General relativity problem sheet 3

1. State Kepler's second law, and give its physical significance. Do you expect a version of it to hold in general relativity?

2. * Show (in Newtonian gravity) that a particle with velocity $\sqrt{2GM/R}$ a distance R from a point mass M will escape to infinity, irrespective of its initial direction, unless it is moving directly towards the mass. Hint: use effective potentials.

3. State the relativity principle. How do the Galilean and Einstein formulations differ?

4. * Show that the equation $\nabla^2\Phi = 4\pi G\rho$ is not invariant under the Lorentz group (use a single Lorentz transformation as a counterexample); what subgroup leaves it invariant?

5. ** The barn paradox: Farmer Alice has a barn of length L_B and a tractor of length $L_T > L_B > L_T/\gamma$, where γ is calculated from the maximum speed of the tractor. She argues that due to length contraction, she can slam the door shut as the tractor enters the barn and it will fit inside. Bob the tractor driver is not so sure. Consider both reference frames and decide whether the door closes before the front of the tractor hits the rear wall of the barn.

6. A particle (in special relativity) moves in uniform circular motion, that is,
$$\vec{x} = (t, r\cos\omega t, r\sin\omega t, 0)$$
(a) Write down its worldline according to an observer moving with velocity v along the y-axis. You will need to use the old time t as a parameter. Hint: this follows directly from the Lorentz transformation.
(b) If the particle at rest decays with half-life $\tau_{1/2}$, what is its observed half-life?
* (c) Show that the proper acceleration α is given by
$$\alpha = \frac{r\omega^2}{1 - r^2\omega^2}$$

7. * The Doppler effect gives the variation of frequency of a wave in the case of moving observers or emitters, for example the lowering of pitch of a siren as the vehicle moves past. Light waves have no medium, so all that matters is the relative motion of the observer and emitter. Use $E = hf$ for photons, and the relativistic dot product to show that a photon observed at frequency f moving in the x direction was emitted at frequency $f\gamma(u)(1 - u_x)$ by a light source with velocity $\mathbf{u}$.