Analogy of PDE classification with classification of conic section

This sheet provides the motivation behind the terminology used for classifying PDE's and parallels the methods used in the lectures. **Note** that this is only an analogy and doesn't replace the theory behind PDE classification. *This is for supplementary information only - it will not be required in an exam.*

The general equation describing a conic section (i.e. hyperbola, parabola or ellipse) is given by

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

where a, b, c, d, e, f are all constants. Dividing by a (assuming $a \neq 0$ w.l.o.g. (without loss of generality))

$$x^2 + \frac{2b}{a}xy + \frac{c}{a}y^2 + \frac{d}{a}x + \frac{e}{a}y + \frac{f}{a} = 0 \tag{1}$$

For general a, b, c, d, e, f how do we tell if it is an ellipse, parabola or hyperbola ? Well - factorise, by completing the square:

$$(x - \omega^+ y)(x - \omega^- y) + O(x, y, 1) = 0 \tag{2}$$

where $O(x, y, 1)$ means terms proportional to x, y and 1. Expanding equation (2) and identifying with (1) gives

$$\left.\begin{aligned} \omega^+ + \omega^- &= -2b/a \\ \omega^+\omega^- &= c/a \end{aligned}\right\}$$

which means that $\omega^\pm$ are the roots of $\omega^2 + (2b/a)\omega + (c/a) = 0$, and so

$$\omega^\pm = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

Three cases arise:

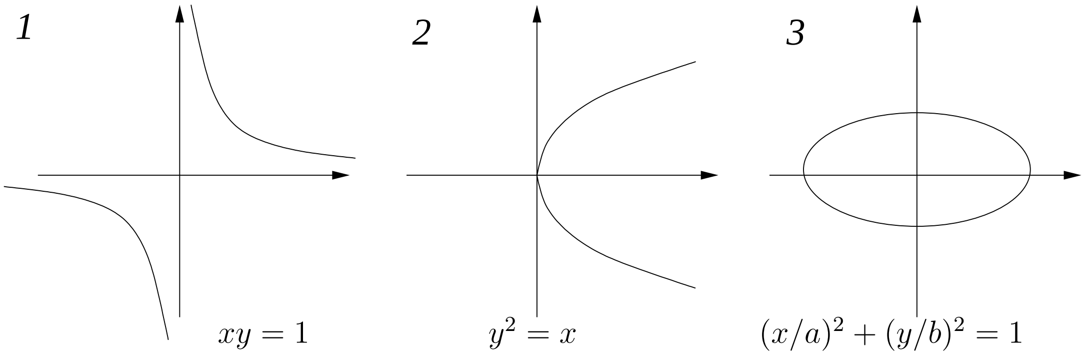
1. If $b^2 - ac > 0$ then $\omega^\pm$ are real and distinct. Change variables from (x, y) to (ξ, η) by writing $\xi = x - \omega^+ y$ and $\eta = x - \omega^- y$ and substitute into (1) to give

   $$\xi\eta + O(\xi, \eta, 1) = 0$$

   or

   $$\xi\eta = A\xi + B\eta + C$$

   for some constants A, B, C which we haven't calculated explicitly. This is the equation of a hyperbola (remember ? ... for example $xy = 1$)



2. If $b^2 = ac$ then $\omega = \omega^+ = \omega^-$, so let $\xi = x - \omega y$ and choose another variable η to be arbitrary, but linearly independent of ξ (i.e. $\xi \neq C\eta$ for any constant C) so that (x, y) can be expressed in terms of (ξ, η). Then from (2),

$$\xi^2 + O(\xi, \eta, 1) = 0$$

or

$$\xi^2 = A\xi + B\eta + C$$

for some A, B, C. This is the equation of a parabola (e.g. $y^2 = x$)

3. If $b^2 - ac < 0$ then the roots $\omega^\pm$ are complex conjugates which we can write as

$$\omega^\pm = \frac{-b \pm \sqrt{ac - b^2}}{a} = s \pm \mathrm{i}t$$

Put this expression into (2) to get

$$(x - sy + \mathrm{i}ty)(x - sy - \mathrm{i}ty) + O(x, y, 1) = 0$$

and here we choose to define $\xi = x - sy$ and $\eta = ty$, which implies

$$(\xi + \mathrm{i}\eta)(\xi - \mathrm{i}\eta) + O(\xi, \eta, 1) = 0$$

or, on expanding

$$\xi^2 + \eta^2 = A\xi + B\eta + C$$

which is the equation of an ellipse (e.g. $x^2 + y^2 = 1$)