General relativity problem sheet 8

1. For the linearised theory, show that
(a)
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$
(b)
$$R_{\alpha\beta\mu\nu} = \frac{1}{2}(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu})$$
and hence
(c)
$$G_{\alpha\beta} = -\frac{1}{2}(\bar{h}_{\alpha\beta,\mu}{}^{\mu} + \eta_{\alpha\beta}\bar{h}_{\mu\nu}{}^{,\mu\nu} - \bar{h}_{\alpha\mu,\beta}{}^{\mu} - \bar{h}_{\beta\mu,\alpha}{}^{\mu})$$

2. Also for the linearised theory, show that if the metric does not depend on the variable θ (in coordinates (t, r, θ, z)), the corresponding conserved momentum p_θ is, to leading order, minus the SR angular momentum $mr^2\gamma d\theta/dt$

3. An object moves in a circular orbit at Schwarzchild radius R around a spherically symmetric mass M. Show that proper time τ is related to coordinate time t by $\tau/t = \sqrt{1 - 3M/R}$. Hint: use the relativistic Kepler's third law derived in the lecture notes.

4. * A particle is in a circular orbit around a black hole. It is perturbed so that the angular momentum is the same, but the energy is slightly increased so there is a small velocity component outwards. Describe and sketch the resulting behaviour, for initial radii $3M$, $4M$, $5M$, $6M$ and $7M$. Hint: you need to consider both the stability of the circular orbit and whether the particle has sufficient energy to escape to infinity.

5. Which of the following properties are indicative of a curvature singularity? (i) the determinant of the metric is zero, (ii) one component of the Riemann tensor diverges, (iii) spacecraft are torn apart by tidal forces, (iv) not all geodesics can be continued to infinite values of the affine parameter, (v) the curvature scalar diverges.

6. ** Consider the Rindler metric
$$ds^2 = x^2dt^2 - dx^2$$
defined for $x > 0$. Discuss the geometry of this two-dimensional spacetime (geodesics, singularities, coordinate extensions, curvature) by analogy with our discussion of the Schwarzchild geometry.