General relativity solution sheet 5

1. (a) We have $A_{\alpha'} = \Lambda^{\gamma}_{\alpha'} A_\gamma$ and $C^{\beta'} = \Lambda^{\beta'}_{\delta} C^\delta$ from the known transformation properties of A and C. Thus we compute:

$$B_{\alpha'\beta'}C^{\beta'} = A_{\alpha'} = \Lambda^{\gamma}_{\alpha'}A_\gamma = \Lambda^{\gamma}_{\alpha'}B_{\gamma\delta}C^\delta = \Lambda^{\gamma}_{\alpha'}B_{\gamma\delta}\Lambda^{\delta}_{\beta'}C^{\beta'}$$

so

$$B_{\alpha'\beta'} = \Lambda^{\gamma}_{\alpha'}\Lambda^{\delta}_{\beta'}B_{\gamma\delta}$$

(b) We have

$$D^{\alpha'\beta'}{}_{\gamma'} = \Lambda^{\alpha'}_{\lambda}\Lambda^{\beta'}_{\mu}\Lambda^{\nu}_{\gamma'}D^{\lambda\mu}{}_{\nu}$$

so

$$D^{\alpha'\beta'}{}_{\alpha'} = \Lambda^{\alpha'}_{\lambda}\Lambda^{\beta'}_{\mu}\Lambda^{\nu}_{\alpha'}D^{\lambda\mu}{}_{\nu} = \delta^{\nu}_{\lambda}\Lambda^{\beta'}_{\mu}D^{\lambda\mu}{}_{\nu} = \Lambda^{\beta'}_{\mu}D^{\lambda\mu}{}_{\lambda}$$

using the fact that $\Lambda^{\alpha'}_{\lambda}$ and $\Lambda^{\nu}_{\alpha'}$ are matrix inverses (since they transform a vector from the unprimed frame to the primed frame and back again).

2. We have

$$V_{\alpha,\gamma} = \left(g_{\alpha\beta}V^\beta\right)_{,\gamma} = g_{\alpha\beta,\gamma}V^\beta + g_{\alpha\beta}V^\beta{}_{,\gamma}$$

but the first term is zero since the metric is constant in SR. In arbitrary coordinates and GR the metric coefficients will depend on position, but the same will hold with the comma replaced by a semicolon ("covariant derivative").

3. (a) The energy density is ρ, and the force acting is tension (a "negative" pressure $-F/A$) along the rod. Otherwise there are no forces or energy fluxes. Thus

$$T = \rho\vec{e}_0 \otimes \vec{e}_0 - \frac{F}{A}\vec{e}_1 \otimes \vec{e}_1$$

assuming the rod is in the x direction.

In the reference frame of an observer with 4-velocity $\vec{u}$, we have

$$T^{00} = T^{\mu\nu}u_\mu u_\nu = \gamma^2\rho - \gamma^2 u_x^2\frac{F}{A}$$

which is most likely to be negative if $u_x \approx 1$, ie to an observer moving with nearly the speed of light along the rod. Thus the weak energy condition implies $\rho > F/A$.

(b) A gas of noninteracting particles with zero velocity has

$$T = \rho\vec{e}_0 \otimes \vec{e}_0 = \rho\vec{u} \otimes \vec{u}$$

where ρ is the mass density in the rest frame (not yet calculated). However, the expression in terms of the 4-velocity $\vec{u}$ is a valid tensor equation, so it is valid in all frames.

We have particles moving in all directions, so the full stress-energy must be

$$T = \rho \int \vec{u} \otimes \vec{u} d\Omega$$

where the integral is over all directions. While apparently a separate integral for each component, symmetry requirements dictate that the off-diagonal terms vanish and the spatial diagonal terms are equal. Thus we find

$$T^{00} = 4\pi\rho\gamma^2$$

$$T^{33} = \rho\gamma^2 \int u_z^2 d\Omega = \rho\gamma^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta (u^2 \cos^2\theta) = \rho\gamma^2 u^2 \frac{4\pi}{3}$$

Now we know that the energy density is given by $\gamma n m$, thus we have $\gamma n m = 4\pi\rho\gamma^2$ and finally

$$T = \gamma n m \quad \text{diag}(1, u^2/3, u^2/3, u^2/3)$$

This is the stress-energy of a perfect fluid, with pressure related to the energy density by $p = T^{00} u^2/3$.

4. We have $\vec{e}_{\mu'} = \Lambda^{\nu}_{\mu'} \vec{e}_{\nu}$ with $\Lambda^{\nu}_{\mu'} = \partial x^{\nu} / \partial x_{\mu'}$.

$$\vec{e}_r = \sin\theta\cos\phi \vec{e}_x + \sin\theta\sin\phi \vec{e}_y + \cos\theta \vec{e}_z$$

$$\vec{e}_\theta = r\cos\theta\cos\phi \vec{e}_x + r\cos\theta\sin\phi \vec{e}_y - r\sin\theta \vec{e}_z$$

$$\vec{e}_\phi = -r\sin\theta\sin\phi \vec{e}_x + r\sin\theta\cos\phi \vec{e}_y$$

Now we have $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$; the calculation is the same as for the kinetic energy in problem 2.1(a), leading to

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

The proper volume element involves $g = \det(g_{\mu\nu}) = r^4 \sin^2\theta$:

$$\sqrt{|g|} dr d\theta d\phi = r^2 \sin\theta dr d\theta d\phi$$

which should look familiar.

5. As above we find $\vec{e}_r = (\cos\theta, \sin\theta)$ and $\vec{e}_\theta = (-r\sin\theta, r\cos\theta)$ in Cartesian coordinates. Thus $\vec{e}_{\hat{r}} = \vec{e}_r$ and $\vec{e}_{\hat{\theta}} = \vec{e}_\theta / r$. We have for the commutator,

$$(\vec{e}_{\hat{r}} \vec{e}_{\hat{\theta}} - \vec{e}_{\hat{\theta}} \vec{e}_{\hat{r}}) f = \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) f - \left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \frac{\partial}{\partial r} f = -\frac{1}{r^2} \frac{\partial f}{\partial \theta}$$

Notice that the commutator is itself a vector (not a second order operator as might be expected).

6. The permutation group has no continuous structure, so it cannot be a manifold. The double pendulum has a metric derived (as in the spherical case) simply from the kinetic energy:

$$g_{\mu\nu} = \begin{pmatrix} 2 & \cos(\alpha-\beta) \\ \cos(\alpha-\beta) & 1 \end{pmatrix}$$

The subset of $\mathbb{R}^2$ consists of a circle together with the two axes. It has singular points wherever these parts cross, in particular, $(-1,0)$, $(0,-1)$, $(0,0)$, $(0,1)$ and $(1,0)$. Along the axes, the metric is given simply by the coordinate, ie $ds^2 = dx^2$ along the x-axis. On the circle, distance is given by $ds^2 = dx^2 + dy^2$ constrained to the circle, ie $d\theta^2$ with $(x,y) = (\cos\theta, \sin\theta)$.

First we must devise coordinates for this class of 4-momenta (the "mass shell"), one possibility being p^1, p^2 and p^3 so that

$$p^\mu = (\sqrt{(p^1)^2 + (p^2)^2 + (p^3)^2 + m^2}, p^1, p^2, p^3)$$

The natural metric is that induced by the ordinary SR metric diag(1,-1,-1,-1), ie $(dp^0)^2 - (dp^1)^2 - (dp^2)^2 - (dp^3)^2$ with

$$dp^0 = \frac{1}{\sqrt{\mathbf{p}^2 + m^2}}(p^1 dp^1 + p^2 dp^2 + p^3 dp^3)$$

Thus we find

$$g_{ij} = -\delta_{ij} + \frac{p_i p_j}{m^2 + \mathbf{p}^2}$$

Spherical coordinates (p, θ, ϕ) are also possible, leading to

$$g_{ij} = \begin{pmatrix} -\frac{m^2}{p^2+m^2} & 0 & 0 \\ 0 & -p^2 & 0 \\ 0 & 0 & -p^2\sin^2\theta \end{pmatrix}$$

A change of sign convention would be natural here.

7. The chart ψ maps an open subset $\mathcal{O}$ of the manifold (represented by (x,y,z)) to an open subset $\mathcal{U}$ of $\mathbb{R}^2$ (represented by (θ,ϕ). Now the usual coordinate ranges $\mathcal{U} = \{(\theta,\phi) : \theta \in (0,\pi), \phi \in [0,2\pi)\}$, while excluding the bad points at the poles, do not correspond to an open set. For example, we could have $\mathcal{U} = \{(\theta,\phi) : \theta \in (0,\pi), \phi \in (0,2\pi)\}$ for which $\mathcal{O} = S^2 - \{(\sqrt{1-z^2}, 0, z) : z \in [-1,1]\}$. Since the map ψ converts (x,y,z) to (θ,ϕ) we need to use the inverse transformation $(\theta = \arctan(\sqrt{x^2+y^2}/z), \phi = \arctan y/x)$ where as usual, the appropriate branches of the arctan function are chosen, corresponding to the signs of x and z, and interpolated when $x = 0$ or $z = 0$.