Mathematical Methods 3 **2006**

Sheet C — Characteristics/Initial conditions/Solutions

In this worksheet we generate solutions, typically to problems supporting characteristics, which convey the initial conditions throughout the entire domain.

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1. The function $u(x,t)$ satisfies $u_t + u_x = 0$ in $x > 0$, $t > 0$ together with the initial condition $u(x,0) = \sin(x)$, $x > 0$ and the boundary condition $u(0,t) = \sin(t)$, $t > 0$. Determine the values of u in the whole of the quarterplane $x > 0$, $t > 0$. Sketch $u(3\pi/2, t)$ for $0 < t < 6\pi$. Is the solution reasonable ?

2. Find the solution of $af_x + f_y = 0$ where $a \neq -1$ which takes the value of

   (a) $f = \mathrm{e}^{-x^2}$ on the line $y = -x$;

   (b) $f = 1$ on the line $y = -x$.

   In both cases discuss what happens when $a = -1$.

3. Find the characteristics of
$$u_x + 2y^{1/2}u_y = xy$$
   and hence sketch the domain where $u(x,y)$ is defined by initial values on the line segment $0 \leq x \leq 2$, $y = 0$. For the initial values
$$u(x,0) = \left\{ \begin{array}{ll} x^2 & \text{for} \quad 0 \leq x \leq 1 \\ 4 - x^2 & \text{for} \quad 1 < x \leq 2 \end{array} \right\},$$
   find the value of $u(1,1)$ and the value of y at which u_x is discontinuous on $x = 2$.

4. (a) Show that $u = \alpha x + \beta t + \gamma$ satisfies $u_{tt} - c^2 u_{xx} = 0$ for any constants α, β, γ. Express the solution in the form $f(x - ct) + g(x + ct)$.

   (b) Show that $u = \sin(kx)\cos(kct)$ satisfies $u_{tt} - c^2 u_{xx} = 0$ for any constant k. Express the solution in the form $f(x - ct) + g(x + ct)$.

   In each case sketch the space-time trajectories of the maximum values of both $f(x - ct)$ and $g(x + ct)$.

5. Consider and example of unidirectional *non-linear* wave motion:
$$u_t + uu_x = 0, \qquad \text{with } u(x,0) = f(x)$$

   (a) Show that the characteristic variable ξ is defined by the implicit equation $x = f(\xi)t + \xi$ and that the solution can also be expressed implicitly in the form
$$u(x,t) = f(x - tu).$$

   (b) Using implicit differentiation, show that $u_x = f'(x - tu)/(1 + tf'(x - tu))$ and hence determine a condition when the solution will break down.

(c) With the example $f(x) = -x$, (i) show that the solution breaks down at $t = 1$; (ii) sketch the characteristic curves for this case and explain your sketch in relation to part (i); and (iii) sketch the solution as a function of time between $0 \le t \le 1$.

(d) Now choose $f(x) = x$, again sketching characteristic curves and how the solution evolves in time. Hence show that this solution does not break down for all $t > 0$.

6. In this question you are asked to solve the initial value problem

$$u_t + uu_x = x, \qquad u(x,0) = f(x)$$

using the method of characteristics.

(a) Using the parameters s and ξ, show that the solution can be expressed as $u(s,\xi) = \frac{1}{2}[f(\xi)+\xi]\mathrm{e}^{s} + \frac{1}{2}[f(\xi)-\xi]\mathrm{e}^{-s}$.

(b) Obtain the solution in the form $u = u(x,t)$ when (i) $f(x) = 1$ and (ii) $f(x) = x$.

7. (Taken from the 2000 exam). Consider the 1st order linear partial differential equation for the function $u(x,y)$ defined in $x \ge 0$ and $y \ge 0$ given by

$$(1+x)\frac{\partial u}{\partial x} + (a+y)\frac{\partial u}{\partial y} = f(x,y) \qquad (1)$$

where $a \ge 0$ and with

$$u(x,0) = g(x).$$

(a) Show that the characteristics for equation (C) may be described by

$$\xi = -1 + \frac{a(1+x)}{(a+y)}$$

and hence sketch the characteristic curves $\xi =$ constant for $a = 1$. Further, show that for any given a all characteristic curves pass through a single point $(-1,-a)$.

(b) By considering the characteristic curves as $a \to 0$ discuss the implications on the nature of the solution when $a = 0$ giving reasons for your answer.

(c) Give the equations of the two lines bounding the region of influence of the data on $y = 0$ for $0 \le x \le 1$ when $a = 1$.

(d) You are now given that $f(x,y) = y + 1$. Show that a suitable parameter along the characteristic is $s = \ln(1 + y/a)$ and that u satisfies

$$\frac{d}{ds}u(\xi,s) = a\mathrm{e}^{s} + (1-a)$$

on $\xi =$ constant and hence show that the solution, $u(x,y)$, of (1) is given by

$$u(x,y) = y + (1-a)\ln\left(1+\frac{y}{a}\right) + g\left(-1+\frac{a(1+x)}{(a+y)}\right).$$