Mathematical Methods 3	**2006**

Sheet B — Classification of PDE's and canonical forms

In this worksheet you are asked to classify PDE's and to find their canonical forms.

1. Classify and find the canonical forms for the following PDE's:

   (a) $u_{xx} - u_{xy} - 6u_{yy} = 0$

   (b) $5u_{xx} - 4u_{xy} = 3u_y$

   (c) $3u_{xy} - 4u_{yy} = 5u_y$

   (d) $4u_{xx} + 12u_{xy} + 9u_{yy} = 3u_x + 4u_y + u$
   (in this case, choose $\eta = wy - x$, as the second characteristic)

   (e) $u_{xx} - 2u_{xy} + 5u_{yy} = u_x - u_y - \frac{1}{4}u$

   Also identify (where appropriate) the characteristic curves, $\xi = C_1, \eta = C_2$.

2. For the canonical forms derived in Q1, use the transformation

$$U(\xi,\eta) = \exp(\alpha\xi + \beta\eta)V(\xi,\eta)$$

   to eliminate as many lower order terms as possible, by choice of the constants α and β.

3. Using the results of Q2 for parts (a), (c), (e) *only*, express the general solution of the corresponding PDE in Q1, each of which should be in terms of arbitrary functions f and g, say. You may find the results of Sheet A, Q2 useful.

4. Classify throughout the whole (x, y) plane, the equations

   (a) $u_{xx} + xu_{yy} = 0$,

   (b) $yu_{xx} + 2xu_{xy} + yu_{yy} = 0$.

   In the first case, obtain the characteristics and the canonical form in the hyperbolic region(s).

5. By reducing the equation

$$u_{xx} + 2u_{xy} - 3u_{yy} + 4u_x - 4u_y = 4\sin(3x - y)$$

   for $u(x, y)$ to its canonical form, find its general solution.

   Hence, find the particular solution which satisfies the conditions $u(x, 0) = 1$, $u_y(x, 0) = 0$.

6. Consider the following equation for $u(x, y)$

$$y^2u_{xx} - x^2u_{yy} - \frac{3}{4}\frac{(y^4 - x^4)}{x^2y^2}u = 0$$

(a) Assuming $x \neq 0$ and $y \neq 0$, show that the characteristics are given by

$$\xi = \tfrac{1}{2}y^2 + \tfrac{1}{2}x^2, \qquad \eta = \tfrac{1}{2}y^2 - \tfrac{1}{2}x^2$$

and sketch these curves in the first quadrant, $x > 0$, $y > 0$ of the (x, y) plane.

(b) Show that the canonical form of the PDE is given by

$$U_{\xi\eta} - \frac{\frac{1}{2}\eta}{\xi^2 - \eta^2} U_\xi + \frac{\frac{1}{2}\xi}{\xi^2 - \eta^2} U_\eta + \frac{\frac{3}{4}\xi\eta}{(\xi^2 - \eta^2)^2} U = 0$$

(c) You are given that this equation may be written in the following form

$$\frac{\partial^2}{\partial\xi\partial\eta}\Big[(\xi^2 - \eta^2)^{1/4} U\Big] = 0$$

Hence find the solution $u(x, y)$ in terms of two arbitrary functions f and g.

7. The Telegrapher's equation for a function $u(x, t)$ is given by

$$u_{tt} - \gamma^2 u_{xx} + 2\lambda u_t = 0, \qquad x > 0,\ t > 0$$

Use a suitable transformation of the dependent variable, u, to show that this equation can be written

$$v_{tt} - \gamma^2 v_{xx} - \lambda^2 v = 0$$

8. *[Hard]* Consider the following equation for $u(x, y)$

$$u_{xx} + \frac{(y^2 - x^2)}{xy} u_{xy} - u_{yy} = 0, \qquad (x, y) \neq (0, 0)$$

It is to be assumed that $u \to 0$ as $x^2 + y^2 \to \infty$ and that $u(x, 0) = h(x)$ for some given function h where $h(x) \to 0$ as $x \to \infty$.

(a) Show that this equation is hyperbolic and that the characteristics are given by

$$\xi = \frac{1}{2}(x^2 + y^2), \qquad \eta = y/x$$

Redefine ξ by $\frac{1}{2}\xi^2$ and η by $\tan\eta$ and hence show that the characteristic curves are just the coordinate lines in polar coordinates.

(b) Using the redefined system, show that the original equation can be reduced to the following canonical form:

$$\frac{\partial^2 U}{\partial\eta\partial\xi} - \frac{1}{\xi}\frac{\partial U}{\partial\eta} = 0$$

(c) Show that the general solution of this equation may be expressed most simply in the form $U(\xi, \eta) = \xi f(\eta) + g(\xi)$ for arbitrary functions ξ and η.

(d) Using the conditions of the problem, determine the solution $u(x, y)$ everywhere in the (x, y)-plane in terms of h.