General relativity problem sheet 4

1. Show that the dot product of 4-vectors satisfies

   Linearity: $\vec{a} \cdot (\alpha\vec{b} + \beta\vec{c}) = \alpha\vec{a} \cdot \vec{b} + \beta\vec{a} \cdot \vec{c}$

   Commutativity: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

   The product rule: $\partial_t(\vec{a} \cdot \vec{b}) = \partial_t\vec{a} \cdot \vec{b} + \vec{a} \cdot \partial_t\vec{b}$

2. * In Compton scattering, a photon (of zero mass) of energy E strikes a stationary electron (of mass m) and is scattered by an angle θ. The wavelength of a photon is given by $\lambda = hc/E$. Show that the final wavelength λ' is given by

   $$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

   Hint: we don't want to know about the final velocity of the electron. Put its 4-momentum on one side of the equation and square both sides.

3. * If nonzero vectors $\vec{u}$ and $\vec{v}$ are orthogonal ($\vec{u} \cdot \vec{v} = 0$) and $\vec{u}$ is timelike, show that $\vec{v}$ is spacelike. Give an example to show that the converse does not hold.

4. If new basis vectors are given by $\vec{e}_{1'} = \vec{e}_1$, $\vec{e}_{2'} = \vec{e}_1 + \vec{e}_2$, find the new dual basis covectors $\tilde{\omega}^{1'}$ and $\tilde{\omega}^{2'}$ in terms of $\tilde{\omega}^1$ and $\tilde{\omega}^2$.

5. * For any (fixed) vector $\vec{x} \in \mathcal{V}$ we can define its dual, the one-form $\tilde{x} : \mathcal{V} \to \mathbb{R}$ by $\tilde{x}(\vec{y}) = g(\vec{x}, \vec{y})$ where g is any $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. (g may for instance represent the metric tensor, but this is not necessary). Show for the components x^μ of $\vec{x}$ with respect to the basis vectors $\vec{e}_\mu$ and for the components x_ν of $\tilde{x}$ with respect to the dual basis $\tilde{\omega}^\nu$ (as defined in lectures) we have: $x^\mu = \tilde{\omega}^\mu(\vec{x})$, $g(\vec{x}, \vec{y}) = x_\mu y^\mu$, and $x_\mu = g_{\nu\mu} x^\nu$. Expand the dual $\tilde{e}_\mu$ of the basis vector $\vec{e}_\mu$ in the dual basis $\tilde{\omega}^\nu$.

6. Show that for any given basis $\vec{e}_\mu$ of $\mathcal{V}$, a tensor T of type $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ can be expanded as

   $$T = T^{\alpha\beta}{}_{\gamma\delta\epsilon} \vec{e}_\alpha \otimes \vec{e}_\beta \otimes \tilde{\omega}^\gamma \otimes \tilde{\omega}^\delta \otimes \tilde{\omega}^\epsilon$$

   where $\tilde{\omega}^\nu$ is the dual basis. How can the components $T^{\alpha\beta}{}_{\gamma\delta\epsilon}$ be calculated?

7. Show that $\nabla \times \nabla F = 0$ for any 3D scalar field F. Hint: use the antisymmetric tensor ϵ_{ijk}.