General relativity problem sheet 5

1. (a) If $A_\alpha = B_{\alpha\beta} C^\beta$, A_α transforms under change of basis in SR as a 1-form and C^β transforms as a vector, show that $B_{\alpha\beta}$ transforms as a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor.
   (b) If $D^{\alpha\beta}{}_\gamma$ transforms as a $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ tensor, show that $D^{\alpha\beta}{}_\alpha$ transforms as a vector.

2. Show in SR that the comma (partial derivative) operator is consistent with raising/lowering an index using the metric. Specifically, if $V_\alpha = g_{\alpha\beta} V^\beta$ show that $V_{\alpha,\gamma} = g_{\alpha\beta} V^\beta{}_{,\gamma}$.

3. (a) Calculate the stress-energy tensor corresponding to a rod of mass density ρ, tension F and cross-sectional area A. What is the constraint on these parameters if the stress-energy must satisfy the "weak energy condition" that $T^{00} > 0$ for all observers?
   (b) ** Calculate the stress-energy tensor corresponding to a gas of n noninteracting particles per unit volume, with rest mass m, and moving with speed v in all directions.

4. Write down the basis vectors $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_\phi$ of spherical coordinates in terms of the Cartesian coordinate basis. Hence find the metric components $g_{\mu\nu}$ in spherical coordinates [Hint: compare with problem 2.6(a)] and calculate the proper volume element.

5. * Write down the orthonormal basis vectors $\vec{e}_{\hat{r}}$ and $\vec{e}_{\hat{\theta}}$ of 2D polar coordinates in terms of the coordinate basis $\vec{e}_r = \vec{\partial/\partial r}$ and $\vec{e}_\theta = \vec{\partial/\partial\theta}$. Regarding these as differential operators (a modern way to think of vectors), compute their commutator, ie
$$\vec{e}_{\hat{r}} \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\theta}} \vec{e}_{\hat{r}}$$
   acting on some scalar field $f(\vec{x})$.

6. * Which of the following are manifolds? For each manifold, remove any singular points if necessary and give a suitable metric: The configuration space (α, β) of the double pendulum in problem 2.6(b). The group of permutations of n objects. The subset of $\mathbb{R}^2$ satisfying
$$xy(x^2 + y^2 - 1) = 0$$
   The set of 4-momenta $\vec{p}$ satisfying $\vec{p} \cdot \vec{p} = m^2$ for some fixed m, and $p^0 > 0$.

7. * Consider the sphere $x^2 + y^2 + z^2 = 1$. Define a chart ψ that gives a parametrisation of an open subset (which one?) of the sphere in terms of the polar coordinates $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$, $z = \cos\theta$.