General relativity problem sheet 2

1. Which of the following are valid equations using the Einstein Summation Convention? For those that are, write them explicitly as one or more equations in terms of individual components. Assume three dimensions.
$$A_i = B_{ji}C_j$$
$$A_iB_j + A_jB_k + A_kB_i = 0$$
$$A_{ii} = 6$$
$$A_i + B_{ij}A_j + C_{kik} = 0$$
$$A_i = B_{ij}C_jD_j$$

2. A matrix M is symmetric if $M_{ij} = M_{ji}$ and antisymmetric if $M_{ij} = -M_{ji}$. Show that for a symmetric matrix S and an antisymmetric matrix A, the trace of AS is zero:
$$A_{ij}S_{ji} = 0$$

3. A hollow sphere has density ρ, inner radius a and outer radius b. Find the gravitational field in the region $r < a$. Suppose now that the sphere were invisible. Could an observer at the centre deduce its existence without leaving the region $r < a$?

4. *(a) Compute the gradient of the gravitational field $\partial g_i/\partial x_j$ (a nine component object) corresponding to a sphere of density ρ and radius R centred at the origin.
**(b) Find a mass distribution $\rho(x, y, z)$ on a bounded domain, that is, zero whenever $x^2 + y^2 + z^2 > R^2$ for some positive constant R; uniformly bounded, that is, $|\rho(x, y, z)| < C$ for some positive constant C independent of position; and for which at least one component of the gradient of the gravitational field is infinite at some point.

5. * Is the rotation group (excluding reflections) abelian (ie commutative) in two dimensions? In three dimensions?

6. Find Lagrangians, conjugate momenta and Hamiltonians for the following situations:
*(a) Kepler problem in three dimensions in spherical coordinates, $x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi, z = r\cos\theta$.
**(b) Double pendulum in a gravitational field, that is, fixed pivot O, light rigid rod OA of unit length at an angle α to the vertical, mass m at A, light rigid rod AB of unit length at an angle β to the vertical, mass m at B, constant gravitational field g downwards.