General relativity problem sheet 6

1. Find metrics for the following surfaces embedded in $\mathbb{R}^3$:
   (a) Torus $(x, y, z) = (\cos\theta(a + b\cos\phi), \sin\theta(a + b\cos\phi), b\sin\phi)$ for $0 \le \theta, \phi < 2\pi$. $a > b$ are constants.
   (b) Cone $(x, y, z) = (r\cos\theta, r\sin\theta, ar)$ for $r > 0$ and $0 \le \theta < 2\pi$. $a > 0$ is a constant.

2. $*$ A manifold is *conformally flat* if the metric takes the form $g_{\mu\nu} = f(x^\mu)\delta_{\mu\nu}$ (more generally we allow some minus signs, as in SR) in some coordinate system, where f is a scalar field. Such a coordinate system preserves angles but not lengths on the manifold. Find such a coordinate system (ψ, ϕ) (called "Mercator's projection") for the 2-sphere

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

   by writing θ in terms of a new variable ψ. Show that a general manifold in three or more dimensions is not conformally flat.

3. $*$ Show that a manifold with constant metric

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

   is actually Minkowski space in disguise. Hint: this is a "cyclic" or "circulant" matrix, so its eigenvectors are of the form $(1, z, z^2, z^3)$ where z is a fourth root of unity.

4. Calculate the connection coefficients $\Gamma^\alpha_{\beta\gamma}$ for 3D spherical coordinates.

5. Derive the following useful formulae that allow the computation of covariant derivatives without explicitly using the connection coefficients (recalling the bracket notation for symmetry and antisymmetry):
   (a) Lie derivative of a vector:

$$[\mathcal{L}_{\vec{u}}(\vec{v})]^\alpha \equiv u^\beta v^\alpha{}_{;\beta} - v^\beta u^\alpha{}_{;\beta} = u^\beta v^\alpha{}_{,\beta} - v^\beta u^\alpha{}_{,\beta}$$

   (b) Exterior derivative of a p-form:

$$[\tilde{d}\rho]_{\alpha\beta\ldots} \equiv \tilde{\nabla}_{[\alpha}\rho_{\beta\ldots]} = \partial_{[\alpha}\rho_{\beta\ldots]}$$

   $**$ (c) Divergence of a vector or *antisymmetric* $\begin{pmatrix} p \\ 0 \end{pmatrix}$ tensor:

$$F^{\alpha\beta\ldots}{}_{;\alpha} = \frac{1}{\sqrt{|g|}}(\sqrt{|g|}F^{\alpha\beta\ldots})_{,\alpha}$$

   where $g = \det(g_{\mu\nu})$ and you may assume the following formula for the derivative of a determinant:

$$(\ln|\det A|)_{,\alpha} = \mathrm{tr}(A^{-1}A_{,\alpha})$$