Transmission and Reflection in the Stadium Billiard: Time-dependent asymmetric transport

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The survival probability of the open stadium billiard with one hole on its boundary is well known to decay asymptotically as a power law. We investigate the transmission and reflection survival probabilities for the case of two holes placed asymmetrically. Classically, these distributions are shown to lose their algebraic decay tails depending on the choice of injecting hole therefore exhibiting asymmetric transport. The mechanism behind this is explained while exact expressions are given and confirmed numerically. We propose a model for experimental observation of this effect using semiconductor nano-structures and comment on the relevant quantum time-scales.

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1. Introduction

Billiards [1] are systems in which a particle alternates between motion in a straight line and specular reflections from the walls of its container, while open billiards contain one or more holes through which particles may escape. Billiards demonstrate a broad variety of behaviours including regular, chaotic and mixed phase space dynamics, depending on the geometry, whilst allowing for mathematical treatment of their properties. Billiard models have been increasingly important in both theoretical and experimental physics, for example as models in statistical mechanics such as the Boltzmann Hypothesis [2], number theory and the Riemann Hypothesis [3], in room acoustics [4], atom optics, where ultracold atoms reflect from laser beams [5], optics in dielectric micro-resonators [6] and in quantum chaos when solving the Helmholtz equation with Dirichlet or Neumann boundary conditions [7]. Open billiards are also a useful model for understanding the close correspondence between classical and quantum mechanics [8].

Quantum open billiards were experimentally realized first in flat microwave resonators in the early 90's [9, 10] and later in semiconductor nano-structures such as quantum dots [11, 12]. Experiments perturbing these systems with small magnetic fields exhibit principal quantum interference effects like weak localization, Altshuler-Aronov-Spivak oscillations and conductance fluctuations, all of which semiclassical theory has arguably succeeded to explain using properties of the underlying classical dynamics [12, 13]. Similarly, in microwave resonators, due to their clean, impurity-free geometry and the tunable coupling strength to the various decay channels, predicted phenomena such as resonance trapping have been experimentally observed [14].

Here we investigate the classical transport of a popular example for the above and other experiments, the stadium billiard with two holes on its boundary placed asymmetrically (see Figure 1). Looking at the phase space of this open system, we find that the predominantly chaotic character of the corresponding closed system is non-trivially affected by the positioning of the holes. In particular we find that the transmission and reflection probabilities, when particles are injected from one of the two holes, are qualitatively different at long times depending on the choice of the injecting hole therefore displaying time-dependent asymmetric transport. We give detailed analytical expressions for these distributions and confirm them numerically. Although investigations through random matrix theory (RMT) regarding the variety of symmetric or asymmetric openings in chaotic systems have been performed [15], to the best of our knowledge there has been no analogous analytic prediction or experimental observation of such an asymmetry in the transport. Hence we conclude with a discussion of a possible experimental model with regards to the relevant quantum time scales.

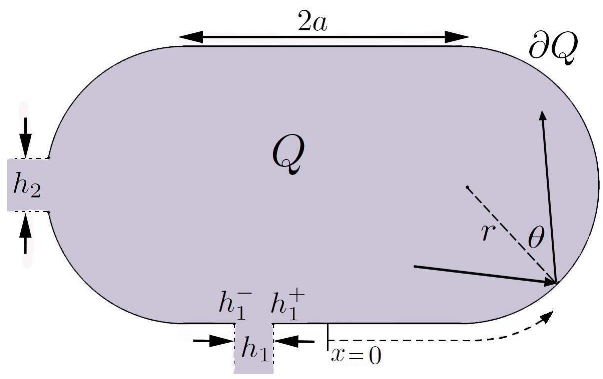

FIG. 1: (Color online) Stadium billiard with two holes H_1 and H_2. The billiard map is parameterized using arc length $0 \leq x < 4a+2\pi r$ and velocity parallel to the boundary $v \sin\theta$ with $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. The hole on the straight segment is such that $-a < h_1^- < h_1^+ < a$.

2. The Stadium and Escape through one hole

The transport problem is closely related to the escape problem for which we also make new observations. The uniform (Liouville) distribution projected onto the billiard boundary has the form $(2|\partial Q|)^{-1}\mathrm{d}x\ \mathrm{d}\sin\theta$ (where $|\partial Q|$ is the perimeter of the billiard while x and θ are defined in Figure 1), and is the most natural choice for an initial distribution of particles. Given such a distribution, the probability $P(t)$ that a particle survives (*i.e.* does not escape through k small holes $H_i \in \partial Q$) in a strongly chaotic billiard up to time t decays exponentially $\sim e^{-\gamma t}$ at long times with the exponent to leading order given by [16]

$$\gamma = \frac{\sum_{i=1}^{k} h_i}{\langle\tau\rangle|\partial Q|}, \tag{1}$$

where $\langle\tau\rangle = \frac{\pi|Q|}{|\partial Q|v}$ is the mean free path for 2D billiards, $h_i = |H_i|$ is the length of each hole, $|Q|$ the area and v the speed of the particles.

The stadium billiard is a chaotic system where the defocusing mechanism guarantees a positive Lyapunov exponent λ (exponential separation rate of nearby trajectories) almost everywhere [17], the exception being a zero-measure family of marginally unstable periodic orbits between the parallel straight segments called 'bouncing ball' orbits. They have been shown to lead to an intermittent, quasi-regular behaviour which effectively causes the closed stadium to display some weaker chaotic properties such as an algebraic decay of correlations [18]. Quantum mechanically they cause scarring [19], the system is not quantum uniquely ergodic [20], an $\hbar$ dependent 'island of stability' appears to surround them [21] and deviations from RMT predictions are observed (especially in the Δ_3-statistics) if not treated appropriately (see [22, 23]).

A small hole of size h_1 placed on the billiard's boundary opens the system and the stadium's survival probability $P(t)$ becomes a useful statistical observable. Due to the intermittency introduced by the bouncing ball orbits, $P(t)$ is found to experience a cross-over from the above exponential decay (1) at short times to an algebraic decay $\sim C/t$ at later times [24]. If the hole is placed on one of the straight segment as in Figure 1, the constant C can be calculated to give [25]

$$C = \frac{(3\ln 3 + 4)\Big((a+h_1^-)^2 + (a-h_1^+)^2\Big)}{4(4a+2\pi r)v} + \mathcal{O}\left(\frac{1}{t}\right), \tag{2}$$

with parameters as defined in Figure 1. We emphasize that this calculation is possible because **the stadium's classical phase space is split by the hole into separate regions occupied by 'fully-chaotic' and 'sticky' orbits, which are responsible for the exponential and algebraic decays respectively**. As an orbit approaches the sticky region in phase space, which surrounds the bouncing ball orbits, it will inevitably escape through H_1 quickly after it obtains an incidence angle $|\theta| < \arctan\left(\frac{h_1}{4r}\right)$. This is a key point that will be discussed further in the two hole case.

We also remark that due to the splitting of the phase space, there is no justification for an intermediate purely exponential decay, as proposed generically for intermittent systems by Altmann *et al.* (see eq. (25) in [26]), but rather a coexistence of exponential and algebraic decay given by:

$$P(t) = \begin{cases} \text{irregular}, & \text{for } t < \hat{t} \\ e^{-\gamma t} + \frac{C}{t}, & \text{for } t > \hat{t}, \end{cases}, \tag{3}$$

where we have neglected terms of order t^{-2} and $\hat{t} \leq \frac{32ar}{h_1}$ can be found as described in [25]. The 'irregular' short-time behaviour is a result of geometry dependent short orbits which become less important if the hole is small. We note that the coefficient of the exponential term in (2) is 1 since for small holes and times greater than $\approx 1/\lambda$, mixing causes the system to forget its initial state and therefore the probability decays as a Poisson process.

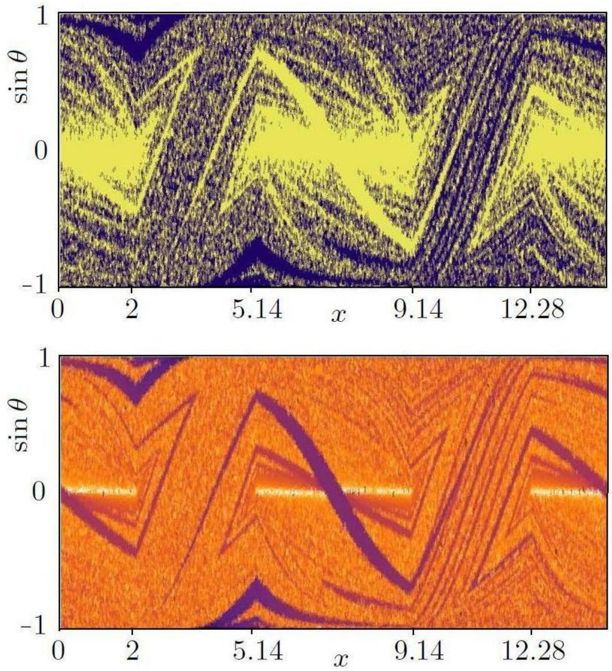

FIG. 2: (Color online) Phase space of open stadium with 2 holes. *Top*: Initial conditions which will escape through hole H_1 are shown in light yellow while those escaping through hole H_2 in dark blue. *Bottom*: Color grading of initial conditions going from purple (dark), to orange, to white corresponding to short, medium and long escape times. ($a = 2$, $r = 1$, $h_i = 0.5$, $h_1^+ = 0.25$).

3. The two hole case: Asymmetric transport

Consider now the case of the stadium with two holes as shown in Figure 1. In Figure 2 we plot in the top panel a picture of the phase space, showing in different colors, the different sets of initial conditions which eventually exit through each hole. The bottom panel shows the time scales of escape as noted in the caption. We notice that the phase space is again separated, as described above, and that the sticky, long-surviving orbits escape only through the hole on the straight segment H_1. Restricting the initial density of particles to one of the holes defines the transport problem and establishes the schematic setup of quantum dots and microwave cavities, where particles/waves are injected through one of the holes and allowed to escape through either, thus creating a direct link with experiment. Looking at the spatial distribution of the final (escape) coordinates (x_f, θ_f) (see Figure 3) we also notice that long surviving orbits entering and subsequently exiting through H_1, unlike in the other possible entry/exit combinations, accumulate on the edges of the hole $x_f = h_1^{\pm} \mp \delta$, $(\delta \ll 1)$ and have small angles θ_f. Note that $(x_f, \theta_f) \to (h_1^{\pm}, 0^{\pm})$ as the time of escape $t_f \to \infty$. This further confirms the splitting of the phase space, but also that the classical *spatial* distribution of exiting particles has a well defined time-dependent character, which only exists in the situation described and plotted in Figure 3.

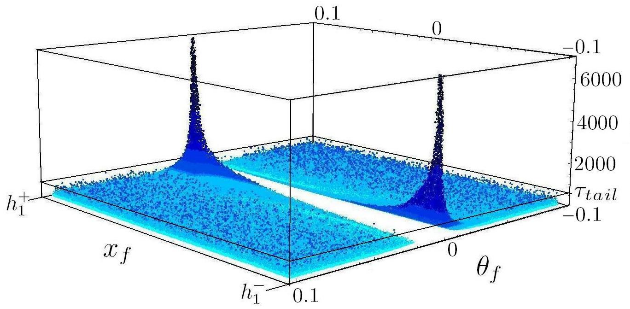

FIG. 3: (Color online) 3D plot of the final (escape) coordinates and time of escape (x_f, θ_f, t_f) for the case of entry and exit through H_1. The color scheme runs from light to dark blue linearly with increasing exit times t_f. Only in this case are the 2 dark spikes observed. ($a = 2$, $r = 1$, $h_i = 0.2$, $h_1^+ = 0.1$). $\tau_{tail} \approx 631.85$ is explained in Figure 4.

In order to quantify our above observations, we define transmission and reflection survival probabilities by $P_i^j(t)$ and $P_i^i(t)$ respectively $(i, j = 1, 2)$, such that

$$P_i^j(t) = P(x_1 \dots x_{\mathcal{N}} \notin H \big| x_0 \in H_i, x_f \in H_j), \qquad (4)$$

where $H = H_1 \cup H_2$, $\mathcal{N}(x_0, t)$ is the number of collisions with the boundary up to time t and x_n denotes the position of the particle at the nth collision. For example, $P_1^2(t)$ is the probability that a particle injected from hole H_1 will survive until time t given that it will escape through hole H_2. It follows from our construction that $P_1^1(t)$ has an algebraic decay tail while the other three possible distributions do not and thus decay purely exponentially with an escape rate given by $\gamma = \frac{h_1 + h_2}{\langle\tau\rangle|\partial Q|}$.

The algebraic tail of $P_1^1(t)$ is due to particles injected near the edges of hole H_1, with small incident angles θ but which do not immediately reflect back into H_1. This extra constraint is described by $|\theta| > \arctan\left|\frac{h_1^{\pm} - x_0}{4r}\right|$ ($\pm$ depending on the sign of θ) gives $P_1^1(t)$ an algebraic tail $\mathcal{O}(t^{-2})$, as expected in integrable scattering problems. As in the one hole escape problem, the phase space is split and fully-chaotic orbits cannot enter the sticky region and therefore do not contribute to the algebraic tail of $P_1^1(t)$. Furthermore, the imposed 'preference' of long surviving particles to escape through H_1 as indicated by Figures 2 and 3, is what denies $P_1^2(t)$ an algebraic tail. In the reverse situation of particles injected through H_2, the splitting of the phase space due to the position of H_1, renders the sticky region surrounding the bouncing ball modes **inaccessible**. Thus both $P_2^1(t)$ and $P_2^2(t)$ do not have algebraic tails. This would not have been be the case if both H_1 and H_2 were placed on a straight [27] or curved segment of the boundary.

In summary the total survival probability $P_i(t)$, where the subscript i indicates the injecting hole, is given by:

$$P_1(t) = e^{-\gamma t} + \frac{D}{t^2} = \wp_1^1 \overbrace{\left(e^{-\gamma t} + \frac{D}{\wp_1^1 t^2}\right)}^{P_1^1(t)} + \wp_1^2 \overbrace{e^{-\gamma t}}^{P_1^2(t)}, \qquad (5)$$

$$P_2(t) = e^{-\gamma t} = \wp_2^2 \overbrace{e^{-\gamma t}}^{P_2^2(t)} + \wp_2^1 \overbrace{e^{-\gamma t}}^{P_2^1(t)}, \qquad (6)$$

for $t > \hat{t}$, where the $\wp_i^j$ are time independent coefficients controlling the $t \to \infty$ reflection and transmission probabilities. Notice that $\wp_i^1 + \wp_i^2 = 1$ due to flux conservation, and $\wp_1^2 = \wp_2^1$ due to time-reversal symmetry. D is given by a similar calculation to [25]:

$$D = \frac{r(3\ln 3 + 4)\Big((a + h_1^-)^2 + (a - h_1^+)^2\Big)}{2h_1 v^2} + \mathcal{O}\left(\frac{1}{t}\right). \qquad (7)$$

In Figure 4, we plot the four conditional distributions $P_i^j(t)$ as functions of time t, and find an excellent agreement with the analytical results summarised in equations (5-7). **This is the simplest possible example where a classically fully chaotic billiard exhibits time-dependent asymmetric transport when opened.** This phenomenon we expect to be shared by many other well studied chaotic or mixed open billiards which display intermittency due to the presence of marginally unstable periodic orbits such as the drivebelt [18] and mushroom [29, 30] billiards. We note that the variety of options with regards to hole positions and sizes and system parameters

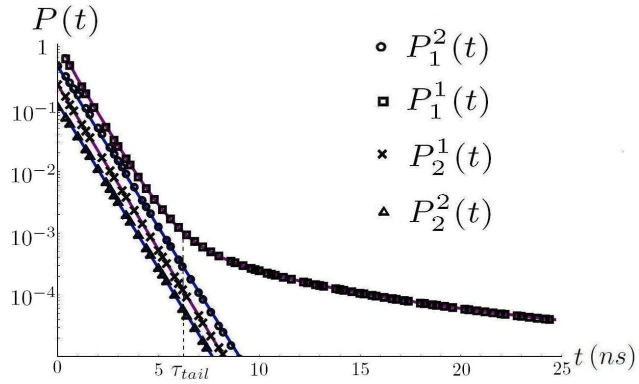

FIG. 4: (Color online) Slightly offset plots comparing numerical simulations (see key) with the analytic expressions (5) and (6) (solid curves) as functions of time t in ns. The simulations consist of 10^9 particles with stadium parameters given by: $a = 2\ \mu m$, $r = 1\ \mu m$, $h_i = 0.2\ \mu m$ and $h_1^- = 0$. $\tau_{tail} \approx 6.315\ ns$ is the large solution of $e^{-\gamma t} = \frac{D}{\wp_1^1 t^2}$, where $\wp_1^1 \approx 0.5594$ was calculated numerically.

offers ways of calibrating and controlling these classical distributions as to achieve faster or slower escape. This point of view relates closely to that of Ref[28]. Also, the exact results obtained here encourage the possibility of experimental observation of the quantum analogue of asymmetric transport in cavities with classically chaotic closed dynamics, which we discuss next.

4. Correspondence in Quantum Dots

At low temperatures ($\sim$ 15mK), electronic transport through the gate electrodes (openings) of a 2D electron gas (quantum dot) in a high quality sample is ballistic [11, 12]. For typical semiconductor nano-structure parameters, the time scale τ_{tail} at which the above observed algebraic tail becomes visible (see Figures 3 and 4) is of the order of a nanosecond (assuming an electron speed $v \approx 10^5\ ms^{-1}$). This is slightly larger than the predicted Ehrenfest time $\tau_E = \lambda^{-1} \ln(1/\hbar)$ for chaotic systems [8] (the time scale at which quantum interference effects become apparent $\approx 0.3\ ns$), and thus at first instance suggests that direct observation of a quantum difference in transmission and reflection survival probabilities is unlikely in existing devices. However, since the nature of chaos lies in orbital instability, the Ehrenfest time varies with the fluctuations of the Lyapunov exponent, which are further intensified by leaks in the system [31]. In the case studied here however, the effect of hole H_1 is crucial since for the sticky, near-bouncing ball subset of the phase space, the finite-time local Lyaponov exponent is zero [32], therefore **leading to a much longer validity and persistence of the classical description**. In fact, this region could be thought of as an h_1-dependent fictitious island of stability in which loss of quantum-to-classical correspondence is much slower, resembling that in mixed systems, such that $\tau_E \propto \hbar^{-1/\beta}$ [33], where β is a scaling parameter characteristic of the system's local phase space structure. Furthermore, we find that by varying the size and hole positions of the dot (while remaining in the ballistic regime) it is possible to calibrate and reduce τ_{tail} by a whole order of magnitude. A good way to do this is by elongating the stadium slightly such that $a/r \approx 5$ and by placing H_1 at the very edge of the straight segment.

Suppose we apply a time-dependent voltage $V(t)$ across the gates of the stadium heterostructure such that the incoming current $I_i^{in}(t)$ through hole H_i is proportional to $V(t)$. Then the charge exiting through each hole will follow the driving current with a lag-time τ which is distributed according to (5) or (6) appropriately. This can be modeled by

$$I_j(t) = (-1)^{i+j+1} \wp_i^j \int_0^\infty I_i^{in}(t-\tau) \frac{\mathrm{d}P_i^j(\tau)}{\mathrm{d}\tau} \mathrm{d}\tau, \qquad (8)$$

where i and j indicate the injecting and exiting hole respectively. The observed, net current through the system is thus given by $I_i^{net}(t) = I_i^{in}(t) + I_1(t) + I_2(t)$. Because the probability density $\frac{\mathrm{d}P_1^1(\tau)}{\mathrm{d}\tau}$ is slightly skewed to the right, relative to the other densities, the two observables $I_1^{net}(t)$ and $I_2^{net}(t)$ will differ by

$$\wp_1^1 \int_0^\infty \frac{\mathrm{d}I^{in}(t-\tau)}{\mathrm{d}\tau} \left(P_1^1(\tau) - P_2^2(\tau)\right) \mathrm{d}\tau. \qquad (9)$$

For experimental observation we propose using a square wave signal $V(t) = V_0 (1 + sign(\sin \omega t))$ such that $\omega > \pi/\tau_{tail}$ as to accentuate the power-law contribution of $P_1^1(t)$. Quantum interference effects such as universal conductance fluctuations may be statistically removed since the skewness of $\frac{\mathrm{d}P_1^1(\tau)}{\mathrm{d}\tau}$ is to leading order geometry dependent through the constant D in (7). In experiments of course, one should make sure that the excess density of charged particles within the dot is always low enough as to avoid a build up of an internal electric field which would effectively destroy the fictitious island of stability (sticky region) enclosing the bouncing ball orbits. For microwave billiards this is not an issue.

5. Discussions and Conclusions

To conclude, we have investigated the classical dynamics of the chaotic stadium billiard with two holes placed asymmetrically. We have found that the transmission and reflection survival distributions can have algebraic and exponential decays observed in the same classically ergodic geometry depending on the choice of injecting hole. We have identified the reason for this being the hole's asymmetric positioning on the straight segment of the billiard, which essentially splits the classical phase space of the system, rendering the sticky region surrounding the bouncing ball orbits inaccessible to chaotic orbits. As a result, the transmission and reflection survival distributions are qualitatively different. Moreover, when injecting from the hole on the curved segment both transmission and reflection distributions decay with a pure exponential. We expect that this observation along with the analytic expressions obtained and confirmed numerically can be appreciated by the (quantum) chaos community. We further propose that observation of this classical result in semiconductor nano-structures (quantum dots) or microwave cavities can improve our understanding of classical to quantum correspondence in transport problems in relation with the different quantum time scales introduced by the classical phenomenon of stickiness. Finally, we expect that the asymmetric transport scenario exhibited here as well as the mechanism described may apply in a similar way to more general open dynamical systems with mixed phase space [34], permitting dynamical trapping of trajectories by suitably placed holes. This shows that long studied systems such as the stadium billiard continue to provide us with interesting new phenomena to study.

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