General relativity solution sheet 8

1. (a) The upper components of the metric correspond to the matrix inverse. Assuming
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$
and
$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$$
we check the result by matrix multiplication. Specifically, expand $g^{\mu\nu}g_{\nu\sigma}$ and show that it is δ^μ_σ to first order. Notice that the perturbation changes sign, ie the upper components $h^{\mu\nu}$ (obtained by raising $h_{\mu\nu}$ with $\eta^{\mu\nu}$) are *not* the perturbations to $g^{\mu\nu}$.

   (b) Using the standard formula for the connection coefficients in terms of the derivatives of the metric, we find
$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\eta^{\alpha\delta}(h_{\delta\beta,\gamma} + h_{\delta\gamma,\beta} - h_{\beta\gamma,\delta})$$
to first order. Then, the expression for the Riemann tensor simplifies because the quadratic terms can be neglected. The remaining terms are
$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + O(h^2) = \frac{1}{2}\eta^{\alpha\epsilon}(h_{\epsilon\beta,\delta\gamma} + h_{\epsilon\delta,\beta\gamma} - h_{\beta\delta,\epsilon\gamma} - h_{\epsilon\beta,\gamma\delta} - h_{\epsilon\gamma,\beta\delta} + h_{\beta\gamma,\epsilon\delta})$$
which leads to the required result by cancelling two terms and lowering the index using $\eta_{\mu\nu}$ (permitted since the tensor is already first order in h).

   (c) This is fairly straightforward index manipulation, keeping in mind that η is used in place of the metric and $h_{\mu\nu}$ is symmetric. Using the definition of the Einstein tensor, we find
$$G_{\alpha\beta} = \frac{1}{2}(h^\mu{}_{\alpha,\beta\mu} + h^\mu{}_{\beta,\alpha\mu} - h_{\alpha\beta,\mu}{}^\mu - \eta_{\alpha\beta}h^{\mu\nu}{}_{,\mu\nu} - h_{,\alpha\beta} + \eta_{\alpha\beta}h_{,}{}^\mu{}_{,\mu})$$
where $h = h^\mu{}_\mu$ is the trace of $h_{\mu\nu}$. Then substitute
$$h_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\bar{h}$$
and
$$\bar{h} = -h$$
noting that $\eta_{\alpha\beta}$ is constant (therefore its derivatives are zero).

2. The linearised theory assumes that $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta = \text{diag}(1, -1, -1, -1)$; we transform to polar coordinates, and find $\tilde{g}_{\mu\nu} = \tilde{\eta}_{\mu\nu} + \tilde{h}_{\mu\nu}$ where $\tilde{h}_{\mu\nu}$ is small (except perhaps for problems near the z-axis). We have $p_\theta = \tilde{\eta}_{\theta\theta}p^\theta$, $\tilde{\eta}_{\theta\theta} = -r^2$ and $p^\theta = md\theta/d\tau = m\gamma d\theta/dt$. To leading order the small $\tilde{h}_{\mu\nu}$ do not contribute.

3. The orbit is described by $x^\mu = (t, R, \pi/2, \omega t)$ where $\omega^2 = M/R^3$ (Kepler's third law) from lecture notes. The proper time is
$$\tau = \int ds = \int \sqrt{g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}}dt = \int \sqrt{1 - \frac{2M}{R} - R^2\omega^2}dt = t\sqrt{1 - \frac{3M}{R}}$$

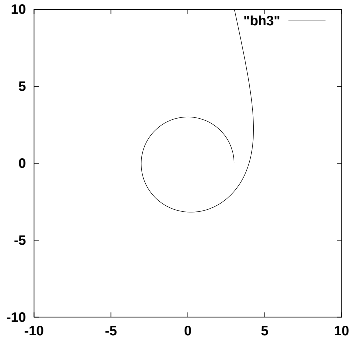
4. We use the values for a circular orbit:
$$\tilde{L}^2 = \frac{Mr^2}{r - 3M}$$

and

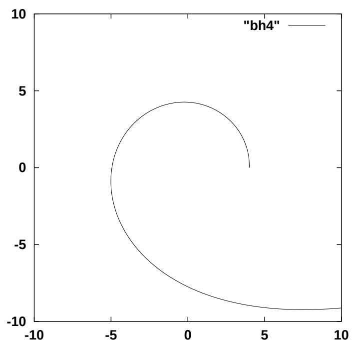
$$\tilde{E}^2 = \frac{1}{r}\frac{(r-2M)^2}{r-3M} + \epsilon$$

which is slightly perturbed by an amount ϵ.

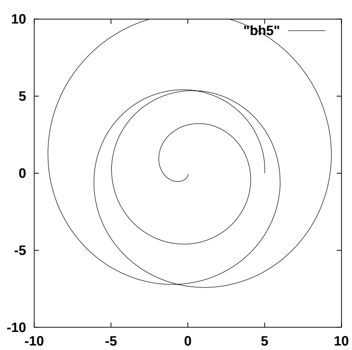
For $r = 3M$ the particle must have zero mass. It escapes with impact parameter $b = L/E = 3\sqrt{3}M$.



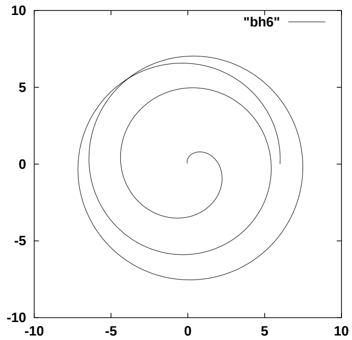
For $r = 4M$ the particle has reduced energy $\tilde{E}$ slightly greater than 1 so it barely escapes to infinity. It has $\tilde{L} = 4$ so its impact parameter will be $b = 4M/u$ for some nonrelativistic velocity $u \ll 1$.



For $r = 5M$ the particle has reduced energy $\tilde{E} = \sqrt{9/10} < 1$ so it cannot escape: it will reach a maximum radius determined by $\tilde{E}^2 = \tilde{V}^2$, that is, $r = 10M$ (it is a cubic equation, but we can divide through by our known solution $r = 5M$). Upon reflection by the effective potential barrier, it will return to $r = 5M$ and pass over it (having slightly more energy than the unstable orbit there), falling into the black hole.



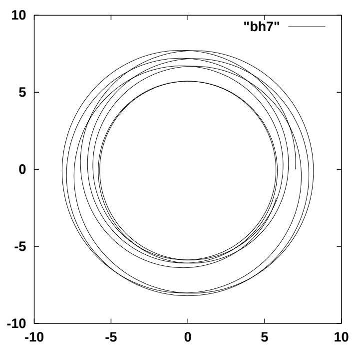
For $r = 6M$ the orbit is marginally unstable, ie the effective potential has a point of inflection. Any perturbation will move outwards slightly before falling into the black hole.



For $r = 7M$ the orbit is stable, so the particle will remain in a stable orbit. The ratio of the oscillation periods in the ϕ and r directions is

$$\frac{T_r}{T_\phi} = \frac{1}{\sqrt{1 - 6M/r}} = \sqrt{7}$$

so the particle will go around about two and a half times between each closest approach.



5. Properties (i) and (ii) could be coordinate singularities, that is, due to a poor choice of coordinate system at a point. (iv) is the geodesic incompleteness definition of a singularity; while this is a useful definition, it is not the same as a curvature singularity since a cone has geodesic incompleteness but no curvature. (iii) denotes a curvature singularity (at least very high curvature) since tidal forces are proportional to curvature, and (v) also denotes a curvature singularity; in both cases the effects are measurable in all coordinate systems, in contrast to (i) and (ii).

6. Let us start by describing the geodesics. We have two conserved quantities, $p_t = g_{tt}p^t = x^2 p^t$ and $m^2 = g_{tt}(p^t)^2 + g_{xx}(p^x)^2 = x^2(p^t)^2 - (p^x)^2$. The null geodesics ($m = 0$) have

$$\frac{dx}{dt} = \frac{p^x}{p^t} = \pm x$$

thus

$$x = e^{\pm t - C}$$

$$t = \pm \ln x + C$$

where C is a constant. We introduce null coordinates

$$u = t - \ln x$$

$$v = t + \ln x$$

so that

$$dt = \frac{du + dv}{2}$$

$$\frac{dx}{x} = d(\ln x) = \frac{dv - du}{2}$$

$$dx = \frac{1}{2} e^{(v-u)/2} (dv - du)$$

$$ds^2 = e^{v-u} du dv$$

Now put $U = -e^{-u}$ and $V = e^v$, then

$$ds^2 = dU dV$$

so with $T = (V + U)/2$ and $X = (V - U)/2$ we find

$$ds^2 = dT^2 - dX^2$$

in other words, 2D Minkowski spacetime! The relationship between the coordinates is

$$x = \sqrt{X^2 - T^2}$$

$$t = \operatorname{arctanh}(T/X)$$

so we have extended a single quadrant (corresponding to x, t coordinates) to the full space (in X, T coordinates). The curvature is clearly zero.