General relativity solution sheet 1

1. Einstein did the hard work (until June 1915) getting an almost correct theory, and discussed it with Hilbert shortly before both men submitted papers with the correct equations. Hilbert's paper was sent five days earlier than Einstein's, and contained some different insights. I would still give Einstein most of the credit, but it goes to show how complicated priority claims can be in scientific research.

2. Einstein cited "Mach's principle" that inertia is due to distant matter in the Universe, so that in an empty Universe with no distant stars, it might be impossible to distinguish between a fixed and a rotating sphere. While the latter example is not a prediction of general relativity, it is clear that the curvature of spacetime by matter is a related concept. This question is complicated by the lack of an explicit principle stated in Mach's own works, and Mach's own negative views on relativity. Mach died before general relativity was well known, let alone experimentally verified.

3. Use the gravitational time dilation factor of $1 + gd/c^2$ where the acceleration of gravity is $g = 10ms^{-1}$, $d = 2m$, and $c = 3 \times 10^8 ms^{-1}$. Thus a clock at your head runs $1 + 2 \times 10^{-16}$ faster than one at your feet. Over a time of, say, 15 years (time spent upright!), or $5 \times 10^8 s$, the time difference comes to approximately $10^{-7}s$.

4. (a) Moving a particle from the pole to the equator (or vice versa) requires no energy: if it did, particles would move spontaneously in one direction or the other. Converting a particle into a photon shows that the photons have the same frequency, hence there is no time dilation.
(b) The velocity of the equator is $v = 2\pi \times 6378km/24hr = 464ms^{-1}$ The special relativistic effect is thus $\gamma = (1 - v^2/c^2)^{-1/2} = 1 + 1.195 \times 10^{-12}$. The general relativistic effect must exactly cancel this, ie $\delta\Phi/c^2 = 1.195 \times 10^{-12}$, $\delta\Phi = 107600m^2s^{-2}$. If we were to use the (false) assumption that the potential is the same as a sphere, we would have calculated $gd = 9.8 * 21000 = 205800m^2s^{-2}$ which is (within errors) twice the correct answer. Note that we have solved this problem without calculating the gravitational potential of a spheroid.

5. The faces of a cube are clearly flat, and the edge connecting two adjacent faces can be opened out to a flat space, however the corners have intrinsic curvature: the triangle formed by joining the centres of the three faces surrounding a corner to each other has angles totalling $3\pi/2$. Similarly the edges of a cylinder have intrinsic curvature, but the faces are flat.

6. We have
$$[G] = M^{-1}L^3T^{-2}$$
and
$$[c] = LT^{-1}$$

while power has dimensions

$$[P] = ML^2T^{-3}$$

Clearly we must have G^{-1} to give the right power of mass, and then we see c^5/G has the right dimensions. Its value in SI units is $(3 \times 10^8)^5/(6.67 \times 10^{-11}) = 3.64 \times 10^{52} W$. This should be compared with the brightest astronomical objects, quasars, with a luminosity (ie power output) of around $10^{40} W$.

The power c^5/G is actually a rough upper limit of the power emitted by any object: In time t, energy $c^5 t/G$ is emitted, within a radius $R = ct$ (since nothing can travel faster than light). The energy corresponds to a mass $M = c^3 t/G$ with escape velocity $\sqrt{2GM/R} = \sqrt{2}c$ of order the speed of light. That is, anything trying to emit this much energy must collapse on itself. Of course the escape velocity is Newtonian, but a full relativistic analysis (later in the course) leads to the same conclusion.