General relativity solution sheet 3

1. Kepler's second law states that the planets move such that the area per unit time that they subtend at the sun is constant. It is a direct consequence of angular momentum conservation, which in turn follows from rotational invariance, ie the Lagrangian does not depend on the azimuthal angle (ϕ in 3D spherical coordinates, θ in 2D polars). This rotational invariance also holds in strong gravitational fields, so a version of Kepler's second law applies in GR.

2. The energy is
$$E = \frac{1}{2}mv^2 - \frac{GmM}{R} = 0$$
Using
$$\tilde{E} = \dot{r}^2/2 + \tilde{V}$$
where $\tilde{V} = \tilde{L}^2/2r^2 - GM/r$ is the effective potential, the equation $\dot{r} = 0$ has one solution in r, ie
$$r_{min} = \tilde{L}^2/2GM$$
and the positivity of $\dot{r}^2$ implies that $r \geq r_{min}$ always. Thus a particle initially moving away from the mass continues to do so, while one moving towards the mass gets reflected at r_{min}, and then always moves away. The only exception is if $\tilde{L} = 0$, in which case a particle moving straight towards the mass will hit it in finite time. Remark: the fact that the particle escapes independent of its initial direction does not hold in GR. Close to the horizon of a black hole, a particle must move almost directly outwards in order to escape.

3. The relativity principle states that the laws of physics are the same in all inertial frames of reference. These are frames in which free particles move with constant velocity, and different frames correspond to observers with constant relative velocity. The Galilean theory also assumes absolute time $t' = t$, while the Einstein theory assumes that the speed of light c is the same in all inertial frames.

4. Let us take the Lorentz transformation corresponding to standard configuration. We have
$$\frac{\partial}{\partial x'} = \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} + \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} = \gamma\partial_x + v\gamma\partial_t$$
so the Laplacian becomes
$$\nabla'^2 = \gamma^2(\partial_x + v\partial_t)^2 + \partial_y^2 + \partial_z^2$$
which now contains time derivatives, so the equation is not invariant. It will remain invariant if $t' = t$, which means the spatial part of the interval (ie $dx^2 + dy^2 + dz^2$) is invariant, leading to rotations and reflections that do not affect the time.

5. In Alice's reference frame the barn is stationary (door and back are vertical in space-time diagram) and the tractor is contracted, and so fits in the barn. In Bob's reference frame the tractor is stationary and much bigger than the contracted barn, so the tractor hits the back wall before the door is shut. There is no contradiction: it turns out that these two events are space-like separated, so their order depends on the choice of reference frame.

The fact that the events are space-like separated means they cannot causally affect each other. Suppose the front of the tractor is stopped by the back of the barn. It cannot send a message faster than light to stop the back of the tractor, or in other words there is no such thing as a rigid body in relativity.

All that remains is to show the separation is in fact spacelike. Let the origin of (both sets of) coordinates be the moment the front of the tractor enters the barn. Working in Alice's (barn) frame, the end of the barn is at $x = L_B$. The front of the tractor is at $x_F = vt$, so the collision with the back is at $(t, x) = (L_B/v, L_B)$. The back of the tractor is at $x_B = vt - L_T/\gamma$ so the door is shut at $(t, x) = (L_T/(\gamma v), 0)$. The relativistic interval between these is $\Delta t^2 - \Delta x^2 = (L_B - L_T/\gamma)^2/v^2 - L_B^2$. Expanding and rearranging this gives $(L_B^2 + L_T^2 - 2L_B L_T \gamma))/(\gamma^2 v^2))$. Now $L_T > L_B > L_T/\gamma$, thus the interval is less than zero, hence spacelike.

6. (a) The Lorentz transformation gives $t' = \gamma(t - vy)$, $y' = \gamma(y - vt)$, $x' = x$, $z' = z$. Thus we have

$$\vec{x'} = (\gamma(t - vr \sin \omega t), r \cos \omega t, \gamma(r \sin \omega t - vt), 0)$$

where t is now a parameter (not time in the new frame).
(b) We have $d\tau = ds = dt/\gamma$, so the observed half-life in the laboratory frame is $t = \tau/\sqrt{1 - \omega^2 r^2} > \tau$ since the velocity is $r\omega$.
(c) We differentiate to find

$$u^\alpha = \frac{dx^\alpha}{d\tau} = \gamma \frac{dx^\alpha}{dt} = \gamma(1, -r\omega \sin \omega t, r\omega \cos \omega t, 0)$$

where $\gamma = 1/\sqrt{1 - r^2\omega^2}$.

$$a^\alpha = \frac{du^\alpha}{d\tau} = \gamma \frac{du^\alpha}{dt} = \gamma^2(0, -r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t, 0)$$

$$\alpha = \sqrt{-\vec{a} \cdot \vec{a}} = \gamma^2 r\omega^2 = \frac{r\omega^2}{1 - r^2\omega^2}$$

This calculation is very straightforward because γ does not depend on t.

7. In the observer's reference frame the photon has 4-momentum $(hf, hf, 0, 0)$ and the source has 4-velocity $(\gamma, \gamma \mathbf{u})$. The dot product of these is $hf\gamma(1 - u_x)$. The dot product is a scalar, and so does not depend on the reference frame. In the source's reference frame its 4-velocity is $(1, 0, 0, 0)$ thus the expression above gives the energy of the photon in that frame, and the frequency is found by dividing by h.

Note that even when $u_x = 0$, ie the motion is transverse, there is a Doppler effect. There is an issue of reciprocity, as in the twin paradox. The transverse Doppler paradox is resolved by noting that while the photon is transverse in the observer's frame in the above calculation, it is not transverse in the source's reference frame due to aberration.