{12} or Λ^{12} as the element of the matrix Λ in row 1, column 2. The transpose of a matrix is simply

$$(\Lambda^T)_{ij} = \Lambda_{ji}$$

Matrix equations can be written explicitly in terms of their indices, for example $A = BC$ is

$$A_{ik} = \sum_j B_{ij} C_{jk}$$

Einstein Summation Convention: Any index appearing twice in a product is automatically summed on. Thus we write simply

$$A_{ik} = B_{ij} C_{jk}$$

In more detail: A **term** is a product of quantities with indices; an **equation** involves equality of (sums of) terms. An index can appear once in a term, in which case it is a **free index** and appears once in every term in an equation. The equation is thus really a set of equations in number three (or the dimension) to the power of the number of free indices. If an index appears twice in a term, it is called a **dummy index** and is summed. An index cannot appear more than twice in a term; if necessary dummy indices need to be relabled to prevent this. When we use upper and lower indices the free indices will be either upper in all the terms, or lower, and the sum will always be over an upper and a lower index. When we use primed indices, the primed and unprimed versions (eg j and j') will be considered distinct. The reasons for these rules will become apparent later.

In the above example, i and k are free indices, together taking a total of $3^2 = 9$ values. j is a dummy index and automatically summed.

Note that the following equations are equivalent:

$$A_{ik} = C_{jk} B_{ij}$$

The components are real numbers, for which multiplication is commutative.

$$A_{ij} = B_{ik} C_{kj}$$

We can represent multiplication by a "column vector" $\mathbf{y} = A\mathbf{x}$,

$$y_i = A_{ij} x_j$$

the dot product of two vectors $c = \mathbf{x} \cdot \mathbf{y}$

$$c = x_i y_i = \delta_{ij} x_i y_j$$

where δ_{ij} ("Kronecker delta") is one if $i = j$ and zero otherwise. The cross product of two vectors $\mathbf{z} = \mathbf{x} \times \mathbf{y}$

$$z_i = \epsilon_{ijk} x_j y_k$$

where both j and k are repeated so they are summed, and ϵ is a "3-index tensor" with values

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$$

$$\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$$

and all others zero.

With this in mind, we introduce a rotation matrix Λ and write

$$x^{i'} = \Lambda^{i'}_j x^j - v^{i'} t - d^{i'}$$

The reason for raising the index of the vector will become clearer when we discuss relativity. Note that the sum is over a lower and an upper index, and the free index is always up, as mentioned above.

Note that the prime goes on the index, not on the vector. This is because we think of the vector as an abstract object, and the three numbers are its components with respect to a particular (primed or unprimed) orientation of coordinates, or "reference frame".

For what rotations Λ is Newtonian gravity

$$M_n \ddot{\mathbf{x}}_n = \sum_{m \neq n} \frac{G M_n M_m}{|\mathbf{x}_m - \mathbf{x}_n|^3} (\mathbf{x}_m - \mathbf{x}_n)$$

is invariant under the generalised Galilean transformation? NB: m and n denote particles here (and below), not spatial components. We have

$$\ddot{x}^{i'} = \Lambda^{i'}_j \ddot{x}^j$$

so the forces transform as

$$F^{i'} = m\ddot{x}^{i'} = \Lambda^{i'}_j F^j$$

Substituting into the equations of motion, we need only that the distance $|\mathbf{x}_m - \mathbf{x}_n|$ is preserved:

$$\begin{aligned}
|\Delta \mathbf{x}'|^2 &= g_{i'j'} \Delta x^{i'} \Delta x^{j'} \\
&= g_{i'j'} \Lambda^{i'}_k \Delta x^k \Lambda^{j'}_l \Delta x^l \\
&= \Lambda^{i'}_k g_{i'j'} \Lambda^{j'}_l \Delta x^k \Delta x^l \\
&= (\Lambda^T g \Lambda)_{kl} \Delta x^k \Delta x^l \\
&= g_{kl} \Delta x^k \Delta x^l
\end{aligned}$$

where $g_{ij} = \delta_{ij}$ is the metric, which is just the unit matrix in Euclidean space, but will become more complicated later. In the last step, we needed the matrix equation

$$\Lambda^T g \Lambda = g$$

but in this case g is just the identity matrix, so we have

$$\Lambda^T \Lambda = I$$

that is, Λ is an orthogonal matrix.

Recall the definition of a *group*: a set G with a binary operation satisfying, for all $a, b, c \in G$:

$ab \in G$ closure

$a(bc) = (ab)c$ associativity

$ae = ea = a$ for some $e \in G$ called the *identity*

$aa^{-1} = a^{-1}a = e$ for some $a^{-1} \in G$ called the *inverse* of a

If, in addition, the operation is commutative, ie $ab = ba$ for all elements a and b, the group is called *Abelian*.

We can easily check that the product of orthogonal matrices is orthogonal; matrix multiplication is associative; the identity matrix is the group identity; the inverse of an orthogonal matrix exists and is orthogonal. Thus the set of orthogonal matrices form a group, with the operation given by matrix multiplication.

Example: To show that matrix multiplication is associative:

$$\begin{aligned}[(AB)C]_{ij} &= (AB)_{ik}C_{kj} = A_{il}B_{lk}C_{kj} \\ &= A_{il}(BC)_{lj} = [A(BC)]_{ij}\end{aligned}$$

Example: To show that δ_{ij} has the identity property:

$$(A\delta)_{ij} = \sum_k A_{ik}\delta_{kj}$$

but the sum has a contribution only when $k = j$, thus

$$(A\delta)_{ij} = A_{ij}\delta_{jj}(\text{no sum}) = A_{ij}$$

and similarly with (δA).

Matrix multiplication does not commute: $AB \neq BA$ in general, so groups of matrices are typically non-Abelian. The matrix determinant satisfies $det(AB) = det(A)det(B)$ and so the determinant of an orthogonal matrix must be ± 1. It is impossible to move continuously from a determinant of 1 to a determinant of -1 so the rotation group is in fact given by two separate pieces; the matrices with positive determinant are rotations while those with negative determinant are reflections.

2.4 Variational mechanics

[Covered in maths unit Mechanics 2. This is used in sections 4.2 and 5.3, both of which are the basis for later sections.]

Lagrangian and Hamiltonian mechanics will be useful to us for a number of reasons:

1. As noted previously, the geodesic equation for the motion of particles in a gravitational field is elegantly formulated as a variational principle, ie finding a curve of extremal length.
2. Lagrangian mechanics is easily formulated in arbitrary coordinate systems, as is general relativity.
3. A Hamiltonian formulation of the geodesic equation will provide a quicker calculation of geodesics than the standard formulation using the connection coefficients.
4. Variational approaches also provide a good introduction to conserved quantities.

The idea is that free particles move in straight lines in space-time, corresponding to minimal length; can curved particle trajectories due to external forces be described as minimising something more complicated than length? The answer (you guessed) is yes.

In Lagrangian mechanics introduce arbitrary coordinates q_i describing the state of the system (for one particle moving in 3 dimensions, $i = 1..3$). Fix the initial and final states of the system, $\mathbf{q}^{(i)}, \mathbf{q}^{(f)}$ then find a stationary point of the action, defined by

$$S = \int_{t_i}^{t_f} L(\mathbf{q}, \dot{\mathbf{q}}, t)dt$$

where the Lagrangian function L is chosen so as to give the desired dynamics, the dot indicates a time derivative, and the stationary point is calculated over the space of all functions $\mathbf{q}(t)$ satisfying the initial and final conditions.

Suppose we vary the trajectory by adding a small perturbation $\delta\mathbf{q}$ vanishing at the endpoints. Then the new action will be

$$S + \delta S = \int_{t_i}^{t_f} L(\mathbf{q} + \delta\mathbf{q}, \dot{\mathbf{q}} + \dot{\delta\mathbf{q}}, t)dt$$

$$\delta S = \int_{t_i}^{t_f} \sum_i (\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q^i}}\dot{\delta q^i})dt$$

Integrating the second term by parts, and noting that the boundary terms vanish by assumption, we find

$$\delta S = \int_{t_i}^{t_f} \sum_i \delta q^i (\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q^i}})$$

This will vanish for any δq^i if

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q^i}} = 0$$

These equations (one for every i) are known as the Euler-Lagrange equations.

Example: Suppose that $\mathbf{q} = (x, y, z)$ and

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x, y, z)$$

where m is a constant and U is an arbitrary function. Then Lagrange's equations read

$$-\frac{\partial U}{\partial x} - m\frac{d}{dt}\dot{x} = 0$$

$$-\frac{\partial U}{\partial y} - m\frac{d}{dt}\dot{y} = 0$$

$$-\frac{\partial U}{\partial z} - m\frac{d}{dt}\dot{z} = 0$$

which is equivalent to Newton's equations with a potential energy function $U(x, y, z)$. Note that the Lagrangian function is kinetic *minus* potential energy.

The statement that the action is invariant under small perturbations makes no mention of any coordinates, therefore, kinetic minus potential energy in any coordinates gives the correct equations of motion.

Example: Kepler problem in two dimensions. We use polar coordinates:

$$x = r\cos\theta$$

$$y = r\sin\theta$$

and so

$$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$

hence

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{mM}{r}$$

(we have set $G = 1$). Lagrange's equations are

$$mr\dot{\theta}^2 - \frac{mM}{r^2} - m\ddot{r} = 0$$

$$-\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

The second equation states that a certain quantity, the angular momentum, is conserved.

We note that in general, if L does not depend on q_i, $\partial L/\partial\dot{q}_i$ is conserved. This allows us to reduce the number of equations to be solved, a very important principle for simplifying the solution to dynamical problems. We give this quantity a new symbol:

$$p_i = \partial L/\partial\dot{q^i}$$

and call it the canonical momentum conjugate to q^i. Thus Lagrange's equations can be written

$$\dot{p_i} = \frac{\partial L}{\partial q^i}$$

Note that the canonical momentum is equal to the usual (*mechanical*) momentum in Cartesian coordinates, but not otherwise. It also differs from the mechanical momentum in the case of a magnetic field (not covered in this course).

We can further capitalise on the canonical momentum by using it instead of the velocity $\dot{q}_i$, as follows: solve to obtain $\dot{q^i} = u^i(q^i, p_i)$ and construct another function, called the Hamiltonian, as

$$H(q^i, p_i, t) = \sum_i p_i u^i(q^i, p_i) - L(q^i, u^i(q^i, p_i), t)$$

then we have Hamilton's equations,

$$\partial H/\partial q^i = -\partial L/\partial q^i = -\dot{p_i}$$

$$\partial H/\partial p_i = \dot{q^i}$$

$$dH/dt = -\partial L/\partial t$$

thus the Hamiltonian is conserved if L is independent of t. Recall that L is kinetic minus potential energy; the Hamiltonian is usually kinetic plus potential energy, ie total energy.

Example: the Hamiltonian of the 2D Kepler problem. We find

$$p_r = \frac{\partial L}{\partial\dot{r}} = m\dot{r}$$

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$$

which we invert (trivially)

$$\dot{r} = p_r/m$$
$$\dot{\theta} = p_\theta/(mr^2)$$

and write

$$\begin{aligned} H &= \sum p_i \dot{q^i} - L \\ &= p_r^2/m + p_\theta^2/(mr^2) - \frac{1}{2m}(p_r^2 + p_\theta^2/r^2) - mM/r \\ &= \frac{1}{2m}(p_r^2 + p_\theta^2/r^2) - \frac{mM}{r} \end{aligned}$$

2.5 Nonrelativistic Kepler problem

[Mechanics 1 (maths) or Mechanics 207 (physics). This is used in section 7.3, particularly the effective potential technique.]

We have from above, two constants of motion:

$$\tilde{L} = r^2\dot{\theta}$$

is the angular momentum divided by the mass, and

$$\tilde{E} = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{M}{r}$$

is the energy divided by the mass. We can write this as

$$\tilde{E} = \dot{r}^2/2 + \tilde{V}$$

where $\tilde{V} = r^2\dot{\theta}^2/2 + \tilde{U} = \tilde{L}^2/2r^2 + \tilde{U}$ is the *effective* potential. Diagram showing effective potential and qualitative description of orbits.

A quantitative description is best obtained in terms of $\rho = 1/r$; we have

$$\dot{r} = \frac{dr}{d\rho}\frac{d\rho}{d\theta}\frac{d\theta}{dt} = -\frac{1}{\rho^2}\frac{d\rho}{d\theta}\tilde{L}\rho^2 = -\tilde{L}\frac{d\rho}{d\theta}$$

and hence

$$\ddot{r} = \frac{d}{d\theta}\left(-\tilde{L}\frac{d\rho}{d\theta}\right)\dot{\theta} = -\tilde{L}^2\rho^2\frac{d^2\rho}{d\theta^2}$$

Substituting this into Lagrange's equation for $\ddot{r}$, we get

$$\frac{d^2\rho}{d\theta^2} + \rho = \frac{M}{\tilde{L}^2}$$

$$\rho = C\cos(\theta - \alpha) + \frac{M}{\tilde{L}^2}$$

which is the polar equation for a conic section. The fact that the planets move in ellipses with the sun as a focus is Kepler's first law. Orbits with positive energy (either a large initial velocity or a repulsive inverse square force) are parabolas or hyperbolas.

Kepler's second law says that the planets trace out equal areas in equal times. The area traced out by a planet is $dA/dt = r^2\dot{\theta}/2$, which is constant by conservation of angular momentum.

We have area $A = \tilde{L}T/2 = \pi ab$ where a and b are the semimajor and semiminor axes respectively. Thus

$$T = \frac{2\pi ab}{\tilde{L}}$$

The constant $M/\tilde{L}^2$ appearing in the equation for the ellipse is given by a/b^2 so

$$T^2 = \frac{4\pi^2 a^2 b^2}{Mb^2/a} = \frac{4\pi^2 a^3}{M}$$

which is a statement of Kepler's third law, the periods of the planets are proportional to the cubes of the major axes of the ellipses.

3 Special relativity

[This is a brief overview of the theory - consult a text on special relativity for more details, eg Rindler, "Introduction to special relativity", OUP, 1991. Most texts on general relativity have a short chapter on special relativity. I have some notes and problems on the web from a former SR unit.]

3.1 Minkowski space-time

The Newtonian description of the physical world, reformulated by Hamilton and Lagrange, did not explain the Michelson-Morley experiment (1880s) which showed that light travelled at the same speed in all directions, irrespective of the motion of the Earth. Einstein resolved the problem by postulating new axioms:

1. The laws of physics are the same in all inertial reference frames
2. The speed of light is the same in all inertial reference frames

and relaxing the previously held conviction that time is the same for all observers. An inertial reference frame is one in which Newton's first law holds, ie bodies not subject to external forces move with constant speed and direction. In general relativity all these definitions will become "local": inertial observers in gravitational fields separated by large distances will accelerate with respect to each other, but on scales small enough to ignore tidal forces, the laws of special relativity are assumed to hold.

In special relativity, each observer makes measurements according to an array of rods and clocks with no relative motion. The clocks are synchronised so that a light beam emitted from the mid-point of a line (as determined by the rods) reaches the ends at the same time. Observers moving with different velocities do not agree on whether two spatially separated events are simultaneous, ie $t' \neq t$ in general. In general relativity, an observer's inertial reference frame is local, ie no attempt is made to define a global coordinate system associated with a given observer at a particular point in space and time.

The reference frames of special relativity are related by the Lorentz transformations, which differ from the Galilean transformations of nonrelativistic physics (above). For a complete derivation, please refer to a text on special relativity. We note that Einstein's axioms imply that the Lorentz transformations should be linear (because bodies move with constant velocity in both frames), and should preserve the interval (note $c = 1$, timelike convention)

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

for example, $\Delta s = 0$ characterises the motion of light, in any inertial reference frame. In general relativity, we still have an invariant spacetime interval, but we demand that it be expressed in local, differential form,

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2$$

as we have not defined large distances in curved space-time.

We define an *event* as a single point in space-time. The interval Δs^2 now splits space-time into three regions, the past, the future, and elsewhere (diagram). An event can only cause effects in its future light cone, and is only affected by events in its past light cone. Intervals in the future or past are called timelike, elsewhere are called spacelike, and on the light cone itself are called lightlike, or null. We continue to use a topology (ie limits, continuity, open sets, etc) based on the usual topology of $\mathbb{R}^4$, quite distinct from the interval. A *world line* is the trajectory of a particle in space-time, given by a timelike (null) curve for massive (massless) particles.

As in the Galilean case we can write the interval in terms of a metric,

$$\Delta s^2 = g_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

where $g_{\mu\nu} = diag(1, -1, -1, -1)$, greek letters take values 0(time),1,2,3 and we sum over repeated indices as usual. We solve for the condition that the interval is preserved:

$$x^{\alpha'} = \Lambda^{\alpha'}_{\beta} x^\beta$$

$$\Delta s'^2 = g_{\mu'\nu'} \Lambda^{\mu'}_{\alpha} \Delta x^\alpha \Lambda^{\nu'}_{\beta} \Delta x^\beta$$

so that the interval is preserved if the matrix equation $\Lambda^T g \Lambda = g$ is satisfied. This means, as in the Galilean case, $\det \Lambda = \pm 1$.

For the case of a 2×2 matrix

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the above equation leads to $a^2 - c^2 = d^2 - b^2 = 1$, $ab - cd = 0$. If we write $b = \pm \sinh \eta$ and $c = \pm \sinh \chi$ then the equations lead to $a = \pm \cosh \chi$, $d = \pm \cosh \eta$, $\chi = \eta$, and an even number of negative signs among all terms. One such possibility is

$$\Lambda = \begin{pmatrix} \cosh \eta & -\sinh \eta \\ -\sinh \eta & \cosh \eta \end{pmatrix}$$

which leads to

$$x' = \cosh \eta (x - t \tanh \eta)$$

from which we interpret $\tanh \eta$ as the relative velocity v in standard configuration, and so $\sinh \eta = v/\sqrt{1 - v^2}$ and

$\cosh\eta = 1/\sqrt{1-v^2} = \gamma \geq 1$. If there is more than one velocity (hence ambiguity) we might use function notation, ie $\gamma(v)$. We thus arrive at the Lorentz transformations in standard configuration (recall: origins coincide at time zero, O' observer moving with speed v in x-direction as observed by O).

$$t' = \gamma(t - vx)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

The parameter η is called rapidity, and is the space-time equivalent of the rotation angle. In other words, we should understand Lorentz transformations, which relate observers in constant relative motion, as rotations in a four dimensional space-time, which preserve space-time "length" given by the interval.

The set of suitable transformations Λ forms a group, which is in four connected pieces: $\det\Lambda = -1$ gives "improper" Lorentz transformations; $\Lambda_0^{0'} < 0$ interchanges the future and the past. Unless otherwise specified, we will restrict ourselves to the proper orthochronous transformations with $\det\Lambda = 1$ and $\Lambda_0^{0'} > 0$.

3.2 Applications of the Lorentz transformations

We understand from the above derivation that the Lorentz transformations are a kind of rotation in four dimensional space-time; but what do they look like in our previous three dimensional conception of reality? We look at some examples.

Example: Two events are simultaneous and a distance L apart along the x axis in a reference frame O. What is the time interval between them in O' corresponding to an observer moving with speed v in the x direction?

Solution: The events are at (t, x, y, z) equal to $(0, 0, 0, 0)$ and $(0, L, 0, 0)$ The situation is in standard configuration, substitute to get $t' = \gamma(t-vx) = -vL/\sqrt{1-v^2}$. According to the moving observer the right most event occurred first. Sketch on a space-time diagram.

Example: A at $x = 0$ sends a message to B at $x = L$ at some speed u faster than light. Show that in some reference frame the two events are simultaneous, ie any such message would violate causality.

Solution: In the original frame we have A at $(0, 0, 0, 0)$ and B at $(L/u, L, 0, 0)$. An observer with speed v along the x axis observes B at $(\gamma L(1/u - v), \gamma L(1 - v/u), 0, 0)$ where $\gamma = 1/\sqrt{1-v^2}$. The events are simultaneous if the time at B is the same as A (ie zero), hence $1/u - v = 0$, $v = 1/u$ which is possible since $u > 1$.

Example: Time dilation: Show that a clock moving with constant velocity runs slowly by a factor γ.

Solution: In the observer's frame, the clock moves with speed v in the x direction, ie its worldline is given by $x = vt$, ie a typical event is $(t, vt, 0, 0)$. In the clock's frame O' the coordinates of this event are $(t(1 - v^2)/\sqrt{1-v^2}, 0, 0, 0)$, ie

$t' = t\sqrt{1-v^2}$ and time measured by the clock is reduced by a factor $1/\gamma$. Remark: This is routinely observed in particle physics experiments, in which moving unstable particles are observed to live longer in our reference frame.

Example: The twin paradox: Alice travels 4 light years at 0.8c and then returns home at the same speed. Her twin Bob remains at home. Bob says that because Alice has moved with respect to him she will be younger than him when she returns; Alice uses the same argument to say he will be younger. What is their age difference when she returns, and in what direction?

Solution: Alice is not in an inertial frame since she changes her direction of motion. Bob is in an inertial frame, and has the correct explanation. The return travel time is $2\times 4/0.8 = 10$ years, $\gamma = 1/\sqrt{1-v^2} = 5/3$, thus Alice has aged 6 years and Bob 10 years. We get this result in any inertial frame, for example in Alice's outgoing frame Bob is always moving, but Alice is moving really close to the speed of light on her return, and the answers are the same. Sketch.

Example: Velocity addition: O' moves with speed v relative to O, and O" moves with speed w relative to O', how fast does O" move with respect to O? All velocities are in the positive x direction.

Solution: O" has world line $x' = wt'$. We invert the Lorentz transformations (ie change the sign of v) to get

$$x = \gamma(x' + vt') = \gamma(w+v)t'$$

$$t = \gamma(t' + vx') = \gamma(1+vw)t'$$

thus the required speed is

$$\frac{x}{t} = \frac{w+v}{1+wv}$$

which you may remember as the addition formula for $\tanh$, ie it is rapidity not velocity that adds, at least in one dimension. We will do the multidimensional version in the next section. Note also that any velocity added to 1 (the speed of light) is 1: light moves at the same speed in all inertial frames.

Example: Length contraction: Show that a moving object contracts by a factor γ in the direction of motion (only), compared to its own reference frame.

Solution: Consider a rod of length L in its own reference frame, ie "proper length". Its ends are at $x' = 0$ and $x' = L$. We have

$$x' = \gamma(x - vt)$$

thus in the lab frame the ends are at

$$x = vt$$

$$x = vt + L/\gamma$$

ie a shortening. Note that in the lab frame we have to compare at fixed t; if we had used t' we would have obtained the wrong answer. In the y and z directions there is no effect, since $y' = y$ and $z' = z$.

Example: A cube of length L is located a large distance along the y axis and moves along the x axis with speed v. How does it appear from the origin?

Solution: In the lab frame it is contracted in the x direction by a factor γ. However we also need to take account of the different travel time of light from the further to the nearer side, a time L. In this time the cube has moved a distance Lv. Thus we see the near face with length $L\sqrt{1-v^2}$ and the rear face with length Lv, ie in total longer than before!

3.3 Scalars and vectors in special relativity

Now we introduce the concept of a (Lorentz) scalar: this is any quantity which is the same in all inertial frames of reference. Examples are m (the "rest" mass of a particle) and ds^2. Clearly the time registered by a moving clock (its *proper* time) is also a Lorentz scalar. This is simply

$$\begin{aligned} d\tau = & \; dt \quad \text{in its reference frame} \\ = & \; ds \quad \text{in its reference frame} \\ = & \; ds \quad \text{in all frames since a scalar} \end{aligned}$$

In a general reference frame we would compute a proper time derivative as

$$\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = \gamma\frac{d}{dt}$$

Here, we computed $dt/d\tau = (ds^2/dt^2)^{-1/2} = (1-\mathbf{u}^2)^{-1/2} = \gamma$. Note that this is the (special relativistic) time dilation effect: moving clocks run slowly by a factor γ.

We define a new object, called the *4-velocity* as

$$u^\mu = \frac{d}{d\tau}x^\mu = (\gamma, \gamma\mathbf{u})$$

This has the same transformation properties as the 4-position x^μ due to the invariance of the proper time:

$$u^{\alpha'} = \frac{d}{d\tau}x^{\alpha'} = \frac{d}{d\tau}\Lambda^{\alpha'}_{\beta}x^\beta = \Lambda^{\alpha'}_{\beta}\frac{d}{d\tau}x^\beta = \Lambda^{\alpha'}_{\beta}u^\beta$$

In full, this reads (for two frames in standard configuration):

$$\begin{aligned} \gamma(u') &= \gamma(v)(\gamma(u) - v\gamma(u)u_x) \\ \gamma(u')u'_x &= \gamma(v)(\gamma(u)u_x - v\gamma(u)) \\ \gamma(u')u'_y &= \gamma(u)u_y \\ \gamma(u')u'_z &= \gamma(u)u_z \end{aligned}$$

Dividing the last three equations by the first, we find

$$u'_x = \frac{u_x - v}{1 - u_x v}$$

$$u'_y = \frac{u_y}{\gamma(v)(1 - u_x v)}$$

$$u'_z = \frac{u_z}{\gamma(v)(1 - u_x v)}$$

which is what you would obtain by differentiating the Lorentz transformations, and reduces to the nonrelativistic $u'_x = u_x - v$; we already did the x component above.

Example: Stellar aberration: A star is located at a large distance and angle θ from the x axis, in the sun's reference frame. The Earth in its motion around the sun is momentarily moving in the x direction with speed v. At what angle is the star observed from the Earth?

Solution: The photon has velocity $u_x = -\cos\theta$, $u_y = -\sin\theta$, thus we have

$$u'_x = \frac{-\cos\theta - v}{1 + v\cos\theta}$$

$$u'_y = \frac{-\sin\theta}{\gamma(v)(1 + v\cos\theta)}$$

$$\theta' = \arctan\left(\frac{u'_y}{u'_x}\right) = \arctan\left(\frac{\sin\theta}{\gamma(v)(\cos\theta - v)}\right)$$

If we put in real numbers we find that $v = 10^{-4}$ (almost exactly). Thus the aberration for $\theta = \pi/2$ is roughly this in radians, ie about 20 arc seconds, well within the range of small telescopes.

Definition: a *4-vector* is a set of four components that transforms the same as x^μ under Lorentz transformations. The 4-velocity is thus a 4-vector. The advantage of using 4-vectors is that the inner product, defined using the metric,

$$\vec{a} \cdot \vec{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3 = g_{\mu\nu} a^\mu b^\nu$$

is a scalar. Proof:

$$g_{\mu'\nu'} a^{\mu'} b^{\nu'} = g_{\mu'\nu'} \Lambda^{\mu'}_\alpha a^\alpha \Lambda^{\nu'}_\beta b^\beta = g_{\alpha\beta} a^\alpha b^\beta$$

since $\Lambda^T g \Lambda = g$. Thus the scalar $\vec{u} \cdot \vec{u}$ is the same in all inertial reference frames. This quantity is

$$\vec{u} \cdot \vec{u} = \gamma^2 - \gamma^2 u^2 = 1$$

so this doesn't tell us anything new, except that $\vec{u}$ is timelike.

Remark on dimensions: it is most natural to give the components of $4 - vectors$ the same dimension, so that, if we want to reintroduce c at some point, we would write $\vec{u} = (\gamma c, \gamma \mathbf{u})$.

The 4-velocity of an observer in their own reference frame is $(1, 0, 0, 0)$, and this fact can be used to great advantage. For example, the γ factor of particle A in particle B's reference frame is simply $\vec{u}_B \cdot \vec{u}_A = (1, 0, 0, 0) \cdot (\gamma, \gamma\mathbf{u})$ in B's reference frame. But this is a scalar, so it is valid in all reference frames.

Example: Particle A moves in the x direction with speed $3/5$ (of the speed of light), and particle B moves in the direction of the line $x = y$ with speed $4/5$. What is their relative velocity?

$$\vec{u}_A = (5/4, 3/4, 0, 0)$$

$$\vec{u}_B = (5/3, 2\sqrt{2}/3, 2\sqrt{2}/3, 0)$$

$$\gamma = \vec{u}_A \cdot \vec{u}_B = 25/12 - 1/\sqrt{2}$$

$$v = \sqrt{1 - \gamma^{-2}} \approx 0.687$$

We can differentiate again with respect to proper time, obtaining the *4-acceleration*:

$$\vec{a} = \frac{d}{d\tau}\vec{u}$$

$$a^\mu = \frac{du^\mu}{d\tau} = (\gamma\frac{d\gamma}{dt}, \mathbf{u}\gamma\frac{d\gamma}{dt} + \gamma^2\mathbf{a})$$

In the instantaneous rest frame of the particle, this is simply $(0, \mathbf{a})$. The "proper acceleration" is thus

$$\alpha = \sqrt{-\vec{a}\cdot\vec{a}}$$

and we note that $\vec{a}$ is spacelike. Since $\vec{u}\cdot\vec{u}$ is constant, we also have

$$\vec{u}\cdot\vec{a} = 0$$

The mass of a particle measured in its reference frame (also called "rest mass") is also a scalar, and will be denoted simply m (some authors use m_0). By analogy with nonrelativistic mechanics, we construct the 4-momentum

$$\vec{p} = m\vec{u} = (\gamma m, \gamma m\mathbf{u})$$

which we interpret as $(E, \mathbf{p})$. Note that $\mathbf{p}$ reduces to the nonrelativistic momentum as $\gamma \to 1$. For energy we need to expand in a power series:

$$E = \gamma m = m + mu^2/2 + 3mu^4/8 + \dots$$

The first term is mc^2 (in usual units), and is called the "rest energy". It is a constant, so it is ignored in nonrelativistic mechanics. The second term is the nonrelativistic kinetic energy, and the remaining terms are relativistic corrections to the kinetic energy. In high energy physics it is possible to change the number of particles, and in this case conservation of energy implies that the first term needs to be taken into account.

We note the following relations:

$$\vec{p}\cdot\vec{p} = m^2$$

$$E^2 = \mathbf{p}^2 + m^2$$

$$\mathbf{u} = \mathbf{p}/E$$

The first holds in the particle's frame, but as the scalar product does not depend on the reference frame, it holds in all frames, and leads to the second relation.

The momentum and energy both diverge as the speed approaches that of light, making it impossible for massive particles to reach or exceed this value. However, the equations above make perfect sense if $m = 0$, $u = c$ and $p = E$. Such particles exist, and always travel at the speed of light. Examples are photons (light particles) and gravitons (quantised gravity waves). For these particles, the 4-velocity is infinite, but the 4-momentum is finite.

Example: a proton strikes a stationary target, and the following reaction takes place: $p + p \to p + p + p + \bar{p}$. The $\bar{p}$ is an antiproton, with the same mass as a proton. What is the minimum kinetic energy of the incoming proton? We write down conservation of 4-momentum:

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_f$$

then square both sides,

$$m^2 + 2m^2\gamma + m^2 = (4m)^2$$

since the right hand side has four stationary particles in the minimum energy case. Thus

$$\gamma = 7$$

so the total energy of the incoming proton is $7m$ and its kinetic energy is $6m$.

3.4 Vectors and covectors: general theory

[Covered in maths units Linear Algebra 2, differentiable manifolds; the latter is also relevant to the next two sections]

We now codify this logic more formally, and extend it to tensors. We note that a 4-vector $\vec{V}$ is defined by its transformation properties,

$$V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^{\beta}$$

where Λ is any Lorentz transformation. It is easy to show that the set of 4-vectors satisfy the axioms for a vector space $\mathcal{V}$.

In particular, a vector space has a set of objects called vectors, and a set of scalars (here real numbers). The vectors have a binary operation ("addition"), under which it is an abelian group (see section 2.3) with the zero vector as the identify element. There is an operation in which a scalar is multiplied by a vector to produce a vector, that satisfies (for $a, b \in \mathbb{R}, \vec{V}, \vec{W} \in \mathcal{V}$):

$$\begin{aligned} a(\vec{V} + \vec{W}) &= a\vec{V} + a\vec{W} \\ (a+b)\vec{V} &= a\vec{V} + b\vec{V} \\ (ab)\vec{V} &= a(b\vec{V}) \\ 1(\vec{V}) &= \vec{V} \end{aligned}$$

A linearly independent set of vectors $\{\vec{V}_i\}$ is one in which the equation

$$\sum_i a^i \vec{V}_i = 0$$

has only the solution $a^1 = a^2 = \ldots = 0$.

A spanning set of vectors $\{\vec{V}_i\}$ is one in which for any vector $\vec{W} \in \mathcal{V}$ there exist real numbers a^i called components of $\vec{W}$ with respect to $\{\vec{V}_i\}$, such that

$$\vec{W} = \sum_i a^i \vec{V}_i$$

A basis is a linearly independent spanning set, and a fundamental result of linear algebra is that if there is a finite basis, all other bases have the same number of elements, called the dimension of the vector space.

The vector space used in special relativity is four dimensional, that is, we can find a basis $\vec{e}_\alpha$ $(\alpha = 0, 1, 2, 3)$. In other words, for any vector $\vec{V}$ we write

$$\vec{V} = V^\alpha \vec{e}_\alpha$$

where V^α are the components of the vector $\vec{V}$ with respect to the basis $\{\vec{e}_\alpha\}$. Any inertial observer defines a basis which is simply given by $\vec{e}_0 = (1, 0, 0, 0)$, $\vec{e}_1 = (0, 1, 0, 0)$ etc. in his/her own reference frame. The above transformation thus defines a transformation of the basis:

$$\vec{V} = V^{\alpha'}\vec{e}_{\alpha'} = \Lambda^{\alpha'}_{\beta}V^{\beta}\vec{e}_{\alpha'} = V^{\beta}\vec{e}_{\beta}$$

if we have

$$\vec{e}_{\beta} = \Lambda^{\alpha'}_{\beta}\vec{e}_{\alpha'}$$

which is easy to remember, since we must match the indices. Specifically, each term has the same index/indices in the same position (a lower β in the above equation); other indices come in up-down pairs and are summed using the Einstein Summation Convention.

Associated with the vector space $\mathcal{V}$ is a dual space $\mathcal{V}^*$, defined as follows: Let $\tilde{\rho} : \mathcal{V} \to \mathbb{R}$ be a linear scalar valued function on $\mathcal{V}$, that is, $\tilde{\rho}(\vec{V})$ is a scalar, and

$$\tilde{\rho}(a\vec{U} + b\vec{V}) = a\tilde{\rho}(\vec{U}) + b\tilde{\rho}(\vec{V})$$

It is clear that the set of such linear functions also forms a vector space, with the obvious addition and multiplication by a scalar,

$$(a\tilde{\rho} + b\tilde{\sigma})(\vec{V}) = a\tilde{\rho}(\vec{V}) + b\tilde{\sigma}(\vec{V})$$

Due to linearity, the action of $\tilde{\rho}$ on any vector is completely determined by its action on the basis vectors,

$$\tilde{\rho}(\vec{V}) = V^{\alpha}\tilde{\rho}(\vec{e}_{\alpha}) = V^{\alpha}\rho_{\alpha}$$

where ρ_{α} as defined above is the component of ρ with respect to a dual basis $\tilde{\omega}^{\alpha}$, if we have

$$\rho_{\alpha} = \tilde{\rho}(\vec{e}_{\alpha}) = \rho_{\beta}\tilde{\omega}^{\beta}(\vec{e}_{\alpha})$$

that is,

$$\tilde{\omega}^{\beta}(\vec{e}_{\alpha}) = \delta^{\beta}_{\alpha}$$

We note that everything is symmetrical between the vectors and the covectors, hence the name "dual vector space". The transformation of covectors is again straightforwardly determined from our above definitions and the fact that a scalar is invariant:

$$V^{\alpha'}\rho_{\alpha'} = \Lambda^{\alpha'}_{\beta}V^{\beta}\rho_{\alpha'} = V^{\beta}\rho_{\beta}$$

if we have

$$\rho_{\beta} = \Lambda^{\alpha'}_{\beta}\rho_{\alpha'}$$

and similarly

$$\tilde{\rho} = \rho_{\alpha}\tilde{\omega}^{\alpha} = \Lambda^{\beta'}_{\alpha}\rho_{\beta'}\tilde{\omega}^{\alpha} = \rho_{\beta'}\tilde{\omega}^{\beta'}$$

if we have

$$\tilde{\omega}^{\beta'} = \Lambda^{\beta'}_{\alpha}\tilde{\omega}^{\alpha}$$

In both cases we need just match the indices. We note that everything is symmetrical between the vectors and the covectors, hence the name "dual vector space". This is completely independent of the scalar product, which we have not yet introduced here. Nomenclature: vectors are also called "contravariant vectors" and upper indices "contravariant components"; covectors are also called 1-forms (see later for the definition of 2-forms etc.) or "covariant vectors" with "covariant components". We will not use the terms contravariant and covariant in this course.

One place you may have already seen a dual vector space is quantum mechanics (bra and ket notation); our work is simpler since we have only a finite dimensional space, and no complex conjugation needed. There is also a geometrical picture, in which vectors are arrows with a point (V^0, V^1, V^2, V^3), and covectors are sets of parallel planes $\rho_0 x^0 + \rho_1 x^1 + \rho_2 x^2 + \rho_3 x^3 = n$. The action of $\tilde{\rho}(\vec{V}) = \vec{V}(\tilde{\rho})$ gives the number of planes pierced by the vector.

3.5 Tensors: metric and Levi-Civita

Tensors are simply linear, scalar valued functions of several vectors, and-or covectors. Let us start with a few examples.

The metric tensor g is a linear function of two vectors, ie

$$g(\vec{U}, \vec{V}) = \vec{U} \cdot \vec{V}$$

It has components

$$g_{\alpha\beta} = g(\vec{e}_\alpha, \vec{e}_\beta) = \vec{e}_\alpha \cdot \vec{e}_\beta$$

which are simply $diag(1, -1, -1, -1)$ if the basis vectors correspond to an inertial observer. In general we say an *orthonormal basis* is one in which the basis vectors are normalised and mutually orthogonal, so the metric is diagonal with diagonal entries ± 1. The above equation leads to the relation

$$\vec{U} \cdot \vec{V} = g_{\alpha\beta} U^\alpha V^\beta$$

We can also expand the metric tensor in terms of a basis, this time of a 16 dimensional vector space $\mathcal{V}^* \otimes \mathcal{V}^*$,

$$g = g_{\alpha\beta} \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$$

where the outer product is defined by

$$(\tilde{\rho} \otimes \tilde{\sigma})(\vec{U}, \vec{V}) = \tilde{\rho}(\vec{U}) \tilde{\sigma}(\vec{V})$$

We obtain the transformation properties as before, to get the result (again match indices...)

$$g_{\alpha'\beta'} = \Lambda^\gamma_{\alpha'} \Lambda^\delta_{\beta'} g_{\gamma\delta}$$

The metric is a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor, so described because of the placement of its indices. It is also symmetric, which may be expressed as either

$$g(\vec{U}, \vec{V}) = g(\vec{V}, \vec{U})$$

or

$$g_{\alpha\beta} = g_{\beta\alpha}$$

This property does not depend on the choice of basis. The symmetry implies that the metric has only 10 independent components (in 4 dimensions).

The metric can also be used to convert vectors to covectors and vice versa, or in component language, raise and lower indices. For any vector $\vec{V}$ we can define a covector $\tilde{V}$ by

$$\tilde{V}(\vec{U}) = g(\vec{V}, \vec{U})$$

with components

$$V_\alpha = g_{\alpha\beta} V^\beta$$

How can we reverse this operation? We need a $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor (also denoted g) so that

$$\vec{V}(\tilde{\rho}) = g(\tilde{V}, \tilde{\rho})$$

that is,

$$V^\alpha = g^{\alpha\beta} V_\beta$$

Combining this with the other equation we find

$$V_\alpha = g_{\alpha\beta} g^{\beta\gamma} V_\gamma$$

which, if true for all vectors $\vec{V}$ implies that $g^{\beta\gamma}$ is the matrix inverse of $g_{\alpha\beta}$. In the basis corresponding to an inertial observer, this is still $diag(1, -1, -1, -1)$. In the geometrical interpretation, raising or lowering indices corresponds to finding a set of parallel lines which is perpendicular to a particular vector.

What happens when we raise one of the indices of the metric itself?

$$g^\alpha_\beta = g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta$$

since the upper and lower forms of g are matrix inverses.

Example (2002 Exam): Given a two dimensional metric

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

and a 1-form $P_\alpha = (4, 5)$ compute the components $g^{\alpha\beta}$ and P^α.

The metric with upper components is simply the matrix inverse:

$$g^{\alpha\beta} = \begin{pmatrix} -3 & 2 \\ 2 & -1 \end{pmatrix}$$

The index of the 1-form is raised using the metric:

$$P^\alpha = g^{\alpha\beta} P_\beta$$

That is, $P^1 = g^{11}P_1 + g^{12}P_2 = -3 \times 4 + 2 \times 5 = -2$, $P^2 = g^{21}P_1 + g^{22}P_2 = 2 \times 4 - 1 \times 5 = 3$.

Example: Calculate all these quantities if $\vec{e}_{1'} = \vec{e}_1 + \vec{e}_2$ $\vec{e}_{2'} = -\vec{e}_1 + 2\vec{e}_2$, and $\vec{V} = 2\vec{e}_1 - \vec{e}_2$.

We have $\vec{e}_{\alpha'} = \Lambda^\beta_{\alpha'} \vec{e}_\beta$ so

$$\Lambda^\beta_{\alpha'} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

noting that the upper index gives the row for the matrix and the lower index gives the column.

The inverse matrix is

$$\Lambda^{\alpha'}_\beta = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$$

which allows us to calculate $\tilde{\omega}^{\alpha'} = \Lambda^{\alpha'}_{\beta}\tilde{\omega}^{\beta}$:

$$\tilde{\omega}^{1'} = \frac{2}{3}\tilde{\omega}^1 + \frac{1}{3}\tilde{\omega}^2$$

$$\tilde{\omega}^{2'} = -\frac{1}{3}\tilde{\omega}^1 + \frac{1}{3}\tilde{\omega}^2$$

Now we have $V^\alpha = (2, -1)$ so we can use $V^{\alpha'} = \Lambda^{\alpha'}_{\beta} V^\beta$ to find $V^{\alpha'} = (1, -1)$. Thus $\vec{V} = \vec{e}_{1'} - \vec{e}_{2'}$.

The metric is $g_{\alpha'\beta'} = \vec{e}_{\alpha'} \cdot \vec{e}_{\beta'}$ so we find

$$g_{\alpha'\beta'} = \begin{pmatrix} 2 & 1 \\ 1 & 5 \end{pmatrix}$$

The inverse metric is the matrix inverse of this:

$$g^{\alpha'\beta'} = \frac{1}{9}\begin{pmatrix} 5 & -1 \\ -1 & 2 \end{pmatrix}$$

The corresponding 1-form $\tilde{V}$ has components $V_\alpha = (2, -1)$ since the metric is diagonal in unprimed coordinates. We use $V_{\alpha'} = g_{\alpha'\beta'} V^{\beta'}$ to obtain $V_{\alpha'} = (1, -4)$. We check that

$$\tilde{V} = \tilde{\omega}^{1'} - 4\tilde{\omega}^{2'} = \frac{2}{3}\tilde{\omega}^1 + \frac{1}{3}\tilde{\omega}^2 + \frac{4}{3}\tilde{\omega}^1 - \frac{4}{3}\tilde{\omega}^2 = 2\tilde{\omega}^1 - \tilde{\omega}^2$$

We also check raising the indices again,

$$V^{\alpha'} = g^{\alpha'\beta'} V_{\beta'}$$

so

$$V^{1'} = \frac{5}{9}1 + (-\frac{1}{9})(-4) = 1$$

$$V^{2'} = \frac{-1}{9}1 + \frac{2}{9}(-4) = -1$$

As a second example, let us consider the cross product of vectors in three dimensions (briefly mentioned before). Like the scalar product, it is linear, so it is described by a tensor. This is a three index (specifically $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ tensor,

$$(\mathbf{a} \times \mathbf{b})(\tilde{\rho}) = \epsilon(\mathbf{a}, \mathbf{b}, \tilde{\rho})$$

or in components

$$(\mathbf{a} \times \mathbf{b})^k = \epsilon_{ij}{}^k a^i b^j$$

with respect to the basis of the 27 dimensional space $\mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}$

$$\epsilon = \epsilon_{ij}{}^k \tilde{\omega}^i \otimes \tilde{\omega}^j \otimes \mathbf{e}_k$$

The transformation properties are

$$\epsilon_{i'j'}{}^{k'} = \Lambda^l_{i'} \Lambda^m_{j'} \Lambda^{k'}_n \epsilon_{lm}{}^n$$

and we know from the properties of the cross product that this tensor is antisymmetric in its first two indices,

$$\epsilon(\mathbf{a}, \mathbf{b}, \tilde{\rho}) = -\epsilon(\mathbf{b}, \mathbf{a}, \tilde{\rho})$$

$$\epsilon_{ij}{}^k = -\epsilon_{ji}{}^k$$

however no symmetry can be defined with respect to the third index, as it is of a different type.

A closely related tensor is obtained when we discuss the triple product of vectors $\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}$ which gives the volume of a parallelepiped, alternatively the determinant of a 3×3 matrix. We have

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = g_{kl}\epsilon_{ij}{}^{k} a^i b^j c^l = \epsilon_{ijl} a^i b^j c^l$$

so the relevant tensor is just the same, with a lowered index. We know that the volume does not depend on a cyclic permutation of the vectors (and hence the indices), and changes sign when the orientation reverses from right to left handed. This means we have

$$\epsilon_{ijk} = \begin{cases} 1 & \text{even permutation of 123} \\ -1 & \text{odd permutation of 123} \\ 0 & \text{otherwise} \end{cases}$$

in an orthonormal right handed basis. This ϵ is completely antisymmetric in each of its indices when they are lowered.

Example: calculate $\epsilon_{ij}{}^{j}$ This is

$$g^{jk}\epsilon_{ijk} = g^{kj}\epsilon_{ijk} = g^{jk}\epsilon_{ikj} = -g^{jk}\epsilon_{ijk}$$

using the symmetry/antisymmetry properties of g and ϵ, and relabeling the indices in the middle step. Since this is equal to the negative of itself, it must be zero.

We can construct a similar completely antisymmetric tensor in four dimensions, called the Levi-Civita tensor, with

$$\epsilon_{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{even permutation of 0123} \\ -1 & \text{odd permutation of 0123} \\ 0 & \text{otherwise} \end{cases}$$

in a right handed orthochronous orthonormal basis. This tensor will be useful in defining a 4-dimensional volume element corresponding to a hyperparallelepiped obtained from four vectors,

$$\epsilon_{\alpha\beta\gamma\delta} a^\alpha b^\beta c^\gamma d^\delta$$

and also a 3-dimensional *volume form*

$$\Sigma_\alpha = \epsilon_{\alpha\beta\gamma\delta} a^\beta b^\gamma c^\delta$$

which gives a 1-form (set of parallel planes) aligned to three 4-vectors of magnitude equal to the volume of the 3-dimensional parallelpiped. The flux of a vector $\vec{v}$ through the volume element is then

$$\Sigma_\alpha v^\alpha$$

Symmetry and antisymmetry: For any $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor $T_{\alpha\beta}$ we can decompose it into a symmetric part $T_{(\alpha\beta)}$ and an antisymmetric part $T_{[\alpha\beta]}$, as follows:

$$T_{\alpha\beta} = T_{(\alpha\beta)} + T_{[\alpha\beta]}$$

$$T_{(\alpha\beta)} = \frac{1}{2}(T_{\alpha\beta} + T_{\beta\alpha})$$

$$T_{[\alpha\beta]} = \frac{1}{2}(T_{\alpha\beta} - T_{\beta\alpha})$$

and note that the 16 independent components have been split into 10 (the symmetric part) plus 6 (the antisymmetric part). This works analagously for $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensors. We can define symmetry and antisymmetry over more than two indices by summing over all permutations (with negative signs for odd permutations in the case of antisymmetry), but there is no simple decomposition in this case.

Example: antisymmetrise a $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$ tensor:

$$T_{[\alpha\beta\gamma]} = \frac{1}{6}(T_{\alpha\beta\gamma} - T_{\alpha\gamma\beta} - T_{\beta\alpha\gamma} + T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta} - T_{\gamma\beta\alpha})$$

A purely antisymmetric $\begin{pmatrix} 0 \\ p \end{pmatrix}$ tensor is called a p-form. An antisymmetric tensor product between p-forms is called the wedge product, a generalisation of the cross product of vectors in 3D.

$$(\rho \wedge \sigma)_{a_1 \ldots a_p b_1 \ldots b_q} = \frac{(p+q)!}{p!q!} \rho_{[a_1 \ldots a_p} \sigma_{b_1 \ldots b_q]}$$

In some ways the calculus of forms is simpler than general tensors, but we will not need them in any essential way in this course.

3.6 Tensors: general theory

Tensors, which describe any linear relation between vectors, have many applications in physics. One example you may have seen in the inertia tensor I_{ij} in rigid body dynamics, that relates the angular momentum and angular velocity,

$$L_i = I_{ij}\omega^j$$

this is also a symmetric tensor. Unlike the metric and antisymmetric tensors, this tensor has components which are generally not 0, 1, or -1 in an orthonormal basis.

We can now summarise a number of properties of tensors: An $\begin{pmatrix} m \\ n \end{pmatrix}$ tensor T, defined as a multilinear real valued function acting on m covectors and n vectors, transforms as

$$\begin{aligned} & T^{\alpha'_1\alpha'_2\ldots\alpha'_m}_{\beta'_1\beta'_2\ldots\beta'_n} \\ = \; & \Lambda^{\alpha'_1}_{\gamma_1}\Lambda^{\alpha'_2}_{\gamma_2}\ldots\Lambda^{\alpha'_m}_{\gamma_m}\Lambda^{\delta_1}_{\beta'_1}\Lambda^{\delta_2}_{\beta'_2}\ldots\Lambda^{\delta_n}_{\beta'_n}T^{\gamma_1\gamma_2\ldots\gamma_m}_{\delta_1\delta_2\ldots\delta_n} \end{aligned}$$

So far we have seen a number of operations on tensors which generate other tensors (proof easy): If A and B are $\begin{pmatrix} m \\ n \end{pmatrix}$ tensors and C is a $\begin{pmatrix} k \\ l \end{pmatrix}$ tensor, we have

- Multiplication by a scalar: cA is a $\begin{pmatrix} m \\ n \end{pmatrix}$ tensor, for example $(cA)^{\alpha\beta}_{\gamma} = cA^{\alpha\beta}_{\gamma}$.
- Addition: $A+B$ is a $\begin{pmatrix} m \\ n \end{pmatrix}$ tensor, for example $(A+B)^{\alpha\beta}_{\gamma} = A^{\alpha\beta}_{\gamma} + B^{\alpha\beta}_{\gamma}$.

- Contraction of indices: $\mathrm{tr}A$ is a $\begin{pmatrix} m-1 \\ n-1 \end{pmatrix}$ tensor. Note that in general there are many ways to calculate a trace, for example a tensor $A^{\alpha\beta}_{\gamma}$ has two traces given by $A^{\alpha\beta}_{\alpha}$ and $A^{\alpha\beta}_{\beta}$; the contraction must be over one upper and one lower index.

- Transpose: A^T is a $\begin{pmatrix} m \\ n \end{pmatrix}$ tensor. As with contraction, there are many way to take a transpose, but the transposed indices must both be up, or both be down; $A^{\alpha\beta\gamma}_{\delta\epsilon}$ has 12 permutations (including the original), for example $A^{\alpha\gamma\beta}_{\epsilon\delta}$.

- Outer product: $A \otimes C$ is a $\begin{pmatrix} m+k \\ n+l \end{pmatrix}$ tensor. For example $(A \otimes C)^{\alpha\beta}{}_{\gamma}{}^{\delta}{}_{\epsilon} = A^{\alpha\beta}{}_{\gamma} C^{\delta}{}_{\epsilon}$.

- Division: If $A \otimes D = C$ then D is a $\begin{pmatrix} k-m \\ l-n \end{pmatrix}$ tensor.

These tensor operations form the basis for the Einstein Summation Convention rules introduced previously; if any equation is written using these rules, and all but one object in the equation are tensors of the specified types, the remaining quantity must also be a tensor.

These properties are straightforward to prove, for example addition:

$$\begin{aligned} A^{\alpha'\beta'} + B^{\alpha'\beta'} &= \Lambda^{\alpha'}_{\gamma}\Lambda^{\beta'}_{\delta}A^{\gamma\delta} + \Lambda^{\alpha'}_{\gamma}\Lambda^{\beta'}_{\delta}B^{\gamma\delta} \\ &= \Lambda^{\alpha'}_{\gamma}\Lambda^{\beta'}_{\delta}(A^{\alpha\beta} + B^{\alpha\beta}) \end{aligned}$$

so the sum is a tensor. But we cannot add tensors of different type, for example

$$A^{\alpha'} + B_{\alpha'} = \Lambda^{\alpha'}_{\beta}A^{\beta} + \Lambda^{\beta}_{\alpha'}B_{\beta}$$

does not transform as any type of tensor.

Example (2002 exam): $T^{\alpha}{}_{\beta}$ is a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor. Show that the trace $T^{\mu}{}_{\mu}$ is a scalar.

By the definition of a tensor, the components of T transform as

$$T^{\alpha'}{}_{\beta'} = \Lambda^{\alpha'}_{\gamma}\Lambda^{\delta}_{\beta'}T^{\gamma}_{\delta}$$

Setting $\beta' = \alpha'$ and summing, we have

$$T^{\alpha'}{}_{\alpha'} = \Lambda^{\alpha'}_{\gamma}\Lambda^{\delta}_{\alpha'}T^{\gamma}_{\delta} = \delta^{\delta}_{\gamma}T^{\gamma}_{\delta} = T^{\gamma}{}_{\gamma}$$

The second step is the statement that $\Lambda^{\alpha'}_{\gamma}$ and $\Lambda^{\delta}_{\alpha'}$ are matrix inverses, and the third step is simply multiplication by the identity matrix. Thus the trace is invariant under Lorentz transformation, and hence a scalar.

Definition: a tensor *field* is a tensor defined at each point in space(-time), ie $T : \mathbb{R}^d \to \mathcal{V}^m \otimes \mathcal{V}^{*n}$. Finally, we can differentiate a tensor field with respect to x. Let us take this in detail. Consider a scalar field $\phi(\vec{x})$, that is, a scalar that depends on position.

$$\frac{\partial\phi}{\partial x^{\alpha'}} = \frac{\partial x^{\beta}}{\partial x^{\alpha'}}\frac{\partial\phi}{\partial x^{\beta}}$$

however we have (for rotations and Lorentz transformations)

$$\frac{\partial x^\beta}{\partial x^{\alpha'}} = \Lambda^\beta_{\alpha'}$$

so $\frac{\partial \phi}{\partial x^\beta}$ is a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor, ie a 1-form. For this reason we use the notation $\phi_{,\alpha}$ for $\partial\phi/\partial x^\alpha$. It is easy to see that the same logic applies to derivatives of more general tensors. Note that here we are assuming that the basis is derived from an inertial frame of reference; in another basis $\phi_{,\alpha}$ will not be the derivative of ϕ with respect to anything in particular. Note also that the equality of the transformation matrix with the partial derivative is more general than rotations or Lorentz transformations, and we will use it for general coordinate transformations when we talk about curvature.

We will use three derivative operators in GR, $\tilde{\partial}$ corresponds to the above partial derivative, $\tilde{d}$ is an antisymmetric derivative for use with forms, and $\tilde{\nabla}$ is a "covariant derivative" which will generate valid tensor equations. All three are the same as applied to scalar fields, and $\tilde{\partial}=\tilde{\nabla}$ when the bases are constant (as we assume in this section).

Example: write the multidimensional Taylor series of a function f using tensor notation.

$$\begin{aligned} f(x^\alpha + a^\alpha) &= f(x^\alpha) + a^\alpha f_{,\alpha}(x^\alpha) + \frac{1}{2}a^\alpha a^\beta f_{,\alpha\beta}(x^\alpha) \\ &+ \frac{1}{6}a^\alpha a^\beta a^\gamma f_{,\alpha\beta\gamma}(x^\alpha) + \ldots \end{aligned}$$

This takes care of all the combinatoric factors, since for example, there are two terms for each $\alpha \neq \beta$, but a single term when $\alpha = \beta$ in the second derivative term.

3.7 Continua in special relativity

We need to find the relativistic equivalent to the mass density ρ which generated the gravitational field in nonrelativistic gravity. We consider a continuous medium, and measure properties in its *Momentarily Comoving Reference Frame* (MCRF), that is, in the reference frame of an observer who is moving at the same velocity as the medium at a particular time, and sees zero net flux of particles. In particular, we are interested in the densities of number of particles, energy-momentum and possibly other conserved quantities.

The number of particles passing through a 3-volume $\tilde{\Sigma}$ is the product of the volume and the "density" however described in special relativity; we thus derive

$$N = \Sigma_\mu J^\mu$$

where J^μ is a vector, the "particle current" or "number flux" density. In the MCRF, the number of particles passing through the surface $\tilde{\Sigma} = \tilde{d}t = (1, 0, 0, 0)$ is simply the proper density n, so $J^\mu = (n, 0, 0, 0) = nu^\mu$ in this frame (where $\vec{u}$ is the MCRF 4-velocity). However this is a tensor equation, thus

$$J^\mu = nu^\mu = n(\gamma, \gamma\mathbf{u})$$

in all reference frames. The γ factors can be understood as the SR length contraction, and density times velocity gives the flux.

Conservation of particles implies that the total flux of particles through a hypercube of size δ is zero. The amount through the $\tilde{dt}$ surfaces is

$$\delta^3 J^0|_{t+\delta/2} - \delta^3 J^0|_{t-\delta/2} = j^0_{,0}\delta^4$$

and similarly for the other surfaces. Finally we obtain

$$J^\mu_{,\mu} = 0$$

or in 3D notation,

$$\frac{\partial}{\partial t}(n\gamma) + \nabla \cdot (n\gamma \mathbf{u}) = 0$$

This is known as the continuity equation. Conservation equations are always expressed in SR as the vanishing of a 4-divergence.

We could also express conservation as a global equation, $dN/dt = 0$ where $N = \int n d^3x$; this is equivalent to the above equation (use the divergence theorem):

$$\frac{dN}{dt} = \int_D \frac{\partial J^0}{\partial t} d^3x = -\int_D \nabla \cdot \mathbf{j} d^3x = -\int_{\partial D} \mathbf{j} \cdot d\mathbf{s} = 0$$

... but of less use in SR as it is frame-dependent.

In a similar manner, the amount of energy-momentum p^μ passing through the volume is also linear; we have

$$p^\mu = \Sigma_\nu T^{\mu\nu}$$

where T is called the *stress-energy* tensor. The elements of this tensor are: T^{00} is energy density, T^{0i} is energy flux, T^{i0} is momentum density and T^{ij} is momentum flux, also called the (NR) stress tensor; in simple cases it is $p\delta^{ij}$ where p is pressure.

We will now argue that T is a symmetric tensor. Energy flux is energy density multiplied by its speed, but momentum is energy times speed (ie $\mathbf{u} = \mathbf{p}/E$), so this is momentum density. For the spatial components, we note that an element of size ϵ exerts a force $F^i(j) = T^{ij}\epsilon^2$ on the element in direction j and minus this in the opposite direction. Diagram. This leads to a torque in the z direction of

$$\tau_z = \sum \mathbf{x} \times \mathbf{F} = (T^{yx} - T^{xy})\epsilon^3$$

however the moment of inertia cMR^2 where c is a constant is proportional to ϵ^5, so this torque leads to infinite angular acceleration $\alpha = \tau/I \sim \epsilon^{-2}$ unless

$$T^{xy} = T^{yx}$$

Thus T is symmetric.

Conservation of each component of the energy-momentum implies, using the same argument as for number of particles, that

$$T^{\mu\nu}_{\quad,\nu} = 0$$

Example: A rod with energy density ρ and compressive stress F and cross-sectional area A along its length in the x direction moves with velocity u in the y direction.

In its rest frame the stress energy tensor has only nonzero components $T^{00} = \rho$ and $T^{11} = F/A$. We can apply a Lorentz transformation to move back the the laboratory frame:

$$T^{\mu'\nu'} = \Lambda^{\mu'}_{\alpha}\Lambda^{\nu'}_{\beta}T^{\alpha\beta}$$

or in matrix notation

$$T' = \Lambda T \Lambda^T =$$

$$\begin{pmatrix} \gamma & 0 & \gamma u \\ 0 & 1 & 0 \\ \gamma u & 0 & \gamma \end{pmatrix} \begin{pmatrix} \rho & 0 & 0 \\ 0 & F/A & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & 0 & \gamma u \\ 0 & 1 & 0 \\ \gamma u & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \rho\gamma^2 & 0 & \rho\gamma^2 u \\ 0 & F/A & 0 \\ \rho\gamma^2 u & 0 & \rho\gamma^2 u^2 \end{pmatrix}$$

omitting the z components. So the energy density appears as an energy flux (no surprise) and a pressure (surprise).

Example: A perfect fluid (no viscosity or heat conduction). The stress-energy in the MCRF is $T = diag(\rho, p, p, p)$ where ρ is the energy density and p is the pressure. We have $T^{0i} = 0$ since the heat conduction is zero, and the spatial components must be a multiple of the identity since this is the only 3-tensor which is diagonal (no shear stress) in all orientations. We can write this in a relativistically invariant manner as

$$\begin{aligned} T^{\mu\nu} &= (\rho + p)u^\mu u^\nu - pg^{\mu\nu} \\ &= \begin{pmatrix} \gamma^2\rho + \gamma^2 u^2 p & \gamma^2\mathbf{u}(\rho + p) \\ \gamma^2\mathbf{u}(\rho + p) & \gamma^2\mathbf{u}\otimes\mathbf{u}(\rho + p) + pI \end{pmatrix} \end{aligned}$$

where I is the identity matrix; this expression is valid in all frames. A *dust* in relativity means a perfect fluid with zero pressure.

Let us write down the conservation of energy-momentum explicitly:

$$T^{\mu\nu}{}_{,\nu} = 0$$

The $\mu = 0$ equation reads

$$\frac{\partial}{\partial t}(\gamma^2\rho + \gamma^2 u^2 p) + \nabla\cdot(\gamma^2(\rho + p)\mathbf{u}) = 0$$

which becomes in the NR limit ($\gamma \to 1$, $\rho c^2 \gg p$ and ρ dominated by the mass density)

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0$$

which is the NR continuity equation. The $\mu = i$ equation reads

$$\frac{\partial}{\partial t}(\gamma^2(\rho + p)\mathbf{u}) + \nabla\cdot(\gamma^2(\rho + p)\mathbf{u}\otimes\mathbf{u} + pI) = 0$$

which in the NR limit becomes

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + pI) = 0$$

that is,

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) + \rho(\mathbf{u}\cdot\nabla)(\mathbf{u}) + \mathbf{u}\nabla\cdot(\rho\mathbf{u}) + \nabla p = 0$$

Applying the continuity equation to two derivatives involving ρ, we obtain

$$(\frac{\partial}{\partial t} + \mathbf{u}\cdot\nabla)\mathbf{u} + \frac{1}{\rho}\nabla p = 0$$

This is the Euler equation, ie the Navier-Stokes equation for a fluid with the viscosity set to zero.

Remarks on the use of the stress-energy tensor in general relativity: The source term for the field equations (determining the curvature from the matter distribution) cannot be the rest mass (a scalar) because this is not conserved: an electron and a positron can annihilate (although Einstein was not aware of this process at the time). It should involve energy (which is conserved), however the energy density is just T^{00} so a relativistically invariant theory must involve the whole of the stress-energy tensor.

We thus conclude that pressure also contributes to gravity. The reason we so not normally observe this is that $p \ll \rho c^2$ except at very high density (eg a neutron star) or very high temperature (relativistic particles). This effect, however means that there is a limit to the mass of any stable object: the larger the pressure at the centre, the larger the gravitational field, etc. so collapse is inevitable.

4 The metric

4.1 Curvilinear coordinates

In our previous discussion of vectors and tensors, we assumed that it was possible to construct a basis which does not depend on position. However, our qualitative description of curvature indicated that it is not possible to define vectors (for example basis vectors) globally, so we must make do with a local definition. We can do this without invoking curvature, by looking at arbitrary coordinate systems.

Consider an arbitrary coordinate system. We demand only that the coordinates cover the space in a (sufficiently) differentiable fashion, and be one-to-one.

For example, polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ are well behaved except at the origin: $x = 0$, $y = 0$ corresponds to an infinite number of values of θ. The origin of polar coordinates is called a *coordinate singularity*, which describes a pathology of the coordinate system but not the underlying space.

We will use x^α to denote the coordinates, for example $x^1 = r$ and $x^2 = \theta$.

A scalar field $\phi(x^\alpha)$ has a gradient in terms of the new coordinates

$$\phi_{,\alpha} = \frac{\partial\phi}{\partial x^\alpha}$$

If we want to discuss this with respect to another coordinate system, the components are related by

$$\phi_{,\alpha'} = \Lambda^\beta_{\alpha'}\phi_{,\beta}$$

where

$$\Lambda^\beta_{\alpha'} = \frac{\partial x^\beta}{\partial x^{\alpha'}}$$

is a transformation matrix which now depends on position (but reduces to the NR rotation matrix and the SR Lorentz transformation). Note, however that the transformation between the components is linear, unlike the transformation between the coordinate systems.

Example: Let x^α denote (x, y) and $x^{\alpha'}$ denote (r, θ). We have

$$\Lambda = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

which is just the Jacobian matrix.

We would like $\phi_{,\alpha}$ to be the components of an abstract 1-form $\tilde{d}\phi$ with respect to a 1-form basis $\tilde{\omega}^\alpha$, ie

$$\tilde{d}\phi = \phi_{,\alpha}\tilde{\omega}^\alpha$$

What is this basis? Well, if ϕ is given by one of the coordinates x^β we get

$$\tilde{d}x^\beta = x^\beta{}_{,\alpha}\tilde{\omega}^\alpha = \delta^\beta_\alpha\tilde{\omega}^\alpha = \tilde{\omega}^\beta$$

Recall that geometrically a 1-form corresponds to a set of parallel planes. The 1-form field $\tilde{d}\phi$ is thus the contour surfaces of the function ϕ, which has a larger magnitude when the surfaces are closer together, ie the gradient of ϕ is larger.

Example: write the 1-form basis for polar coordinates in terms of the 1-form basis for Cartesian coordinates.

$$\tilde{\omega}^r = \tilde{d}r = r_{,\alpha}\tilde{\omega}^\alpha = \frac{x}{\sqrt{x^2+y^2}}\tilde{\omega}^x + \frac{y}{\sqrt{x^2+y^2}}\tilde{\omega}^y$$

$$\tilde{\omega}^\theta = \tilde{d}\theta = \theta_{,\alpha}\tilde{\omega}^\alpha = -\frac{y}{x^2+y^2}\tilde{\omega}^x + \frac{x}{x^2+y^2}\tilde{\omega}^y$$

Diagram showing $\tilde{d}\theta$ in polar coordinates.

Now we define vectors as linear functions of 1-forms (as before). In particular the basis vectors satisfy

$$\vec{e}_\alpha(\tilde{\omega}^\beta) = \delta^\beta_\alpha$$

What is the geometrical interpretation of these basis vectors? A vector (as before) is a line with an arrow at one end. This will cut through only one set of surfaces $\tilde{d}x^\alpha$ if is pointing along the x^α curve, ie the curve on which x^α varies but all others are constant. Thus $\vec{e}_\alpha$ is a "tangent" vector to the x^α curve.

More precisely, a *curve* is a mapping $\mathbb{R} \to \mathbb{R}^d$ described by a function $x^\alpha(s)$ from a parameter $s \in \mathbb{R}$ to a coordinate value $x^\alpha \in \mathbb{R}^d$. Note that it is possible to parametrise the same path by many different curves, just by choosing a different parameter. Now a scalar function ϕ will vary along the curve as described by

$$\frac{d\phi}{ds} = \phi_{,\alpha}\frac{dx^\alpha}{ds} = \tilde{d}\phi(\frac{\vec{\partial}}{\partial s})$$

Here we have used the fact that $d\phi/ds$ does not depend on the coordinates (ie it is a scalar), thus the linear function acting on $\tilde{d}\phi$ must be a vector $\vec{V} = \frac{\vec{\partial}}{\partial s}$ called the "tangent vector to the curve $x^\alpha(s)$" with coordinates

$$V^\alpha = \frac{dx^\alpha}{ds}$$

In the special case of the coordinate curve α, with $s = x^\alpha$ and $\phi = x^\beta$, we have

$$\delta^\beta_\alpha = \frac{d\phi}{ds} = \tilde{d}x^\beta(\frac{\vec{\partial}}{\partial x^\alpha})$$

so $\frac{\vec{\partial}}{\partial x^\alpha} = \vec{e}_\alpha$ as required.

Vectors transform as before:

$$V^{\alpha'}\rho_{\alpha'} = V^{\alpha'}\Lambda^\beta_{\alpha'}\rho_\beta = V^\beta\rho_\beta$$

as long as

$$V^\beta = V^{\alpha'}\Lambda^\beta_{\alpha'}$$

Note that x^α is no longer a vector, since it obeys a different transformation law. However, an infinitesimal displacement $\vec{dx}$ is,

$$dx^{\alpha'} = \frac{\partial x^{\alpha'}}{\partial x^\beta}dx^\beta$$

4.2 Length, angle and volume

Tensors are defined by complete analogy with SR, in particular, the element of length (a scalar) defines the metric tensor

$$ds^2 = g(\vec{dx}, \vec{dx})$$

Its components in a general coordinate system are given by the usual tensor transformation law,

$$g_{\alpha'\beta'} = \Lambda^\gamma_{\alpha'}\Lambda^\delta_{\beta'}g_{\gamma\delta} = \sum_\gamma \frac{\partial x^\gamma}{\partial x^{\alpha'}}\frac{\partial x^\gamma}{\partial x^{\beta'}}$$

if the unprimed coordinates correspond to Cartesian coordinates.

Example: polar coordinates again. Using the above equation we find

$$\begin{aligned} g_{\alpha'\beta'} &= \begin{pmatrix} \left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 & \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} \\ \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} & \left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \end{aligned}$$

which is usually written

$$ds^2 = dr^2 + r^2 d\theta^2$$

Note that this is exactly what we calculated when we wrote down the kinetic energy $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ in arbitrary coordinate systems for the Lagrangian formalism.

The inverse metric is given by the matrix inverse:

$$g^{\alpha'\beta'} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

in polar coordinates.

NR variational mechanics in tensor language: if we have

$$L = \frac{m}{2} g_{ij}\dot{x}^i\dot{x}^j - U$$

then

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = m g_{ij}\dot{x}^j$$

and note that p_i is a 1-form according to tensor operations. We invert these equations to obtain

$$\dot{x}^i = \frac{1}{m} g^{ij} p_j$$

where the inverse metric g^{ij} performs the inversion operation (which is linear in this case). Finally we have

$$H = p_i\dot{x}^i - L = \frac{1}{2m} g^{ij} p_i p_j + U$$

Remark: This (without the potential, and a new definition of "time") is just the Lagrangian/Hamiltonian we will use to generate geodesics.

The angle between two vectors at a point is defined as usual using the metric,

$$\cos\theta = \frac{g(\vec{u},\vec{v})}{\sqrt{g(\vec{u},\vec{u})g(\vec{v},\vec{v})}}$$

these expressions are valid in arbitrary coordinate systems. In special relativity we know that "$\cos\theta'' = \gamma$ if both vectors are timelike. In general "$\cos\theta''$ can be greater or less than one, or also imaginary. The angle between two curves is defined the angle between their tangent vectors, but does not depend on the parametrisation. Note that the angle is invariant under transformations which scale the metric,

$$g_{\mu\nu} \to \phi(x^\alpha) g_{\mu\nu}$$

These are called conformal transformations.

The volume is determined as before using a totally antisymmetric tensor ϵ with as many indices as dimensions. For example, in four dimensions we have

$$\epsilon_{\alpha\beta\gamma\delta} dx^\alpha dx^\beta dx^\gamma dx^\delta$$

for the volume of a parallelepiped. The tensor is completely antisymmetric in all coordinate systems, but its components are only zero, one and minus one in an orthonormal basis. An infinitesimal volume element can be defined by an alternative formula,

$$\left|\det(\Lambda^\alpha_{\beta'})\right| dx^{0'} dx^{1'} dx^{2'} dx^{3'}$$

where the unprimed coordinate system is the original Cartesian coordinates. There is an even more useful expression in terms of the determinant of the metric. We write down the expression for the transformation of the metric,

$$g_{\mu'\nu'} = \Lambda^\alpha_{\mu'}\Lambda^\beta_{\nu'} g_{\alpha\beta}$$

and take the determinant,

$$g' = \det(\Lambda)^2 g$$

where $g = \det(g_{\mu\nu})$ is a common notation. Now we have $g = \pm 1$ so the volume element is (now omitting the primes)

$$\sqrt{g} d^3 x$$

for 3D Euclidean space with a general metric, and

$$\sqrt{-g} d^4 x$$

for SR in an arbitrary coordinate system. In general

$$\sqrt{|g|} d^n x$$

In polar coordinates we have $ds^2 = dr^2 + r^2 d\theta^2$ so $g = r^2$ and the proper volume element is

$$r dr d\theta$$

Finally we remark that instead of a basis directly derived from a coordinate system, it is sometimes useful to use an orthonormal basis. For example in polar coordinates we have

$$\vec{e}_{\hat{r}} = \frac{\vec{e}_r}{|\vec{e}_r|} = \vec{e}_r$$

$$\vec{e}_{\hat{\theta}} = \frac{\vec{e}_\theta}{|\vec{e}_\theta|} = \frac{1}{r} \vec{e}_\theta$$

In this new basis we have

$$g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta = \delta_{\alpha\beta}$$

which looks simpler than the coordinate basis derived earlier. We will not use an orthonormal basis much, however, because an expression like

$$\phi_{,\alpha}$$

makes sense as a partial derivative, but an expression

$$\phi_{,\hat{\alpha}}$$

while it can be defined by its transformation properties, is no longer the partial derivative with respect to anything. Nor is the 1-form basis

$$\tilde{\omega}^{\hat{\mu}} = \frac{\tilde{\omega}^\mu}{|\tilde{\omega}^\mu|} = \frac{\tilde{d}x^\mu}{|\tilde{\omega}^\mu|}$$

nor the transformation matrices Λ. Specifically, we see that we need

$$\Lambda^{\hat{\mu}}_{\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1/\mathrm{r} \end{pmatrix}$$

the zeros imply that the putative $\hat{r}$ and $\hat{\theta}$ coordinates depend only on r and θ, respectively. However this is not consistent with

$$\frac{\partial \hat{\theta}}{\partial \theta} = \frac{1}{r}$$

thus $\hat{\theta}$ and $\hat{r}$ do not exist.

4.3 Differentiable manifolds

Finally we are in a position to describe curved spaces. A *differentiable manifold* is a general space described below, which can be defined without reference to a metric tensor. With a positive definite metric it is a *Riemannian manifold*, an indefinite metric a *pseudo-Riemannian manifold* and a SR (+—) type metric a *Lorentzian manifold*. The precise definition will become clear when we define the signature of the metric, below.

We want space-time to be locally like SR in its topology and metric structure, but differ in terms of curvature properties at finite distances (which we discussed qualitatively in section 1), and possibly its global topological properties. For example, a sphere has a different topological structure to a plane, in that a closed curve can be deformed to a point either "inwards" or "outwards". In order to consider only local topology, we do not demand that the whole manifold can be described by a single coordinate system (in fact spherical coordinates are ill-defined at the poles), but by a collection of overlapping coordinate systems.

A n dimensional manifold is defined as follows: It is a set $\mathcal{R}_n$ together with subsets $\{\mathcal{O}_\alpha\}$ with the following properties:

1. Each $p \in \mathcal{R}_n$ lies in at least one $\mathcal{O}_\alpha$, ie the $\mathcal{O}_\alpha$ cover $\mathcal{R}_n$.

2. For each α, there is a one-to-one onto map $\psi_\alpha : \mathcal{O}_\alpha \to \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha$ is an *open* set of $\mathbb{R}^n$. These maps are called "charts" or "coordinate systems". They are necessarily open so that their intersections are large enough to define parallel transport, etc. uniquely from one chart to the next. The collection of charts is sometimes called an *atlas*.

3. If two sets $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ overlap, we can consider the map $\psi_\beta \circ \psi_\alpha^{-1}$ which takes points in the relevant subset of $\mathcal{U}_\alpha$ to the relevant subset of $\mathcal{U}_\beta$. This map must be sufficiently smooth (typically C^∞ although we will use only about four derivatives.).

We induce a topology of $\mathcal{R}_n$ from $\mathbb{R}^n$ (requiring it to be preserved in the maps $\psi_\beta \circ \psi_\alpha^{-1}$), that is, we define open sets in $\mathcal{R}_n$ as the image of open sets by the ψ^{-1}, and define limits, continuity and so forth from there.

A *scalar field* is a map $\Phi : \mathcal{R}_n \to \mathbb{R}$. In coordinate notation we would write $\phi(x^\alpha) = \Phi(\psi^{-1}(x^\alpha))$.

In order to define vectors and 1-forms we have a choice: we can define vector fields as first order differential operators

$$\vec{V} = V^\alpha \frac{\vec{\partial}}{\partial x^\alpha}$$

and 1-forms as linear functions on vectors. Alternatively, we can define 1-forms as gradients of scalars

$$\tilde{\rho} = \rho_\alpha \tilde{d} x^\alpha$$

and vectors as linear functions on 1-forms. In either case, we must remember that these vectors and 1-forms live in a vector space defined separately at each point in the manifold: $\mathcal{V}_x$

or $\mathcal{V}_x^*$ with $x \in \mathcal{R}_n$. We cannot directly compare vectors at different points in the manifold, nor think of a "displacement vector" as an arrow joining two points. The "tangent space" $T\mathcal{R}_n$ is the set of all local vector spaces, ie the set of $\{x, \vec{v}\}$ with $x \in \mathcal{R}_n$ and $\vec{v} \in \mathcal{V}_x$.

All we have defined for SR tensors holds at each point in the manifold, and all that we discussed regarding curvilinear coordinate systems holds, for example the basis vectors and 1-forms, curves, the metric structure.

Theorem: There is a basis which reduces any symmetric invertible metric $g_{\mu\nu}$ to the form of a diagonal matrix with entries ± 1 at one point.

Proof: In matrix notation, the change of basis formula for the metric looks like $g' = \Lambda^T g \Lambda$. It is a theorem of linear algebra that such a matrix Λ exists which diagonalises any real symmetric matrix. The elements can be reduced to ± 1 or zero by suitable diagonal Λ (ie rescaling the basis vectors) and permuted by permutation matrices (ie permuting the basis vectors). If the metric is invertible, none of the diagonal entries is zero.

Note that we cannot change the number of ± 1 entries in a metric by change of basis. The invariant quantity given by the sum of the diagonal elements is called the *signature* of the metric, ie 4 for Euclidean 4-space and -2 for GR (with the timelike convention).

Note also that the definition of the proper volume element depends on the existence of coordinates in which $g_{\mu\nu}$ takes the usual SR form.

For the case of a manifold embedded in a higher dimensional flat space, it is possible to calculate the induced metric.

Example: A sphere in the usual spherical coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$ and $z = r\cos\theta$, with r constant (not a coordinate). We have

$$ds^2 = dx^2 + dy^2 + dz^2$$

where

$$dx = \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi = r\cos\theta\cos\phi d\theta - r\sin\theta\sin\phi d\phi$$

etc. leading to

$$ds^2 = r^2(d\theta^2 + \sin^2\theta d\phi^2) = r^2 d\Omega^2$$

where $d\Omega$ is shorthand for solid angle.

Of course, there are many possible coordinate systems for a sphere, with correspondingly different metrics.

5 The connection

5.1 Curvilinear coordinates

Now we return to flat space for a discussion of the differentiation of vectors. In general coordinate systems, we cannot just differentiate the components of a vector, we need to also differentiate the basis vectors. For example:

$$\frac{\partial}{\partial x^\beta}\vec{V} = \frac{\partial}{\partial x^\beta}(V^\alpha \vec{e}_\alpha) = V^\alpha_{\ ,\beta}\vec{e}_\alpha + V^\alpha \frac{\partial}{\partial x^\beta}\vec{e}_\alpha$$

In polar coordinates, we find for example

$$\begin{aligned}
\frac{\partial}{\partial r}\vec{e}_r &= \frac{\partial}{\partial r}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = 0 \\
\frac{\partial}{\partial \theta}\vec{e}_r &= \frac{\partial}{\partial \theta}(\cos\theta\vec{e}_x + \sin\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y \\
&= \frac{1}{r}\vec{e}_\theta \\
\frac{\partial}{\partial r}\vec{e}_\theta &= \frac{\partial}{\partial r}(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y) = -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y \\
&= \frac{1}{r}\vec{e}_\theta \\
\frac{\partial}{\partial \theta}\vec{e}_\theta &= \frac{\partial}{\partial \theta}(-r\sin\theta\vec{e}_x + r\cos\theta\vec{e}_y) = -r\cos\theta\vec{e}_x - r\sin\theta\vec{e}_y \\
&= -r\vec{e}_r
\end{aligned}$$

In general, the term

$$\frac{\partial}{\partial x^\beta}\vec{e}_\alpha$$

is a vector, so it can be written as a linear combination of basis vectors. We introduce a new symbol $\Gamma^\alpha_{\beta\gamma}$ called a "Christoffel symbol", or "connection coefficient", and write

$$\frac{\partial}{\partial x^\beta}\vec{e}_\alpha = \Gamma^\mu_{\alpha\beta}\vec{e}_\mu$$

Since Γ is not a tensor (it is zero in Cartesian coordinates and nonzero in other coordinate systems), we do not bother with spacing its indices.

In our above calculation, we have obtained the connection coefficients for polar coordinates:

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$$

$$\Gamma^r_{\theta\theta} = -r$$

and all the rest zero.

Let us write the equation for the derivative of a vector in a different manner by interchanging index labels:

$$\frac{\partial\vec{V}}{\partial x^\beta} = \left(V^\alpha_{,\beta} + V^\mu\Gamma^\alpha_{\mu\beta}\right)\vec{e}_\alpha$$

We define new notation

$$V^\alpha_{;\beta} = V^\alpha_{,\beta} + V^\mu\Gamma^\alpha_{\mu\beta}$$

We also define the *covariant derivative* of a vector field $\vec{V}$ as

$$\tilde{\nabla}\vec{V} = \frac{\partial\vec{V}}{\partial x^\beta}\otimes\tilde{\omega}^\beta = V^\alpha_{;\beta}\vec{e}_\alpha\otimes\tilde{\omega}^\beta$$

which is a $\begin{pmatrix}1\\1\end{pmatrix}$ tensor. In Cartesian coordinates, all the Christoffel symbols are zero, so $V^\alpha_{,\beta} = V^\alpha_{;\beta}$ but in other coordinate systems (and generally in curved manifolds) this will not be the case.

We can also define "directional" covariant derivatives:

$$\nabla_\alpha\vec{V} = V^\beta{}_{;\alpha}\vec{e}_\beta$$

$$\nabla_{\vec{U}}\vec{V} = U^\alpha V^\beta{}_{;\alpha}\vec{e}_\beta$$

where clearly

$$\nabla_\alpha = \nabla_{\vec{e}_\alpha}$$

We already know the derivative $\tilde{d}f$ of a scalar field f is a tensor, so we define

$$\tilde{\nabla} f = \tilde{d}f$$

We can also compute the divergence of a vector field, which is a scalar,

$$V^\mu{}_{;\mu}$$

In polar coordinates this is

$$V^\mu{}_{;\mu} = V^\mu{}_{,\mu} + \Gamma^\alpha_{\mu\alpha} V^\mu$$

with

$$\Gamma^\alpha_{r\alpha} = \Gamma^r_{rr} + \Gamma^\theta_{r\theta} = \frac{1}{r}$$

$$\Gamma^\alpha_{\theta\alpha} = \Gamma^r_{\theta r} + \Gamma^\theta_{\theta\theta} = 0$$

Thus

$$V^\mu{}_{;\mu} = V^\mu{}_{,\mu} + \frac{1}{r} V^r = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial \theta} V^\theta$$

which is the usual formula for divergence in polar coordinates. The Laplacian operator ∇^2 is obtained by taking the gradient of a scalar, converting it into a vector $(\phi_{,r}, \phi_{,\theta}/r^2)$, and using the above formula:

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

What is the covariant derivative of a 1-form? Take the equation

$$\phi = \rho_\alpha V^\alpha$$

where ϕ is a scalar, $\tilde{\rho}$ is a 1-form and $\vec{V}$ is a vector. Then

$$\phi_{,\beta} = \rho_{\alpha,\beta} V^\alpha + \rho_\alpha V^\alpha{}_{,\beta} = \rho_{\alpha,\beta} V^\alpha + \rho_\alpha (V^\alpha{}_{;\beta} - \Gamma^\alpha_{\mu\beta} V^\mu)$$

Rearranging terms and relabling indices, we find

$$\phi_{,\beta} = (\rho_{\alpha,\beta} - \rho_\mu \Gamma^\mu_{\alpha\beta}) V^\alpha + \rho_\alpha V^\alpha{}_{;\beta}$$

Now everything except the parenthesised term is a tensor, thus

$$\rho_{\alpha;\beta} = \rho_{\alpha,\beta} - \rho_\mu \Gamma^\mu_{\alpha\beta}$$

must be a tensor. Then we have

$$(\rho_\alpha V^\alpha)_{;\beta} = \rho_{\alpha;\beta} V^\alpha + \rho_\alpha V^\alpha{}_{;\beta}$$

which is just the product rule for differentiation. We can remember these formulae using the rule that the differentiated index appears last in Γ, the sign is negative for 1-forms, and everything else follows from the Einstein summation convention. The same procedure leads to the following for tensors:

$$T_{\mu\nu;\rho} = T_{\mu\nu,\rho} - T_{\alpha\nu}\Gamma^\alpha_{\mu\rho} - T_{\mu\alpha}\Gamma^\alpha_{\nu\rho}$$

$$T^\mu_{\nu;\rho} = T^\mu_{\nu,\rho} + T^\alpha_\nu \Gamma^\mu_{\alpha\rho} - T^\mu_\alpha \Gamma^\alpha_{\nu\rho}$$

$$T^{\mu\nu}{}_{;\rho} = T^{\mu\nu}{}_{,\rho} + T^{\alpha\nu}\Gamma^\mu_{\alpha\rho} + T^{\mu\alpha}\Gamma^\nu_{\alpha\rho}$$

where we see that each index leads to a Γ term, like that of a vector or a 1-form depending on its position.

Now we will show that Γ is symmetric on its lower two indices *in a coordinate basis*. For a scalar field ϕ we have

$$\phi_{,\alpha\beta} = \phi_{,\beta\alpha}$$

since partial derivatives commute (in a non-coordinate basis this doesn't work, since the comma operator is no longer a derivative). In cartesian coordinates, this is identical to the valid tensor equation

$$\phi_{;\alpha\beta} = \phi_{;\beta\alpha}$$

which must therefore hold with respect to all bases (coordinate or not). This reads

$$\phi_{,\alpha\beta} - \phi_{,\mu}\Gamma^{\mu}_{\alpha\beta} = \phi_{,\beta\alpha} - \phi_{,\mu}\Gamma^{\mu}_{\beta\alpha}$$

In a coordinate basis, we can cancel the second derivative terms, and find

$$\Gamma^{\mu}_{\alpha\beta} = \Gamma^{\mu}_{\beta\alpha}$$

How does the metric relate to the connection? In Cartesian coordinates we have

$$g_{\mu\nu;\rho} = g_{\mu\nu,\rho} = 0$$
$$g^{\mu\nu}{}_{;\rho} = g^{\mu\nu}{}_{,\rho} = 0$$

so the covariant derivative must be zero in all bases. This means we can raise and lower indices of covariant derivatives of tensors. That is

$$g_{\mu\nu}V^{\nu}_{;\rho} = (g_{\mu\nu}V^{\nu})_{;\rho} = V_{\mu;\rho}$$

and so forth. In addition, we can compute the connection from the metric. We write down $g_{\mu\nu;\rho} = 0$ in three ways:

$$g_{\alpha\beta,\mu} = \Gamma^{\nu}_{\alpha\mu}g_{\nu\beta} + \Gamma^{\nu}_{\beta\mu}g_{\alpha\nu}$$
$$g_{\alpha\mu,\beta} = \Gamma^{\nu}_{\alpha\beta}g_{\nu\mu} + \Gamma^{\nu}_{\mu\beta}g_{\alpha\nu}$$
$$-g_{\beta\mu,\alpha} = -\Gamma^{\nu}_{\beta\alpha}g_{\nu\mu} - \Gamma^{\nu}_{\mu\alpha}g_{\beta\nu}$$

next we add these equations, grouping terms using the symmetry of g:

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} =$$
$$(\Gamma^{\nu}_{\alpha\mu} - \Gamma^{\nu}_{\mu\alpha})g_{\beta\nu} + (\Gamma^{\nu}_{\alpha\beta} - \Gamma^{\nu}_{\beta\alpha})g_{\nu\mu} + (\Gamma^{\nu}_{\beta\mu} + \Gamma^{\nu}_{\mu\beta})g_{\alpha\nu}$$

Now, in a coordinate basis, we can use the symmetry of Γ to cancel the first two terms, leaving

$$g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha} = 2g_{\alpha\nu}\Gamma^{\nu}_{\beta\mu}$$

Now we use the fact that $g^{\alpha\gamma}$ is the matrix inverse of $g_{\alpha\nu}$, and find

$$\frac{1}{2}g^{\alpha\gamma}(g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}) = \Gamma^{\gamma}_{\beta\mu}$$

The fact that this formula only works in coordinate bases gives another reason for working with coordinate bases. We note that the connection coefficients are linear in the first derivative of the metric, but nonlinear in the metric itself.

Example: In polar coordinates,

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2}g^{\alpha\theta}(g_{\alpha r,\theta} + g_{\alpha\theta,r} - g_{r\theta,\alpha})$$

with $g^{r\theta} = 0$ and $g^{\theta\theta} = r^{-2}$ we have

$$\Gamma^{\theta}_{r\theta} = \frac{1}{2r^2}(g_{\theta r,\theta} + g_{\theta\theta,r} - g_{r\theta,\theta}) = \frac{1}{2r^2}g_{\theta\theta,r} = \frac{1}{2r^2}(r^2)_{,r} = \frac{1}{r}$$

5.2 Curved manifolds

In order to use the results of the previous section, we need to understand to what extent a general curved manifold is "locally flat". In other words, can we construct a coordinate system near an arbitrary point with properties similar to the Cartesian coordinates. From the picture of a "tangent plane" we expect that the curvature should show up at the second derivative. Let us count components:

A change of coordinates relates the 10 components of $g_{\mu\nu}$ to the 16 components of $\Lambda^{\alpha}_{\beta'}$. We have already seen that we can find a coordinate system in which $g_{\mu\nu}$ takes the SR form, assuming that the original metric had one positive and three negative eigenvalues. The remaining 6 degrees of freedom correspond to the parameters of the Lorentz group (3 boosts, 3 rotations).

Differentiating this relation, we have 40 components of $g_{\mu\nu,\alpha}$ which are related to 40 components of $\Lambda^{\alpha}_{\beta',\gamma'}$. In both cases we have a 3-index tensor that is symmetric on a pair of indices.

Differentiating again, we find that the 100 components of $g_{\mu\nu,\alpha\beta}$ cannot fixed by the 80 components of $\Lambda^{\alpha}_{\beta',\gamma'\delta'}$: the remaining 20 components will turn out to correspond to the Riemann curvature tensor.

Formally we have the "local flatness theorem": at each point of a Riemannian manifold, we can find a coordinate system in which

$$g_{\mu\nu} = \eta_{\mu\nu} + O(x^2)$$

where $\eta_{\mu\nu}$ is a diagonal matrix with only ± 1 entries. This means that we can use the results of the previous section, which required $\Gamma^{\alpha}_{\beta\gamma} = 0$ in some coordinate system. In particular, we can show that $g_{\mu\nu;\alpha} = 0$ and $\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}$ in a coordinate basis.

The physical interpretation of this result is that there exist local inertial frames, in which the SR laws of physics can be formulated. However, we must be careful to restrict ourselves to one covariant derivative at this stage: two covariant derivatives (except of a scalar field) lead to derivatives of Γ, which are generally nonzero. Thus $\tilde{\nabla}$ satisfies all the usual rules for differential operators, except that ∇_α and ∇_β do not commute (more on this later...) Having formulated physical laws in the local inertial frame, we follow the simple prescription of "comma goes to semicolon", to give a result valid in all coordinate systems.

Example: conservation of energy-momentum in GR takes the form

$$T^{\mu\nu}{}_{,\nu} = 0$$

in SR, and hence (by the equivalence principle) in a local inertial frame (coordinate system in which $\Gamma^{\alpha}_{\beta\gamma} = 0$ locally) but this is equivalent to

$$T^{\mu\nu}{}_{;\nu} = 0$$

in that coordinate system. The latter equation is valid in all coordinate systems.

Remark: it is possible to define manifolds with a connection, but no metric. Of course we will always use a metric, from which the connection can be derived.

5.3 Parallel transport and geodesics

Parallel transport of a vector along a curve simply says that, in the local inertial frame, the vector remains constant. Recall that a curve $x^\alpha(\lambda)$ (with parameter λ) has tangent vector $u^\alpha = dx^\alpha/d\lambda$. Parallel transport of a vector $\vec{V}$ in a local inertial frame is thus

$$0 = \frac{dV^\alpha}{d\lambda} = u^\beta V^\alpha_{,\beta} = u^\beta V^\alpha_{;\beta}$$

the last of which applies generally. Other notations are

$$u^\beta V^\alpha_{;\beta} = [\nabla_{\vec{u}}\vec{V}]^\alpha = \left[\frac{D\vec{V}}{d\lambda}\right]^\alpha$$

Parallel transport preserves the scalar product of vectors (ie their lengths and angles):

$$\begin{aligned}\frac{d}{d\lambda}(g_{\mu\nu}v^\mu w^\nu) &= u^\alpha(g_{\mu\nu}v^\mu w^\nu)_{;\alpha} \\ &= u^\alpha g_{\mu\nu}(v^\mu_{;\alpha}w^\nu + v^\mu w^\nu_{;\alpha}) = 0\end{aligned}$$

if both vectors are parallel transported.

In a flat space, a vector that is parallel transported back to its original point remains unchanged. Let us try this in polar coordinates along the curve $r = a$, $\theta = \lambda$. We have

$$u^\alpha = \frac{dx^\alpha}{d\lambda} = (0, 1)$$

A vector $\vec{V}$ is transported as

$$0 = \frac{D\vec{V}}{d\lambda} = \nabla_{\vec{u}}\vec{V} = \nabla_\theta\vec{V}$$

or in coordinates

$$0 = V^\alpha_{;\theta} = V^\alpha_{,\theta} + \Gamma^\alpha_{\beta\theta}V^\beta$$

We recall the connection coefficients $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$, $\Gamma^r_{\theta\theta} = -r$ with all the rest zero. Thus we have

$$V^\theta_{,\theta} = -V^r/a$$

$$V^r_{,\theta} = aV^\theta$$

and so

$$V^\theta_{,\theta\theta} = -V^\theta$$

and hence

$$V^\theta = A\sin\theta + B\cos\theta$$

$$V^r = -aV^\theta_{,\theta} = -aA\cos\theta + aB\sin\theta$$

where the constants A and B are determined bu the initial values of V^r and V^θ. Note that a complete rotation $\Delta\theta = 2\pi$ has no effect on $\vec{V}$, which we expect for a flat space.

A *geodesic* curve is one that "looks locally flat", ie one that parallel transports its tangent vector. Thus its equation is

$$\begin{aligned}0 &= \frac{Du^\alpha}{d\lambda} = u^\beta u^\alpha_{;\beta} \\ &= u^\beta(u^\alpha_{,\beta} + \Gamma^\alpha_{\gamma\beta}u^\gamma) = \frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\gamma\beta}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda}\end{aligned}$$

Note that this equation is invariant under transformations $\lambda' = a\lambda + b$. A parameter λ for which this equation is satisfied is called an "affine" parameter; the same path with a non-affine parameter is technically not a geodesic.

Let us measure length along a geodesic:

$$\left(\frac{ds}{d\lambda}\right)^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = g_{\mu\nu}u^\mu u^\nu$$

but the magnitude of a vector is unchanged under parallel transport, so this is constant: an affine parameter is proportional to length, or else the length is zero. In any case, a geodesic which begins timelike, spacelike or null will remain that way. Hence we can talk about timelike, spacelike or null geodesics. In the case of massless particles, it makes sense to choose a parameter λ such that the tangent vector is the (finite) 4-momentum $\vec{p}$.

The path of a free massive (massless) particle in GR will be a timelike (null) geodesic according to the equivalence principle, if we assume that the mass is sufficiently small not to affect the spacetime in which it moves.

A geodesic can also be defined as the shortest path between two points in spacetime. Let us derive the geodesic equation this way:

$$L(x^\mu, u^\mu) = \sqrt{g_{\mu\nu}u^\mu u^\nu}$$

where $u^\mu = dx^\mu/ds$ and the parameter s is for the present arbitrary. Lagrange's equations are

$$\frac{\partial L}{\partial x^\mu} = \frac{d}{ds}\frac{\partial L}{\partial u^\mu}$$

that is,

$$\frac{g_{\mu\nu,\sigma}u^\mu u^\nu}{2\sqrt{g_{\mu\nu}u^\mu u^\nu}} = \frac{d}{ds}\frac{g_{\sigma\nu}u^\nu}{\sqrt{g_{\mu\nu}u^\mu u^\nu}}$$

Knowing that the total length is independent of the choice of parameter s, we now fix it to proper time, so $\sqrt{g_{\mu\nu}u^\mu u^\nu} = 1$. Thus we have

$$\frac{1}{2}g_{\mu\nu,\sigma}u^\mu u^\nu = g_{\sigma\nu,\mu}u^\mu u^\nu + g_{\sigma\nu}\frac{du^\nu}{ds}$$

and using the symmetry of μ and ν for the $g_{\sigma\nu,\mu}$ term, and the inverse metric, we find

$$\frac{du^\rho}{ds} + \frac{1}{2}g^{\rho\sigma}(g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma})u^\mu u^\nu = 0$$

which is the geodesic equation.

It is easy to show that we could have used the simpler

$$L = \frac{m}{2}g_{\mu\nu}u^\mu u^\nu$$

as the Lagrangian. The canonical momentum for this Lagrangian is

$$p_\mu = \frac{\partial L}{\partial u^\mu} = mg_{\mu\nu}u^\nu = mu_\mu$$

which is why we added the factor of $m/2$. The Hamiltonian is

$$H = p_\mu u^\mu - L = \frac{1}{2m}g^{\mu\nu}p_\mu p_\nu$$

For massless particles we would prefer to use a parameter given by the limit of τ/m. This has the effect of dropping the m in the above Hamiltonian.

From this we deduce constants of motion: the Hamiltonian itself, which has the value $m/2$. Also, if the metric does not depend on one of the coordinates x^α (say, the time, or an angle), then p_α (say the energy, or angular momentum) is conserved. We can show this explicitly: the geodesic equation can be written

$$p^\alpha p_{\beta;\alpha} = 0$$

or

$$m\frac{dp_\beta}{d\tau} - \Gamma^\gamma_{\beta\alpha} p^\alpha p_\gamma = 0$$

now the connection term is

$$\begin{aligned}\Gamma^\gamma_{\alpha\beta} p^\alpha p_\gamma &= \frac{1}{2} g^{\gamma\nu}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) p^\alpha p_\gamma \\ &= \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu}) p^\alpha p^\nu\end{aligned}$$

but two terms cancel due to the symmetry of α and ν so we finally have

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

which makes the conservation obvious. Alternatively, we could have used Hamilton's equation

$$\frac{dp_\alpha}{d\tau} = -\frac{\partial H}{\partial x^\alpha} = -\frac{1}{2m} g^{\mu\nu}{}_{,\alpha} p_\mu p_\nu$$

Example: the sphere,

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

where r is constant and θ and ϕ are the coordinates. We have $\Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \cot\theta$ and $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$ (see problem 6.4). Thus the geodesic equation reads:

$$\frac{du^\phi}{d\lambda} + 2\cot\theta u^\theta u^\phi = 0$$

$$\frac{du^\theta}{d\lambda} - \sin\theta\cos\theta (u^\phi)^2 = 0$$

where $u^\alpha = dx^\alpha/d\lambda$, λ is an affine parameter, ie a linear function of distance. This form of the equation is not very helpful in finding a solution. Let us instead use the Hamiltonian, and set $m = 1$, so that the parameter λ will be simply distance s:

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu = \frac{1}{2r^2}(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta})$$

Hamilton's equations give:

$$u^\theta = \frac{d\theta}{d\lambda} = \frac{\partial H}{\partial p_\theta} = p_\theta / r^2$$

$$u^\phi = \frac{d\phi}{d\lambda} = \frac{\partial H}{\partial p_\phi} = p_\phi/(r^2 \sin^2\theta)$$

$$\frac{dp_\theta}{d\lambda} = -\frac{\partial H}{\partial \theta} = \frac{\cos\theta}{r^2 \sin^3\theta} p_\phi^2$$

$$\frac{dp_\phi}{d\lambda} = -\frac{\partial H}{\partial \phi} = 0$$

thus we have two conserved quantities, the Hamiltonian itself, and p_ϕ. It is easy to check the original form of the geodesic equation:

$$\frac{du^\phi}{d\lambda} = \frac{d}{d\lambda}\frac{p_\phi}{r^2 \sin^2\theta} = \frac{p_\phi}{r^2}\frac{-2\cos\theta}{\sin^3\theta}u^\theta = -2\cot\theta u^\theta u^\phi$$

$$\frac{du^\theta}{d\lambda} = \frac{1}{r^2}\frac{dp_\theta}{d\lambda} = \frac{\cos\theta}{r^4\sin^3\theta}p_\phi^2 = \sin\theta\cos\theta(u^\phi)^2$$

Let us consider a geodesic which starts from the point $(\theta,\phi) = (\theta_0, 0)$ and is directed east, ie parallel to $\vec{e}_\phi$. Its tangent vector u^α is thus proportional to $(0,1)$, but will be normalised so that $1 = g_{\mu\nu}u^\mu u^\nu = r^2\sin^2\theta_0(u^\phi)^2$ that is, $\vec{u} = (0, 1/(r\sin\theta_0))$ initially. Geometrically, we see that closer to the pole, the geodesic spans many lines of longitude per unit distance.

Thus the conserved quantities are

$$p_\phi = r^2\sin^2\theta u^\phi = r\sin\theta_0$$

$$\frac{1}{2r^2}(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}) = \frac{r^2}{2}((u^\theta)^2 + \sin^2\theta(u^\phi)^2) = \frac{1}{2}$$

thus at general angle θ we have

$$u^\phi = \frac{\sin\theta_0}{r\sin^2\theta}$$

$$u^\theta = \pm\frac{1}{r}\sqrt{1 - \frac{\sin^2\theta_0}{\sin^2\theta}}$$

from which we can calculate the angle between the tangent vector and the basis vector $\vec{e}_\phi$:

$$\cos\psi = \frac{\vec{u}\cdot\vec{e}_\phi}{|\vec{u}||\vec{e}_\phi|} = \frac{r\sin\theta_0}{1.r\sin\theta} = \frac{\sin\theta_0}{\sin\theta}$$

If we construct a spherical triangle from a segment of geodesic and two segments to the pole, we see this formula is just the sine rule for spherical triangles, where A etc are angles and a etc are the lengths of the opposite sides in units of the radius (ie angles subtended at the centre of the sphere):

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

There is also a cosine rule

$$\cos c = \cos a\cos b + \sin a\sin b\cos C$$

obtained by an appropriate (messy) integration of the u^ϕ and u^θ equations. We get the usual (Euclidean) forms of these rules using the small angle approximations of sine and cosine.

We can also use the Lagrangian formulation of the geodesic equation to compute the connection coefficients. For example in polar coordinates we have

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$$

from which we find

$$r\dot{\theta}^2 = \frac{d}{dt}(\dot{r})$$

$$0 = \frac{d}{dt}(r^2\dot{\theta})$$

These give us

$$\ddot{r} - r\dot{\theta}^2 = 0$$
$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = 0$$

from which we read off

$$\Gamma^r_{\theta\theta} = -r$$
$$\Gamma^\theta_{r\theta} = \frac{1}{r}$$

as before.

Now we are in a position to fully answer the question posed earlier: how to construct a coordinate system near a point for which all the connection coefficients are zero. The following construction is called "Riemann normal coordinates". Take a point X^α (in an arbitrary cordinate system) and a vector V^α. Solve the geodesic equation with initial conditions $x^\alpha(0) = X^\alpha$, $dx^\alpha/d\lambda(0) = V^\alpha$ (in general λ is not proper length since V^α may not be normalised). Label the point at $\lambda = 1$ on the geodesic by the components V^α: these become the new coordinates. In these coordinates, all geodesics passing through the original point are straight lines with constant tangent vector by definition. Thus all the connection coefficients are zero at that point.

Let us try this for the sphere, using the north pole $\theta = 0$ as the initial point. Locally, we choose coordinates (ξ, η) such that $\xi \sim \cos\phi$ and $\eta \sim \sin\phi$. All geodesics from the north pole all lines of constant ϕ, with distance given simply by θ. Thus the new coordinates are related to the old ones by

$$\xi = \theta\cos\phi$$
$$\eta = \theta\sin\phi$$

The metric is

$$\begin{aligned} ds^2 &= d\theta^2 + \sin^2\theta d\phi^2 \\ &= \frac{d\xi^2}{\theta^4}(\xi^2\theta^2 + \eta^2\sin^2\theta) + \frac{d\eta^2}{\theta^4}(\eta^2\theta^2 + \xi^2\sin^2\theta) \end{aligned}$$

where $\theta = \sqrt{\xi^2 + \eta^2}$. The metric reduces to $\delta_{\alpha\beta}$ and is clearly an even function of ξ and η, so the connection coefficients vanish at $\xi = \eta = 0$ as expected.

Finally, we can easily incorporate non-gravitational forces in GR: for example, in SR electromagnetism we have

$$ma^\mu = qF^\mu{}_\nu u^\nu$$

where m is rest mass, $\vec{a}$ is 4-acceleration, q is charge, F is the electromagnetic field tensor, and $\vec{u}$ is the 4-velocity. With an appropriate choice of the components of F, this is equivalent to the usual Lorentz force law

$$\frac{d\mathbf{p}}{dt} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

This becomes:

$$m(\frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma) = qF^\mu{}_\nu u^\nu$$

which is another example of the comma-to-semicolon rule.

6 Curvature

6.1 The curvature tensor

As we have seen, the non-flat aspects of a manifold become apparent at the second derivatives of the metric, which cannot all be set to zero at some point generally (in contrast to the first derivatives) by a change of coordinates.

Let us consider a vector $\vec{V}$ which is parallel transported around a coordinate rectangle $(x^1, x^2) = (a, b) \to (a + \delta a, b) \to (a + \delta a, b + \delta b) \to (a, b + \delta b) \to (a, b)$. Points denoted A,B,C,D,A. From the parallel transport law

$$\nabla_1 \vec{V} = 0$$

we conclude

$$V^\alpha_{,1} = -\Gamma^\alpha_{\mu 1} V^\mu$$

so at the end of the first segment

$$V^\alpha(B) = V^\alpha(A) - \int_{AB} \Gamma^\alpha_{\mu 1} V^\mu dx^1$$

and similarly for the other segments. The total change in V^α around the loop is

$$\begin{aligned} \delta V^\alpha &= -\int_{AB} \Gamma^\alpha_{\mu 1} V^\mu dx^1 - \int_{BC} \Gamma^\alpha_{\mu 2} V^\mu dx^2 \\ &+ \int_{CD} \Gamma^\alpha_{\mu 1} V^\mu dx^1 + \int_{DA} \Gamma^\alpha_{\mu 2} V^\mu dx^2 \end{aligned}$$

where the positive sign is due to the negative direction. These do not cancel, because Γ and V are not constant: to first order we get

$$\delta V^\alpha \approx \int_a^{a+\delta a} \delta b (\Gamma^\alpha_{\mu 1} V^\mu)_{,2} dx^1 - \int_b^{b+\delta b} \delta a (\Gamma^\alpha_{\mu 2} V^\mu)_{,1} dx^2$$

$$\delta V^\alpha \approx \delta a \delta b [(\Gamma^\alpha_{\mu 1} V^\mu)_{,2} - (\Gamma^\alpha_{\mu 2} V^\mu)_{,1}]$$

Now we write the derivatives of V^α in terms of the Γ as at the beginning, and relabel indices, to find:

$$\delta V^\alpha = \delta a \delta b [\Gamma^\alpha_{\mu 1,2} - \Gamma^\alpha_{\mu 2,1} + \Gamma^\alpha_{\nu 2} \Gamma^\nu_{\mu 1} - \Gamma^\alpha_{\nu 1} \Gamma^\nu_{\mu 2}] V^\mu$$

Now the difference δV is a vector that depends linearly on three vectors, $\vec{V}$, and the infinitesimal displacements $\delta a \vec{e}_1$ and $\delta b \vec{e}_2$. Thus, it corresponds to a $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ tensor, called the *Riemann curvature tensor*:

$$\delta \vec{V}(\tilde{\rho}) = R(\tilde{\rho}, \vec{V}, \vec{\delta a}, \vec{\delta b})$$

$$\delta V^\alpha = R^\alpha{}_{\beta\gamma\delta} V^\beta (\delta a)^\gamma (\delta b)^\delta$$

where

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\mu\gamma} \Gamma^\mu_{\beta\delta} - \Gamma^\alpha_{\mu\delta} \Gamma^\mu_{\beta\gamma}$$

Remark: there are unfortunately countless conventions on the sign and placement of indices in the Riemann tensor. Note that the Riemann tensor is linear in the first derivative of the connection, but nonlinear in the connection itself. Thus it is linear in the second derivative of the metric, but nonlinear in the metric itself and in its first derivative. The above equation is what we will use to compute the Riemann tensor. Slightly shorter methods exist, but are beyond the scope of the course. This is a good candidate for computer algebra packages.

We have two equivalent statements: a (simply connected) manifold is "flat" as defined by globally defined vectors, and the Riemann tensor is zero everywhere.

The other significant place the Riemann tensor appears is when we try to commute two covariant derivatives. In a local inertial frame (connection is zero but its derivatives are not) we have

$$\nabla_\alpha \nabla_\beta V^\mu = (\nabla_\beta V^\mu)_{,\alpha} = V^\mu{}_{,\beta\alpha} + \Gamma^\mu_{\nu\beta,\alpha} V^\nu$$

and so the commutator

$$[\nabla_\alpha, \nabla_\beta] V^\mu \equiv \nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu = (\Gamma^\mu_{\nu\beta,\alpha} - \Gamma^\mu_{\nu\alpha,\beta}) V^\nu$$

since the partial derivatives commute. Now this is equivalent to the tensor equation

$$[\nabla_\alpha, \nabla_\beta] V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu$$

which is thus true in any basis. The commutator of covariant derivatives is, in effect, performing the same operations as the parallel transport around a rectangle.

Remark: we can explicitly add "curvature coupling" terms to GR equations we have obtained using the comma-to-semicolon rule. This does not affect the SR limit. Whether such terms are in fact correct depends on experiment, or failing that, aesthetics. Such curvature terms may appear naturally as a result of combining two equations with covariant derivatives, or by demanding a special symmetry, such as conformal invariance.

Let us compute the Riemann tensor in a local inertial frame. We have

$$\Gamma^\alpha_{\beta\gamma,\delta} = \frac{1}{2} g^{\alpha\mu} (g_{\mu\beta,\gamma\delta} + g_{\mu\gamma,\beta\delta} - g_{\beta\gamma,\mu\delta})$$

and so

$$\begin{aligned} R^\alpha_{\beta\gamma\delta} &= \frac{1}{2} g^{\alpha\mu} (g_{\mu\beta,\delta\gamma} + g_{\mu\delta,\beta\gamma} - g_{\beta\delta,\mu\gamma} \\ &- g_{\mu\beta,\gamma\delta} - g_{\mu\gamma,\beta\delta} + g_{\beta\gamma,\mu\delta}) \\ &= \frac{1}{2} g^{\alpha\mu} (g_{\mu\delta,\beta\gamma} - g_{\beta\delta,\mu\gamma} - g_{\mu\gamma,\beta\delta} + g_{\beta\gamma,\mu\delta}) \end{aligned}$$

by commuting partial derivatives. Lowering the first index we find

$$R_{\alpha\beta\gamma\delta} = \frac{1}{2} (g_{\alpha\delta,\beta\gamma} - g_{\beta\delta,\alpha\gamma} - g_{\alpha\gamma,\beta\delta} + g_{\beta\gamma,\alpha\delta})$$

This leads to the following symmetries:

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} = -R_{\alpha\beta\delta\gamma} = R_{\gamma\delta\alpha\beta}$$

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\gamma\delta\beta} + R_{\alpha\delta\beta\gamma} = 0$$

which must therefore hold in all bases. Due to these identities, the Riemann tensor has only 20 independent components in four dimensions. From our previous argument, it thus completely expresses the curvature.

We obtain the 20 components as follows: an antisymmetric second rank tensor has 6 components. The Riemann tensor is composed of two such tensors, in a symmetric way, thus it is like a symmetric second rank tensor in a 6 dimensional space - this makes 21 components. Finally, the last relation is trivial unless all four indices are different. When they are different, it relates the three components of the Riemann tensor which have four different indices, so provides one extra condition, leaving 20 independent components.

In 2D polar coordinates we could have only one independent component, say

$$R^r{}_{\theta r\theta}$$

We recall that $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$, $\Gamma^r_{\theta\theta} = -r$. Thus we have contributions only from

$$R^r{}_{\theta r\theta} = \Gamma^r_{\theta\theta,r} - \Gamma^r_{\theta\theta}\Gamma^\theta_{\theta r} = -1 - (-r)(1/r) = 0$$

as expected.

Differentiating the expression for the Riemann tensor and considering a local inertial frame at a point P,

$$R^\alpha{}_{\beta\gamma\delta,\epsilon} = \Gamma^\alpha_{\beta\delta,\gamma\epsilon} - \Gamma^\alpha_{\beta\gamma,\delta\epsilon}$$

from which we show that at P,

$$R_{\alpha\beta\gamma\delta,\epsilon} + R_{\alpha\beta\delta\epsilon,\gamma} + R_{\alpha\beta\epsilon\gamma,\delta} = 0$$

which is equivalent to

$$R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta} = 0$$

which is thus valid in all bases and at all points. This last relation is called the *Bianchi identities*.

6.2 The Einstein field equations

We have discussed the formulation of physical laws in curved spacetime, and now turn to the question as to what should replace the Newtonian equation (with $G = 1$):

$$\nabla^2\Phi = 4\pi\rho$$

We already discussed the right hand side of the equation - the relativistic generalisation is the stress energy tensor $T^{\mu\nu}$ which is symmetric and divergence-free. For the left hand side we need a second rank tensor with the following properties

1. It should be symmetric
2. It should be divergence free (cf Maxwell's equations which automatically imply conservation of charge: $F^{\mu\nu}{}_{,\nu} = 4\pi J^\mu$ in SR).
3. It should be linear (for simplicity) in the second derivatives of the metric.

We will now construct the most general such tensor using the Riemann tensor (which contains all the non-redundent information about the second derivatives of the metric) and the metric tensor itself. The Bianchi identities

$$R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta} = 0$$

give us hope that something constructed from the Riemann tensor can be divergence free - but there are too many indices (and no "divergence" as such yet). We contract on α and γ:

$$g^{\alpha\gamma}(R_{\alpha\beta\gamma\delta;\epsilon} + R_{\alpha\beta\delta\epsilon;\gamma} + R_{\alpha\beta\epsilon\gamma;\delta}) = 0$$

Note that we didn't have any option about the choice of indices: If we contract on α and β, or on any two of the last three indices we get zero from the symmetry. If we choose any other pair, we get the same expression or its negative, again from the symmetry. This leads to

$$R_{\beta\delta;\epsilon} + R^{\alpha}{}_{\beta\delta\epsilon;\alpha} - R_{\beta\epsilon;\delta} = 0$$

where

$$R_{\beta\delta} = R^{\alpha}{}_{\beta\alpha\delta}$$

is the Ricci tensor, obtained by (again) the only possible contraction of the Riemann tensor (up to a sign). The Ricci tensor is symmetric (using the symmetries of the Riemann tensor). It can be defined without a metric, since the original Riemann tensor was of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ type. We still have a four-index tensor, so we contract again,

$$g^{\beta\delta}(R_{\beta\delta;\epsilon} + R^{\alpha}{}_{\beta\delta\epsilon;\alpha} - R_{\beta\epsilon;\delta}) = 0$$

Again, the other possible contractions lead to the same equation or $0 = 0$. We find

$$R_{;\epsilon} - R^{\alpha}{}_{\epsilon;\alpha} - R^{\alpha}{}_{\epsilon;\alpha} = 0$$

where $R = R^{\alpha}{}_{\alpha}$ is the Ricci scalar, the only scalar linear in the curvature. We have

$$(Rg^{\alpha}{}_{\epsilon} - 2R^{\alpha}{}_{\epsilon})_{;\alpha} = 0$$

which leads finally the divergence a symmetric second rank symmetric tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$$

which is called the Einstein tensor.

Can we construct any other tensors from the metric and its first derivative? The first covariant derivative is zero, so this is no use. The metric itself is symmetric and divergence free. Thus we are led to the most general form satisfying the above requirements:

$$G^{\mu\nu} - \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}$$

Here, the 8π is a constant, which will be justified later by taking the Newtonian limit. Λ is called the cosmological constant. A positive constant (with our sign conventions...) leads to a uniform effective positive energy density and negative pressure, making the Universe tend to expand.

The cosmological constant has a colourful history - at first Einstein included it in order to stop the Universe (then thought static) from collapsing in on itself. In 1928 Hubble discovered that the galaxies were receding from each other, making the cosmological constant unnecessary - at that point Einstein called it "his greatest blunder". Much more recently, astronomical evidence is in favour of a small cosmological constant.

There are also arguments from particle physics for a cosmological constant: each type of particle has "vacuum fluctuations" that give rise to a stress-energy of the vacuum proportional to the metric. However, realistic estimates (barring possible almost cancelation) lead to a value for the cosmological constant which is about 10^{120} too high, possibly the most embarrassing result of theoretical physics.

We will ignore the cosmological constant in what follows. For systems much smaller than the Universe, its effect is negligible.

The Einstein field equations related the ten independent compoenents of the Einstein tensor to those of the stress-energy tensor. Actually there are only six equations, since arbitrary coordinate transformations can set conditions on four of the components. This arbitrary freedom can be put to good use in solving the equations, but it also complicates the analysis in, for example, variational approaches. Note that the field equations do not determine the Riemann tensor (20 components): there are solutions to the equations in vacuum (ie no stress-energy), both outside massive objects, and as propagating gravitational waves. Because the equations are numerous and nonlinear, they are very difficult to solve in general.

Another important point is that energy-momentum conservation is guaranteed by the field equations. This means that, for example, the two centre problem is well defined in Newtonian gravity (the masses are held by unspecified non-gravitational forces) and in electromagnetism (unspecified nonelectromagnetic forces), but in GR the forces explicitly appear, and are described by the stress-energy tensor. There is in fact a solution of the GR field equations involving two black holes with enough charge to balance the gravitational attraction, with the stress-energy given by the appropriate electric field.

There are also more complicated theories of gravity. We could have theories which involve the square of the curvature, or theories where curvature is supplemented by additional (eg scalar) fields. There are two considerations for acceptance of these theories. The first is observational. GR as formulated by Einstein and Hilbert agrees with all observations so far, but has yet to be tested directly in the strong field regime. The second consideration is aesthetic. GR is simpler than other theories, so it has found more acceptance to the present day.

There is as yet, no generally accepted theory of quantum gravity, although calculations can be performed that involve only weak gravitational fields, or quantum matter in curved spacetime. Discussion of these theories is beyond the scope of the course, and an active area of current research.

6.3 Weakly curved spacetimes

We need to check that the Einstein field equations do indeed reproduce Newtonian gravity in the NR limit, and in particular that the constant 8π is correct. Let us suppose that SR is "nearly correct" in that we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is the usual SR metric, and $|h_{\mu\nu}| \ll 1$ is a small correction. Of course, there are only a restricted set of spacetime manifolds that can be reduced to this form in any coordinate system, and for those that can, only in a restricted set of coordinate systems. In this case we have used the complete coordinate freedom of GR to write the metric in a form in which the calculations will be simplest.

Let us consider the effect of coordinate transformations that preserve this form. We have Lorentz transformations

$$x^{\mu'} = \Lambda^{\mu'}_{\nu} x^{\nu}$$

where Λ is a constant metrix given by a SR Lorentz transformation (ie $\Lambda^T \eta \Lambda = \eta$): this is indeed a highly restricted set of coordinate transformations. The metric transforms as

$$\begin{aligned} g_{\alpha'\beta'} &= \Lambda^{\mu}_{\alpha'} \Lambda^{\nu}_{\beta'} g_{\mu\nu} = \Lambda^{\mu}_{\alpha'} \Lambda^{\nu}_{\beta'} \eta_{\mu\nu} + \Lambda^{\mu}_{\alpha'} \Lambda^{\nu}_{\beta'} h_{\mu\nu} \\ &= \eta_{\alpha'\beta'} + h_{\alpha'\beta'} \end{aligned}$$

where η takes the same form as before, and

$$h_{\alpha'\beta'} = \Lambda^{\mu}_{\alpha'} \Lambda^{\nu}_{\beta'} h_{\mu\nu}$$

that is, $h_{\mu\nu}$ transforms like a SR $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. Thus we can think of the full GR equations as being written in terms of SR tensors. Note that for sufficiently extreme Lorentz transformations, the components of $h_{\mu\nu}$ will violate the smallness condition.

Another type of transformation that preserves the "almost flat" form of the metric are "infinitesimal" transformations, also called "gauge transformations",

$$x^{\alpha'} = x^{\alpha} + \xi^{\alpha}(x^{\beta})$$

where we insist $|\xi^{\alpha}{}_{,\beta}| \ll 1$. Thus

$$\Lambda^{\alpha'}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} = \delta^{\alpha}_{\beta} + \xi^{\alpha}{}_{,\beta}$$

$$\Lambda^{\alpha}_{\beta'} = \delta^{\alpha}_{\beta} - \xi^{\alpha}{}_{,\beta} + \dots$$

where we ignore quadratic terms. Then we find

$$\begin{aligned} g_{\alpha'\beta'} &= \Lambda^{\gamma}_{\alpha'} \Lambda^{\delta}_{\beta'} g_{\gamma\delta} = (\delta^{\gamma}_{\alpha} - \xi^{\gamma}{}_{,\alpha})(\delta^{\delta}_{\beta} - \xi^{\delta}{}_{,\beta})(\eta_{\gamma\delta} + h_{\gamma\delta}) \\ &= \eta_{\alpha\beta} + h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha} \end{aligned}$$

where we define

$$\xi_{\alpha} = \eta_{\alpha\beta} \xi^{\beta}$$

Thus we have transformed

$$h_{\alpha'\beta'} = h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$$

Note that the primes in these equations denote the coordinate system, but we assume that α' takes the same numerical value as α. These are obviously not valid tensor equations, since they relate quantities in different coordinate systems, rather than general laws valid for all coordinate systems.

We will now derive the weak field Einstein equations, regarding the GR tensors as SR tensors (ie under Lorentz transformations), and raising/lowering indices using η: this is permitted for *small* tensors, since we drop quadratic terms. The gauge transformations are not Lorentz transformations, but lead to equations with the same physical content.

We can calculate the Riemann tensor to first order,

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$$

independent of the gauge, since a small change of coordinates will not affect these already small quantities.

We define the trace

$$h = h^{\alpha}{}_{\alpha}$$

and trace reverse tensor

$$\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$$

so that

$$\bar{h} = -h$$

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h}$$

thus we find

$$G_{\alpha\beta} = -\frac{1}{2}(\bar{h}_{\alpha\beta,\mu}{}^{\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{\mu} - \bar{h}_{\beta\mu,\alpha}{}^{\mu})$$

We would like to use our four arbitrary functions ξ^{μ} to require four conditions on $h_{\mu\nu}$, namely the *Lorentz gauge condition*,

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0$$

Under a gauge transformation, we find

$$\bar{h}_{\mu'\nu'} = \bar{h}_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} + \eta_{\mu\nu}\xi^{\alpha}{}_{,\alpha}$$

then

$$\bar{h}^{\mu'\nu'}{}_{,\nu'} = \bar{h}^{\mu\nu}{}_{,\nu} - \xi^{\mu,\nu}{}_{\nu}$$

so that we can make $\bar{h}^{\mu'\nu'}{}_{,\nu'} = 0$ by setting

$$\Box\xi^{\mu} \equiv \xi^{\mu,\nu}{}_{\nu} = \bar{h}^{\mu\nu}{}_{,\nu}$$

the 4D Laplacian operator, $\Box$ is given explicitly as

$$\Box = \frac{\partial^2}{\partial t^2} - \nabla^2$$

and it turns out that a given ξ^{μ} can always be found to solve this equation. Thus, in the Lorentz gauge, we write

$$G^{\alpha\beta} = -\frac{1}{2}\Box\bar{h}^{\alpha\beta}$$

so that Einstein's equations become

$$\Box\bar{h}^{\mu\nu} = -16\pi T^{\mu\nu}$$

In Newtonian gravity, we also make a nonrelativistic (low velocity) assumption, that $T^{00} = \rho$ and all other components are small. Also, time derivatives are going to be small compared to space derivatives so we have

$$\nabla^2\bar{h}^{00} = 16\pi\rho$$

Comparing this with Newtonian gravity

$$\nabla^2\Phi = 4\pi\rho$$

we need

$$\bar{h}^{00} = 4\Phi$$

now since all other components of $\bar{h}$ are negligible, we compute

$$\bar{h} = 4\Phi$$

and using

$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h}$$

we find

$$h_{\alpha\beta} = 2\Phi\delta_{\alpha\beta}$$

in other words

$$ds^2 = (1+2\Phi)dt^2 - (1-2\Phi)(dx^2+dy^2+dz^2)$$

We can immediately recognise the (weak field) gravitational redshift

$$d\tau = ds = (1+\Phi)dt$$

but there is also a scaling of lengths evident in this equation, corresponding to a curved spatial metric.

We can consider the motion of relativistic particles (including photons) in this metric, but for the moment we are interested in nonrelativistic particles. The geodesic equation in lower components reads

$$m\frac{dp_\beta}{d\tau} = \frac{1}{2}g_{\nu\alpha,\beta}p^\nu p^\alpha$$

In the NR approximation, we have $\tau = t$ and the only significant component of p^α on the RHS is $p^0 = m$. Thus we have

$$\frac{1}{2}g_{\nu\alpha,\beta}p^\nu p^\alpha \approx \frac{1}{2}m^2 g_{00,\mu} = m^2\Phi_{,\mu}$$

The geodesic equation then reduces to

$$\frac{dp_0}{dt} = m\frac{\partial\Phi}{\partial t}$$

$$-\frac{dp_i}{dt} \approx \frac{dp^i}{dt} = -m\nabla_i\Phi$$

which are just a statement of conservation of energy (if the gravitational potential is constant in time) and Newton's second law. The second statement is obvious; for the first we have

$$m^2 = g_{\mu\nu}p^\mu p^\nu = (1+2\Phi)(p^0)^2 - (1-2\Phi)\mathbf{p}^2$$

thus

$$p^0 = \sqrt{\frac{m^2}{1+2\Phi} + \mathbf{p}^2(1-2\Phi)} \approx (1-\Phi)\sqrt{m^2+\mathbf{p}^2}$$

and so

$$p_0 = g_{00}p^0 \approx (1+\Phi)\sqrt{m^2+\mathbf{p}^2} \approx m + m\Phi + \mathbf{p}^2/2m$$

which is the total (gravitational plus kinetic) energy.

This is energy of a particle - it is possible to define energy of a system as a whole? In general we have trouble because vectors (such as $\vec{p}$) cannot be compared at different places in a curved manifold. However, if we assume that there is an isolated system surrounded by a stationary weak gravitational field, we have

$$\nabla^2 \bar{h}^{\mu\nu} = 0$$

which has solution (vanishing at infinity)

$$\bar{h}^{\mu\nu} = A^{\mu\nu}/r + O(r^{-2})$$

for some constant $A^{\mu\nu}$. In addition we must have the gauge condition

$$0 = \bar{h}^{\mu\nu}{}_{,\nu} = \bar{h}^{\mu j}{}_{,j} = -A^{\mu j} n_j / r^2 + O(r^{-3})$$

where $n_j = x_j/r$. But this is true for all x^i, thus

$$A^{\mu j} = 0$$

and hence only $\bar{h}^{00}$ survives. Thus far from a source, we can use the previous Newtonian approximation, even if the field is strong at the source. By comparing with Newtonian far field, we find $A^{00} = -4M$. Thus we can define the mass of a stationary strong field solution by the motion of geodesics at large distances. Note that this is *not* the mass obtained by adding up constituent particles etc, or energy density.

6.4 Gravitational waves

In vacuum ($T^{\alpha\beta} = 0$) the linearised equations in the Lorentz gauge become:

$$\Box \bar{h}^{\alpha\beta} = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) h^{\alpha\beta} = 0$$

which is the wave equation for each component of $\bar{h}^{\alpha\beta}$ It has solutions of the form

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{-ik_\alpha x^\alpha}$$

for some constant tensor $A^{\alpha\beta}$, and the wave-vector $k^\mu = (\omega, \mathbf{k})$.

Let us align the axes so that the spatial component is in the z-direction:

$$k^\mu = (\omega, 0, 0, k)$$

Then the solution is

$$\bar{h}^{\alpha\beta} = A^{\alpha\beta} e^{i(kz-\omega t)} = A^{\alpha\beta} e^{ik(z-ut)}$$

which is a plane wave moving with speed $u = \omega/k$ in the positive z direction. The wavelength is $\lambda = 2\pi/k$ and the frequency is $f = \omega/(2\pi)$. The surfaces of constant phase correspond the geometrical description of the one-form $\tilde{k}$.

Substituting into the equation we find

$$0 = \eta^{\mu\nu} \bar{h}^{\alpha\beta}{}_{,\mu\nu} = -\eta^{\mu\nu} k_\mu k_\nu \bar{h}^{\alpha\beta}$$

which means that

$$k_\mu k^\mu = 0$$

in other words

$$\omega = |\mathbf{k}|$$

so that the velocity of the wave is one, ie the same as light.

We also require the Lorentz gauge condition, so we have

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0$$

that is

$$A^{\alpha\beta} k_\beta = 0$$

which gives four conditions on $A^{\alpha\beta}$. We can actually require another four conditions:

$$A^\alpha{}_\alpha = 0$$
$$A_{\alpha\beta} u^\beta = 0$$

for a certain arbitrarily chosen timelike vector $\vec{u}$ for example the 4-velocity of an observer. These conditions are imposed by using the remaining gauge freedom allowed within the Lorentz condition. The first condition is that A is traceless, and the second is that A is transverse.

Thus this is called the transverse traceless gauge. In the observer's frame, we have

$$A^{\rm TT}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_{xx} & A_{xy} & 0 \\ 0 & A_{xy} & -A_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which has two degrees of freedom for the z direction. The trace condition implies

$$\bar{h}^{TT}_{\alpha\beta} = h^{TT}_{\alpha\beta}$$

Now the geodesic equation for a free particle initially at rest is

$$\begin{aligned} \frac{du^\alpha}{d\tau} &= -\Gamma^\alpha_{\mu\nu} u^\mu u^\nu = -\Gamma^\alpha_{00} \\ &= -\frac{1}{2}\eta^{\alpha\beta}(h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}) = 0 \end{aligned}$$

so that stationary particles are not accelerated, relative to this coordinate system. Note that this does not imply that the gravitational wave has no physical meaning - it just means that the coordinate system is attached to particles at rest in the observer's frame. The proper distance between two of these particles will change with time, at least in the x and y directions. Diagram of circular ring of particles with both polarisations.

Let us consider a gravitational wave impinging on a spring in the x direction, with mass m at either end, natural length l_0, spring constant k and damping constant ν. Without the gravitational wave, we have

$$m\ddot{x}_1 = -k(x_1 - x_2 + l_0) - \nu(\dot{x}_1 - \dot{x}_2)$$
$$m\ddot{x}_2 = -k(x_2 - x_1 - l_0) - \nu(\dot{x}_2 - \dot{x}_1)$$

now defining

$$\xi = x_2 - x_1 - l_0$$
$$\omega_0^2 = 2k/m$$
$$\gamma = \nu/m$$

we have

$$\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = 0$$

which is the usual damped harmonic oscillator.

Now because the masses are slow moving, the geodesic equation is unchanged to lowest order, so there is no direct force on the masses due to the gravitational wave. There are forces, however, due to the fact that the spring differs from its natural length. Its proper length is

$$l(t) = \int_{x_1(t)}^{x_2(t)} \sqrt{1 + h^{\rm TT}_{xx}(t)}dt$$

and the spring equations are

$$m\ddot{x}_1 = -k(l_0 - l) - \nu\frac{d}{dt}(l_0 - l)$$
$$m\ddot{x}_2 = -k(l - l_0) - \nu\frac{d}{dt}(l - l_0)$$

We define ω_0 and γ as before, then

$$\xi = l - l_0 = x_2 - x_1 - l_0 + \frac{1}{2}h^{\rm TT}_{xx}(x_2 - x_1)$$

which we can solve to get

$$x_2 - x_1 = l_0 + \xi - \frac{1}{2}h^{TT}_{xx} l_0$$

ignoring higher order terms. Finally we have

$$\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = \frac{1}{2}l_0\ddot{h}^{\rm TT}_{xx}$$

which gives a sinusoidal forcing to the spring. In order to detect gravitational waves, we should adjust the natural frequency ω_0 as close as possible to the expected frequency of the gravitational waves, and make the damping γ as small as possible.

What sort of gravitational waves might we expect to detect? Suppose we have some oscillating strong gravitational field: this might be two solar mass neutron stars or black holes in close orbit. Knowing that solutions to the wave equation decay as $1/r$ (we can also derive this for conservation of energy since all waves have energy density proportional to the square of their amplitude), we expect roughly

$$h_{xx} \sim \frac{R_S}{R_D}$$

where R_S is the radius of the source (in the above case a few kilometres) and R_D is the distance to the detector (a typical galactic distance is about $10^{20}m$) so we expect deviations of length by 10^{-17}. Actually this is somewhat optimistic: more detailed calculations give more common events as 10^{-20}. Major efforts are underway to detect these gravitational fields (using laser interferometers), but none have succeeded yet. One indirect detection mechanism involves observing a close binary spiralling together due to loss of energy in gravitational waves. This has been observed.

We have mentioned energy in gravitational waves, but there are a few difficult problems of principle here: this energy is not in $T^{\mu\nu}$ since gravitational waves can travel in a vacuum. In the linearised theory it makes sense to talk of conservation of gravitational and nongravitational energy by constructing an effective $T^{\mu\nu}$ from $\bar{h}^{\mu\nu}$, but this fails in general: conservation of energy itself does not make sense in nonstationary spacetimes.

7 Spherical symmetry

7.1 Spherical symmetry in GR

In the general, nonlinear case, our only hope of finding solutions to the Einstein field equations is to impose symmetry, in a similar fashion to Gauss' theorem in Newtonian gravity. One possiblity of such a symmetry is planar symmetry of gravitational waves. The other case of high symmetry is that of spherical symmetry.

Spherical symmetry implies that the spacetime is comprised of spheres with metric $ds^2 = R^2 d\Omega^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2)$ where R is the "radius" as defined by measurements on the sphere; in particular it is the circumference divided by 2π. Theorem: there is no spherically symmetric vector field on the sphere. Thus any other basis vectors $\vec{e}_r$ and $\vec{e}_t$ must be orthogonal to each sphere, hence $g_{r\theta} = g_{r\phi} = g_{t\theta} = g_{t\phi} = 0$.

Note that R determines distances only on the sphere - it does not determine the distance between neighbouring spheres. It does not determine the distance to a centre of symmetry, or even that such a centre of symmetry exists: a torus has circular symmetry, but no point of the torus can be considered the "centre".

For Schwarzchild coordinates we choose $r = R$ (OK when R is monotonic - not obvious) and $\vec{e}_t$ orthogonal to $\vec{e}_r$, so $g_{tr} = 0$.

Thus we can write

$$ds^2 = g_{tt}dt^2 - |g_{rr}|dr^2 - r^2 d\Omega^2$$

as the most general spherically symmetric spacetime.

An alternative to Schwarzchild coordinates are "isotropic coordinates", for which the radial coordinate is redefined so that

$$ds^2 = g_{tt}dt^2 - |g_{rr}|(dr^2 - r^2 d\Omega^2)$$

or by the usual transformation back to Cartesian coordinates,

$$ds^2 = g_{tt}dt^2 - |g_{rr}|(dx^2 + dy^2 + dz^2)$$

These coordinates reduce naturally to the almost flat spacetime we discussed in the previous section.

Another symmetry we can impose is that of *stationarity*, that is, the metric does not depend on time. This time variable t could be different to the time variable needed to remove the g_{rt} component, so we write

$$ds^2 = g_{tt}dt^2 + 2g_{rt}drdt - |g_{rr}|dr^2 - r^2 d\Omega^2$$

with $g_{tt,t} = g_{rt,t} = g_{rr,t} = 0$ for this case.

In addition, we say that a spacetime is *static*, if it is stationary and time reversal invariant, which kills the g_{rt} term. Thus we end up with

$$ds^2 = g_{tt}dt^2 - |g_{rr}|dr^2 - r^2 d\Omega^2$$

with $g_{tt,t} = g_{rr,t} = 0$ for this case. A rotating star is an example of a spacetime which is stationary but not static: under time reversal, the direction of rotation is reversed. However, a rotating star is not spherically symmetric.

The physical interpretation of the metric components is as follows: the distance between two points along a radial line is determined as

$$s = \int_a^b \sqrt{|g_{rr}|}dr$$

which is generally not r itself.

The g_{tt} component tells the rate of clocks at different positions. The time measured (in coordinate time t) by a stationary clock (at constant (r, θ, ϕ) is given by

$$\frac{d\tau}{dt} = \sqrt{g_{tt}}$$

thus in the stationary case, the gravitational time dilation can be defined by this component. We can also obtain the redshift effect using energy arguments: $p_t = g_{tt}p^t$ is conserved for a particle (possibly a photon) moving along a geodesic. The energy observed by a stationary observer is simply $\vec{p}\cdot\vec{u}$ where $\vec{u}$ is the 4-velocity of such an observer. This 4-velocity is simply

$$u^\alpha = \frac{dx^\alpha}{d\tau} = (1/\sqrt{g_{tt}}, 0, 0, 0)$$

which is also obviously normalised, $\vec{u}\cdot\vec{u} = 1$. Thus

$$E_{\rm local} = \vec{p}\cdot\vec{u} = p_t u^t = p_t/\sqrt{g_{tt}}$$

which gives the gravitational redshift. Photons look "bluer" as they move towards a massive object.

We will also demand "asymptotic flatness" by which g_{tt} and g_{rr} approach unity as $r \to \infty$. This ensures that the t coordinate gives time as registered by a distant clock, and that the spatial geometry approaches that of flat spacetime.

7.2 Spherical stars

We write the spherically symmetric metric in the conventional form

$$ds^2 = e^{2\Phi}dt^2 - e^{2\Lambda}dr^2 - r^2 d\Omega^2$$

where Φ and Λ are functions of r that approach zero at infinity. In the Newtonian limit, Φ becomes the usual gravitational potential. The Einstein tensor takes the form

$$G_{tt} = \frac{e^{2\Phi}}{r^2}\frac{d}{dr}[r(1-e^{-2\Lambda})]$$

$$G_{rr} = -\frac{1}{r^2}e^{2\Lambda}(1-e^{-2\Lambda}) + \frac{2}{r}\Phi'$$

$$G_{\theta\theta} = r^2 e^{-2\Lambda}[\Phi'' + (\Phi')^2 + \Phi'/r - \Phi'\Lambda' - \Lambda'/r]$$

$$G_{\phi\phi} = \sin^2\theta G_{\theta\theta}$$

with off diagonal components zero.

We equate this to a static stress-energy tensor for a perfect fluid, in order to describe a spherical star. Neutron stars have very strong gravitational fields since $r \approx 4M$ and so a GR description is essential.

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}$$

where $\vec{u}$ has only a time component (for static matter). Thus

$$u_t = e^\Phi$$

for normalisation ($u_t u_t g^{tt} = 1$). Then we have

$$T_{tt} = \rho e^{2\Phi}$$

$$T_{rr} = p e^{2\Lambda}$$

$$T_{\theta\theta} = p r^2$$

$$T_{\phi\phi} = \sin^2\theta T_{\theta\theta}$$

Energy-momentum conservation $T^{\mu\nu}{}_{;\nu} = 0$ is trivial except $\mu = r$ which gives

$$(\rho + p)\frac{d\Phi}{dr} = -\frac{dp}{dr}$$

In the Newtonian limit this is just

$$\rho|\mathbf{g}| = -\frac{dp}{dr}$$

which says that the pressure increases inwards at a rate given by the extra force required to hold up the local matter, $\rho|\mathbf{g}|$.

Two of Einstein's equations are nontrivial: the time-time component gives

$$\frac{dm(r)}{dr} = 4\pi r^2 \rho$$

where

$$m(r) = \frac{1}{2}r(1-e^{2\Lambda})$$

In the Newtonian case $m(r)$ is simply the mass within radius r, but in GR it is not: we did not use the *proper* volume element, involving $\sqrt{|g|}$. The difference is the gravitational potential energy, which has the effect of making the total mass of a star less than its constituent particles. Note that we have set $m(0) = 0$, which follows from the fact that the space-time should be locally flat at the centre.

The $r - r$ component gives

$$\frac{d\Phi}{dr} = \frac{m(r) + 4\pi r^3 p}{r(r - 2m(r))}$$

which reduces in the Newtonian limit to

$$|\mathbf{g}| = -\frac{m(r)}{r^2}$$

which is just the usual expression for the gravitational field in a spherical object.

Eliminating $d\Phi/dr$ from this equation and the equation of hydrostatic equilibrium, we obtain the Oppenheimer-Volkov equation,

$$\frac{dp}{dr} = -\frac{(\rho + p)(m + 4\pi r^3 p)}{r(r - 2m)}$$

which, together with an equation of state $p = p(\rho)$ can be solved from an initial pressure $p(0) =$ outwards. The edge of the star occurs when the pressure drops to zero.

Using equations of state derived from quantum statistical mechanics (for white dwarfs: degenerate electrons with massive nuclei, neutron stars: degenerate neutrons), it is found that despite a pressure approaching infinity at the centre, the total mass is limited to approx 1.4 times the mass of the sun. This is called the Chandrasekhar limit. It is due to the apprearance of the pressure on the right hand side of the OV equation, ie pressure effectively contributes to the gravitational field. The observation of collapsed objects with greater than this mass is evidence for black holes (discussed below).

Outside the star, we have $\rho = p = 0$, so $m(r)$ is a constant M.

[Alternative text if small text is omitted:]

Setting $G^{tt} = 0$ by Einstein's equations for a vacuum, we find

$$e^{-2\Lambda} = 1 - 2M/r$$

where M is an unknown constant (which will turn out to be the mass).

[end alternative text]

The remaining equation is then

$$\frac{d\Phi}{dr} = \frac{M}{r(r - 2M)}$$

$$\Phi = \frac{1}{2}\ln(1 - 2M/r)$$

so we have

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2 d\Omega^2$$

from which we deduce that M is indeed the mass as determined by measurements at large distances. This is the Schwarzchild solution. For ordinary spherical objects, the solution continues until the surface of the object; for complete gravitational collapse, we treat the spacetime as being described entirely by this solution (in these or other coordinates). Such an object is called a black hole.

It turns out that we could drop the static assumption: the only solution of the vacuum Einstein equations that is spherically symmetric is the Schwarzchild solution.

Dimensional remark: Recall that ct and GM/c^2 both have dimensions of length.

7.3 Orbits in Schwarzchild spacetime

The Schwarzchild solution clearly has interesting behaviour at $r = 2M$, but we will leave this to the next section.

First, we consider stationary observers, that is, observers at fixed (r, θ, ϕ). We have already calculated the gravitational time dilation: a clock measures time

$$d\tau = \sqrt{1 - 2M/r}dt$$

where t is time measured by a distant, stationary observer.

What is the acceleration of gravity at position r? This is just given by the magnitude of the 4-acceleration of the stationary observer. The 4-velocity is

$$u^\mu = \frac{dx^\mu}{d\tau} = ((1 - 2M/r)^{-1/2}, 0, 0, 0)$$

The 4-acceleration is

$$a^\mu = \frac{Du^\mu}{d\tau} = \frac{du^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta$$

Now the first term is zero (the components u^μ do not depend on time) and the only contribution we will get is from connection coefficients of the form Γ^μ_{tt}. Now the metric is diagonal, so we have

$$\Gamma^\mu_{tt} = \frac{1}{2} g^{\mu\mu}(g_{t\mu,t} + g_{t\mu,t} - g_{tt,\mu})$$

from which only

$$\Gamma^r_{tt} = \frac{1}{2}(1 - 2M/r)(2M/r^2) = \frac{M}{r^2}(1 - 2M/r)$$

Thus

$$a^\mu = (0, M/r^2, 0, 0)$$

This looks like just the standard Newtonian result, but we must remember to compute the proper acceleration

$$\alpha = \sqrt{-\vec{a} \cdot \vec{a}} = \sqrt{-g_{rr} a^r a^r} = \frac{M}{r^2}(1 - 2M/r)^{-1/2}$$

In general, to compare with local observer's measurements, we could use the orthonormal basis:

$$a^{\hat{r}} = (0, (M/r^2)(1 - 2M/r)^{-1/2}, 0, 0)$$

Note also that the sign of the acceleration is positive: the stationary observer is accelerating outwards with respect to a local inertial frame. The acceleration (and hence the force required to maintain a stationary position) is greater than the usual Newtonian result.

Next, we consider orbits of massive particles. Without loss of generality, we consider orbits in the equatorial plane $\theta = \pi/2$: thus $u^\theta = 0$ and $\sin\theta = 1$. We have three "instant" constants of motion, the energy p_t, the angular momentum p_ϕ and the mass squared $\vec{p} \cdot \vec{p}$. Dividing by the masses, we have

$$\tilde{E} = E/m = u_t = g_{tt} u^t = (1 - 2M/r) u^t = (1 - 2M/r) \frac{dt}{d\tau}$$

$$\tilde{L} = L/m = -u_\phi = -g_{\phi\phi} u^\phi = r^2 u^\phi = r^2 \frac{d\phi}{d\tau}$$

$$1 = \vec{u} \cdot \vec{u} = (1 - 2M/r)(u^t)^2 - (1 - 2M/r)^{-1}(u^r)^2 - r^2 (u^\phi)^2$$

At infinity, the energy and angular momentum revert to their SR values:

$$\tilde{E} = \frac{dt}{d\tau} = \gamma$$

$$\tilde{L} = r^2 \frac{d\phi}{d\tau} = \gamma u b$$

where b is the impact parameter.

Thus we find

$$(u^r)^2 = \tilde{E}^2 - \tilde{V}^2$$

where

$$\tilde{V}^2 = (1 - 2M/r)(1 + \tilde{L}^2/r^2)$$

is the effective potential (refer back to section 2.5). We determine where orbits go as in the Newtonian case: for a given $\tilde{L}$ plot the effective potential. Then, for fixed $\tilde{E}$ the particle can move at the radii determined by where $\tilde{E}^2 > \tilde{V}^2$.

For zero angular momentum (radial motion), the effective potential is just $1-2M/r$ so the particle travels from infinity to the black hole without restriction. Notice that $\tilde{E} > 0$ for any $r > 2M$: the mass energy is always greater in magnitude than the (negative) gravitational potential energy. What is the time taken to fall to $r = 2M$?

$$\frac{dr}{d\tau} = u^r$$

is always finite, so the proper time is finite (from some finite radius). The coordinate time is given by

$$\frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = \frac{u^r}{u^t} = \frac{u^r}{\tilde{E}}(1-2M/r)$$

which when integrated,

$$t \approx \frac{\tilde{E}}{u^r}\int\frac{rdr}{r-2M}$$

gives a logarithmic singularity: infall takes infinite coordinate time.

In the case of large angular momentum, there is a high peak in the effective potential: there are Newtonian bound orbits in the shallow trough, and orbits very close to the black hole which cannot escape.

Circular orbits occur where $u^r = 0$, ie the stationary points of the effective potential. The Newtonian circular orbit is stable, while the orbit close to the black hole is unstable. In detail:

$$\frac{d}{dr}\tilde{V}^2 = 0$$

$$\frac{2M}{r^2}(1+\frac{\tilde{L}^2}{r^2}) - (1-\frac{2M}{r})\frac{2\tilde{L}^2}{r^3} = 0$$

$$Mr^2 - \tilde{L}^2 r + 3M\tilde{L}^2 = 0$$

$$r = \frac{\tilde{L}^2 \pm \sqrt{\tilde{L}^4 - 12M^2\tilde{L}^2}}{2M} = \frac{\tilde{L}^2}{2M^2}(1 \pm \sqrt{1-12M^2/\tilde{L}^2})$$

The two solutions are combined when the argument of the square root is zero:

$$\tilde{L}^2 = 12M^2$$

for which

$$r = 6M$$

This is the last stable orbit. For smaller angular momenta, there are no circular orbits.

We can find the angular momentum in terms of the radius:

$$\tilde{L}^2 = \frac{Mr^2}{r-3M}$$

which shows that there are no circular orbits (even unstable ones) with $r < 3M$. Thus

$$\begin{aligned}\tilde{E}^2 &= \tilde{V}^2 = (1-2M/r)(1+\tilde{L}^2/r^2) \\ &= (1-2M/r)(1+M/(r-3M)) = \frac{1}{r}\frac{(r-2M)^2}{r-3M}\end{aligned}$$

The "angular velocity at infinity",

$$\omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \frac{\tilde{L}}{r^2}\frac{(1-2M/r)}{\tilde{E}}$$

$$\omega^2 = \frac{M}{r^2(r-3M)}\frac{r-3M}{r} = \frac{M}{r^3}$$

This is just Kepler's third law (only for circular orbits). The period (in coordinate time) is

$$T_\phi = 2\pi/\omega = 2\pi\sqrt{r^3/M}$$

We can calculate the perihelion precession for slightly perturbed circular orbits. For small oscillations about the minimum in the effective potential, we use

$$\begin{aligned}\frac{d^2\tilde{V}^2}{dr^2} &= \frac{-4Mr^2+6\tilde{L}^2r-24M\tilde{L}^2}{r^5} \\ &= \frac{2Mr-12M^2}{r^3(r-3M)} \sim \frac{2M}{r^3}\end{aligned}$$

substituting the value of $\tilde{L}^2$ and making a Newtonian limit for comparison. Recall the NR harmonic oscillator:

$$v^2 = 2\frac{E}{m} - \frac{kx^2}{m}$$

with period

$$2\pi\sqrt{m/k}$$

Thus the period of oscillations in the radius is

$$\Delta\tau = 2\pi\sqrt{\frac{r^3(r-3M)}{Mr-6M^2}} \sim 2\pi\sqrt{r^3/M}$$

Note that in the Newtonian case, it is just the period of the orbit, so there is no precession. In the relativistic case we need to convert this to coordinate time

$$\begin{aligned}T_r &= u^t\Delta\tau = \tilde{E}(1-2M/r)^{-1}\Delta\tau = \sqrt{\frac{r}{r-3M}}\Delta\tau \\ &= \frac{2\pi r^2}{\sqrt{Mr-6M^2}} = \frac{T_\phi}{\sqrt{1-6M/r}}\end{aligned}$$

Example: a massive particle orbits a black hole in the unstable orbit at $r = 7M/2$. It is perturbed so that it escapes: what velocity and impact parameter does it have at infinity?

For circular orbits we have

$$\tilde{L}^2 = \frac{Mr^2}{r-3M} = 49M^2/2$$

$$\tilde{E}^2 = \frac{1}{r}\frac{(r-2M)^2}{r-3M} = 9/7$$

The reduced energy $\tilde{E}$ is simply γ at infinity, thus we have $u = \sqrt{2}/3$ and $\gamma u = \sqrt{2/7}$. The reduced angular momentum $\tilde{L}$ is $\gamma u b$ where b is the impact parameter. Thus we have

$$b = \frac{\tilde{L}}{\gamma u} = \frac{7M/\sqrt{2}}{\sqrt{2/7}} = \frac{M7\sqrt{7}}{2}$$

Example: a particle in the stable circular orbit at $r = 8M$ is perturbed so that it is no longer quite circular. Draw the orbit.

The oscillations in r have period $T/\sqrt{1-6M/r} = 2T$. Thus the particle returns to its initial radius after orbiting the central mass twice (diagram).

Recall that another of the classical tests of GR was the bending of light in the gravitational field of the sun. We now turn to massless particles. We now avoid dividing by the mass:

$$E = (1 - 2M/r)p^t$$

$$L = r^2 p^\phi$$

$$0 = (1 - 2M/r)(p^t)^2 - (1 - 2M/r)^{-1}(p^r)^2 - r^2(p^\phi)^2$$

which becomes

$$(p^r)^2 = E^2 - (1 - 2M/r)L^2/r^2$$

These are momenta not velocities, so they do not correspond to derivatives with respect to proper time (which does not now exist), but we can still for example, write a derivative like

$$\frac{dr}{d\phi} = \frac{p^r}{p^\phi} = \pm\sqrt{r^4E^2/L^2 - r(r-2M)}$$

Thus the shape of the orbit depends only on L/E, which from SR we can compute is simply the impact parameter b.

The effective potential still has a stationary point, corresponding to an unstable circular orbit. We have

$$\frac{dV^2}{dr} = \frac{d}{dr}\left[\frac{L^2}{r^2}(1 - \frac{2M}{r})\right] = -\frac{L^2}{r^3}(2 - 6M/r)$$

which gives the orbit at

$$r = 3M$$

and

$$V^2 = (1 - 2M/r)L^2/r^2 = \frac{L^2}{27M^2}$$

Thus any photon with $b = L/E < 3\sqrt{3}M$ will be captured by the black hole. A photon with just this impact parameter will spiral around, approaching the circular orbit. A photon with slightly more than this impact parameter will spiral for a while, finally escaping in any direction. Thus shining a light towards a black hole gives a "target" effect.

Let us calculate the directions a photon must travel in order to escape from a black hole, relative to a stationary observer. Such an observer will measure the momentum of the photon using an orthonormal basis:

$$p_{\hat{\phi}} = \sqrt{|g^{\phi\phi}|}p_\phi = -L/r$$

This is also

$$p_{\hat{t}} \sin\delta$$

where δ is the angle to the radial direction and

$$p_{\hat{t}} = \sqrt{|g^{tt}|}p_t = (1 - 2M/r)^{-1/2}E$$

is the local energy, which is also the magnitude of the momentum, $\sqrt{p_{\hat{\phi}}^2 + p_{\hat{r}}^2}$. We now have two cases: in order for a photon at $r > 3M$ not to cross the barrier, it can move outwards, or inwards, such that

$$E^2 < V_{\max}^2 = \frac{L^2}{27M^2}$$

thus

$$\sin^2\delta = \frac{L^2}{r^2E^2}(1 - 2M/r) > 27\frac{M^2}{r^2}(1 - 2M/r)$$

If the observer is at $r < 3M$ the photon must move outwards, so that

$$\sin^2\delta < 27\frac{M^2}{r^2}(1 - 2M/r)$$

To calculate the amount of light deflection in the limit of large distances, we need to integrate the equation relating r and ϕ. We introduce $u = M/r$ and $u_b = M/b$, then we have

$$\left(\frac{du}{d\phi}\right)^2 + (1 - 2u)u^2 = u_b^2$$

now differentiating with respect to u we have

$$\frac{d^2u}{d\phi^2} + u = 3u^2$$

for which the right term is small far from the central mass. Expanding in powers of a parameter $\epsilon \ll 1$

$$u = \epsilon u_1 + \epsilon^2 u_2$$

and equating coefficients we find

$$\frac{d^2u_1}{d\phi^2} + u_1 = 0$$

$$\frac{d^2u_2}{d\phi^2} + u_2 = 3u_1^2$$

The solution of the first equation is

$$u_1 = A\cos\phi$$

using the initial condition we find $A = u_b$ and shifting ϕ by a constant if necessary. This is just a straight line. Substituting into the second equation we write

$$\frac{d^2u_2}{d\phi^2} + u_2 = \frac{3A^2}{2}(1 + \cos 2\phi)$$

$$u_2 = \frac{A^2}{2}(3 - \cos 2\phi)$$

Thus we have

$$u = A\cos\phi + \frac{A^2}{2}(3 - \cos 2\phi) + \ldots$$

which has zeros at

$$\pm(\pi/2 + 2A)$$

thus the total deflection of light is $4A = 4M/b$ to this approximation.

Deflection of light can be observed experimentally, not just in the gravitational field of the sun, but also of distant objects, so called "gravitational lensing". Thus the light of a quasar has been observed to bend in the gravitational field of a galaxy on the line of sight, leading to a double image (diagram). On a smaller scale, observations of light curves of stars in nearby galaxies can be used to measure dark massive objects in our own galaxy.

Remark: differentiating the equation in this way is also necessary for numerical work, since the original equation in the form derivative equals squareroot omits the sign information. The second order equation has two initial conditions, and is thus not independent of L and E.

7.4 The event horizon, singularity and extensions

What is the nature of the surface $r = 2M$? This is a subtle question that was not answered properly until 1960. We know from polar coordinates on the plane and spherical coordinates on the sphere that the metric can break down (specifically develop zero or infinite eigenvalues) without a pathology in the underlying manifold. Such a singularity is called a *coordinate singularity*, and is completely resolved by an appropriate change of coordinates.

What kind of more pathological singularities can exist? It is very difficult to give a watertight definition, but two of the more common approaches are as follows:

A *curvature singularity* is where scalar quantities calculated from the curvature diverge. We include here not only the Ricci scalar, but also quantities quadratic in the Riemann tensor, etc. which contain other curvature information than the Ricci scalar.

However, this definition is problematic. Consider a cone: it is flat everywhere except at the apex. Thus we cannot define it to have a curvature singularity - all curvature scalars are zero near the apex. Another example of this is the weak singularities obtained from ordinary matter such as the sequence of spheres in problem 2.3(b): they should not cause a problem since geodesics (and other physical equations) are well behaved.

This leads us to the better concept is that of *geodesic incompleteness* in which geodesics (the worldlines of free particles) cannot be continued beyond a certain point, at finite affine parameter (ie proper time for massive particles). The geodesic incompleteness definition gives a singularity in the conical case, but no singularity in the weak singularity case, as we would like. The definition is not perfect: it is possible for geodesics to be complete, but accelerated observers to be incomplete, etc. but it will suffice here.

For our purposes, we will be content to show that the curvature diverges, as this is a conceptually simpler calculation. For the case of Schwarzchild and other commonly encountered GR spacetimes, this will be equivalent to the geodesic incompleteness criterion.

With these ideas in mind, we turn to the Schwarzchild geometry. We have seen that a freely falling observer reaches $r = 2M$ at finite proper time but infinite coordinate time. Let us assume for the moment that this pathology is due to a poor choice of coordinates. If we compute the curvature components in an orthonormal frame, we find that all of them are CM/r^3 where $C = 0, \pm 1, \pm 2$. Thus the surface $r = 2M$ seems well behaved, from the point of view of an infalling observer. We already noted that a stationary observer has singular properties in this limit.

The fact that $g_{tt} \to 0$ at $r = 2M$ indicates that this surface does not have finite 3-volume: either it is lower dimensional, or null. The fact that $|g_{rr}| \to \infty$ does not indicate an infinite distance: a radial line has distance

$$s = \int \sqrt{|g_{rr}|}dr = \int \frac{dr}{\sqrt{1-2M/r}}$$

which is finite.

Suppose that a geodesic makes it past $r = 2M$: what next? In the original coordinates, we see that t has become a spacelike coordinate, while r is timelike! Thus all timelike and null trajectories are drawn towards $r = 0$, where the metric again breaks down. The curvature components in the orthonormal basis are now infinite; the Ricci tensor is of course zero since this is a vacuum solution, but the quantity

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48M^2}{r^6}$$

is clearly a scalar, and diverges at $r = 0$ so we have a curvature singularity there.

Does a timelike geodesic hit this singularity in finite proper time? We use the effective potential for zero angular momentum as before,

$$(u^r)^2 = \tilde{E}^2 - (1 - 2M/r)$$

Now for $r < 2M$ the RHS is greater than $\tilde{E}^2$, thus the particle must hit the singularity in a proper time given by

$$\tau = \int_0^{2M} \frac{dr}{\sqrt{\tilde{E}^2 + (2M/r - 1)}} < 2M/\tilde{E}$$

For the case $\tilde{E} = 1$ (particle falling from infinity), we have

$$\tau = \int_0^{2M} dr\sqrt{r/2M} = 4M/3$$

We clearly need a well behaved coordinate system to cover $r = 2M$. We proceed as follows: Ignore the angular variables. Write down expressions for the null radial geodesics. This is

$$0 = (1 - 2M/r)(u^t)^2 - (1 - 2M/r)^{-1}(u^r)^2$$

so that

$$\frac{dt}{dr} = \frac{u^t}{u^r} = \pm(1 - 2M/r)^{-1}$$

The solution is

$$t = \pm r_* + const$$

where the "Regge-Wheeler tortoise coordinate" r_* is

$$r_* = r + 2M \ln(r/2M - 1)$$

so that $dr_*/dr = (1 - 2M/r)^{-1}$. Thus we can define null coordinates by

$$u = t - r_*$$

$$v = t + r_*$$

in terms of which, the metric takes the form

$$ds^2 = (1 - 2M/r)dudv$$

where r is now viewed as a function of u and v, obtained as

$$r + 2M \ln(r/2M - 1) = r_* = (v - u)/2$$

We can now rewrite the metric in a nonsingular form, multiplied by functions of u and v,

$$ds^2 = \frac{2Me^{-r/2M}}{r} e^{(v-u)/4M} dudv$$

The singularity is still present here: our coordinates u and v are infinite at $r = 2M$. Now we transform again, reparametrising the null geodesics,

$$U = -e^{-u/4M}$$

$$V = e^{v/4M}$$

to finally obtain a nonsingular metric

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} dUdV$$

To rewrite the metric in terms of a spacelike and a timelike variable, write

$$T = (U + V)/2$$

$$X = (V - U)/2$$

to finally obtain the simplest "good" coordinate system Kruskal-Szekeres:

$$X = (r/2M - 1)^{1/2} e^{r/4M} \cosh(t/4M)$$

$$T = (r/2M - 1)^{1/2} e^{r/4M} \sinh(t/4M)$$

when $r > 2M$ and

$$X = (1 - r/2M)^{1/2} e^{r/4M} \sinh(t/4M)$$

$$T = (1 - r/2M)^{1/2} e^{r/4M} \cosh(t/4M)$$

when $r < 2M$ in terms of which

$$ds^2 = (32M^3/r)e^{-r/2M}(dT^2 - dX^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where

$$(r/2M - 1)e^{r/2M} = X^2 - T^2$$

$$\frac{t}{2M} = \ln(\frac{X + T}{X - T}) = 2\tanh^{-1}(T/X)$$

Note that the coordinate transformation is singular at $r = 2M$ as it must be. Let us draw a spacetime diagram in terms of the spacelike coordinate X and the timelike coordinate T (suppressing θ and ϕ). Null curves are always at 45 degree angles on such a diagram, by construction. This is in contrast to Schwarzchild coordinates (diagram). Constant r corresponds to $X^2 - T^2 = const$ which are hyperbolas. In particular, $r = 0$ corresponds to $T^2 = X^2 + 1$ and $r = 2M$ corresponds to $X = \pm T$: we see this is actually a null surface. Constant t corresponds to $X/T = const$ which are radial lines emanating from the centre of the diagram. We see that all timelike or null curves that go to $r < 2M$ hit the singularity. The surface $r = 2M$ is called the *event horizon* since it marks the boundary of what can or cannot escape from the black hole.

Perhaps the most interesting aspect of the new coordinates is that the spacetime is extended: not only do we have region I (external) and region II (future), but also region (IV) past and region III (elsewhere). Thus the Schwarzchild coordinates are only a chart on a limited part of the full spacetime. Unfortunately it is impossible to get to region III: it can only be achieved by spacelike curves.

If a star undergoes gravitational collapse, the spacetime will initially be well behaved, but eventually an event horizon will form, and the matter of the star fall onto a newly created singularity (diagram). This excludes regions III and IV of the fully extended system. The region outside the star is guaranteed to be Schwarzchild, since this is the only spherically symmetric solution to the Einstein vacuum equations. To an outside observer, the star will appear to be "frozen" at the point $r = 2M$, with a few highly redshifted photons emitted (also some that orbit at $r = 3M$). However, a later observer going near the event horizon cannot explore the surface of the star - it is in the past, according to the spacetime diagram.

One obvious question at this point is whether singularities are inevitable, or a consequence of making special assumptions such as spherical symmetry. It turns out there are very general theorems, forcing singularities whenever there is an event horizon, although not determining that all timelike/null curves must pass through such a singularity (eg Reissner-Nordstrom). Whether singularities can exist without an event horizon is called the "cosmic censorship conjecture" and is unknown. A "naked singularity" would of course be very interesting to study, although quantum effects might make it unobservable.

It is quite possible that in a full quantum theory of gravity, the singularities are removed (quantum mechanics does quite a good job at removing singularities from NR mechanics), but this is also conjecture.

8 Experimental tests of GR

The equivalence principle

The equivalence principle is comprised of a number of testable ideas. It is very general, and common to metric theories of gravity (ie the idea that spacetime is a curved manifold). Some tests that have been carried out include checking that the acceleration of different bodies in a gravitational field is the same (to 10^{-12}), SR Lorentz invariance (tested daily in particle accelerators), and measurements to ensure that the speed of light, mass and nuclear energy levels (to 10^{-22}!) are independent of direction.

SR tests include the Michelson Morley experiment (isotropy of the speed of light), the observation of time dilation (longer half lives for moving unstable particles), and conservation of 4-momentum as defined in SR.

Weak field, slowly varying

PPN formalism Next, we consider slowly varying weak field effects. There are alternative (more complicated) theories of gravity which differ for some of these effects. Since all gravitational theories must reproduce Newtonian gravity, it is possible to enumerate all the possible deviations of "post-Newtonian" quantities using a limited number of parameters, without specifying the full theory. This approach is called the Parametrised Post-Newtonian (PPN) formalism.

Classical solar system tests: Light bending: we calculated this previously. It was first observed in a solar eclipse in 1919, being the first successful prediction of the new theory. Using Very Long Baseline Interferometry, it is possible to pinpoint astronomical sources to milliseconds of arc, leading to light bending checks to order 10^{-3}. This corresponds to the PPN parameter γ, which measures the amount of space-time curvature associated with a given mass. Light bending has also been observed leading to gravitational lensing in astronomical contexts.

Light delay: the time taken for a light beam to travel through a gravitational field is increased from the Newtonian value. This is a consequence of the geodesics in the Schwarzchild metric that we studied, but the derivation is more complicated and has been omitted. This has been tested to about 10^{-2}. This is also associated with the PPN parameter γ.

Gravitational redshift: this is a consequence of the equivalence principle. This is tested by a number of experiments to an accuracy of 10^{-4}. In addition, the GPS navigation system depends on this value being correct.

Perihelion precession: we calculated this for circular orbits; of course this can be extended to weak field elliptical orbits. In the case of Mercury, the precession is 5600 seconds per century, mostly due to non-GR preturbations from Jupiter etc. The GR term is 43 seconds. This was one of the few experimental results available to Einstein while he was constructing the theory. This effect depends on the PPN parameter β, which describes the amount of nonlinearity in superposed gravitational fields. This has been measured to within 10^{-3}.

Other effects: More recently investigated effects include geodetic precession, in which a gyroscope in orbit about a mass should precess, and gravitomagnetic precession, in which additional precession is obtained from the mass itself spinning. Again PPN parameters are restricted to GR values to at least 10^{-3}. Current measurements are being carried out by Gravity Probe B (final report due Dec 2007).

Weak field: rapidly varying

We can also have quickly varying weak field effects - these are called gravitational waves: too weak for direct detection so far in terrestrial laboratories, but see below. The current laser interfermeter LIGO and VIRGO detectors (4km in size) are on the limit of detecting probable gravitational waves, with a sensitivity of about 10^{-21}. A space-based version, LISA, (5 million km!) may become operational in about 2015, and be more sensitive still.

Compact objects

Neutron stars Now we turn to astrophysical evidence. Pulsars emit regular pulses of radio waves (and sometimes light and X-rays) once in a few seconds up to several hundred times a second. Because nothing as large as the Earth can pulsate or rotate this fast, they must be extremely small and dense. From a theoretical point of view, they are best described as rapidly rotating neutron stars, with radii about 10km and masses up to about 2 solar masses. Because the pulses are very precisely timed (in fact equal to our best atomic clocks), pulsars in a binary system (orbiting another star) have their motion very precisely measured using the time delay effects (ie Doppler shift for motion and gravitational time delay). In 1974, the first "binary pulsar" PSR1913+16 was discovered, consisting of two neutron stars in a close elliptical orbit with a period of 7.75 hours. This has allowed tests of the energy loss due to gravitational radiation (ruling out a number of theories of gravity which predict more radiation than GR due to strong field effects), and strong field perihelion precession. Several other binary pulsars have been discovered since. In March 2005, the first system in which emissions from both pulsars can be detected was discovered; this should provide even more stringent tests.

Black holes Black hole binaries have also been observed, at least a very small star of mass greater than any theory will permit for a neutron star. Both neutron stars and black holes can have accretion disks, ie matter from the other (larger) star in orbit around the star. At about $r = 6M$ it spirals in as predicted by GR; in the case of the neutron star it can be observed hitting the surface at very high velocity. For both neutron stars and black holes, relativistic collisions in orbiting matter leads to very high temperatures and X-ray emission.

Larger mass black holes with accretion disks have been observed in the centres of star clusters and galaxies. They emit highly Doppler shifted radiation from accreting material, and sometimes jets along the axes of rotation. Our own galaxy has a 3.6 million solar mass black hole at the centre which stars orbit with periods as low as 14 years. In some galaxies, the central black hole is absorbing so much matter that it radiates more strongly than its galaxy; this is the most likely explanation for "active galactic nucleii", and, at larger distances and earlier times, quasars.

Cosmology

GR has detailed predictions for the Universe as a whole - cosmology. This is covered in the parallel unit: relativistic cosmology; see also the notes for the postgraduate lectures I gave on cosmology. Unfortunately, the uncertainties in the rate of expansion and the mass density have made it difficult to verify its predictions; recent observations are just starting to connect surveys of galaxies with that of inhomogeneities in the cosmic background radiation, the red shifted black body radiation arising from the big bang. Current evidence is for an acclerated expansion, which requires a cosmological constant, or perhaps a "dark energy" field with bizarre properties. These results (from WMAP and other observations) also give the age of the Universe as 13.7 billion years, and the content of the energy density as about 4% atoms, 22% dark matter and 74% dark energy. So what we know is far less than what we don't know.

