Mathematical Methods 3 2006

Sheet E Fourier Transforms for PDEs

In this worksheet we apply the Fourier Transform method to some physically relevant PDE problems

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1. Consider the intial value problem for the wave equation

$$\frac{\partial^2}{\partial t^2}u(x,t) = c^2\frac{\partial^2}{\partial x^2}u(x,t), \qquad -\infty < x < \infty, \quad t > 0$$

with initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial}{\partial t}u(x,0) &= g(x) \end{aligned}$$

for $-\infty < x < \infty$. Using a Fourier transform in x, show that the solution is

$$u(x,t) = \frac{1}{2}\Big(f(x-ct) + f(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')dx'$$

[HINT: Use the result of Sheet D, Q1(b) to help with inversion. Also you will need to consider the result of $\displaystyle\int_{x-ct}^{x+ct} \mathrm{e}^{-iku}du$]

2. (a) Solve Laplace's equation in the half-plane

$$\frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y) = 0, \qquad -\infty < x < \infty, \quad y > 0$$

subject to the conditions

$$\phi(x,0) = f(x), \qquad \text{and} \qquad \phi \to 0, \quad \text{as } y \to \infty$$

using convolutions [HINT: You will need a result from section 2.4 of the lectures]

(b) Evaluate the solution when

$$f(x) = \begin{cases} 1, & |x| < a, \\ 0, & |x| > a. \end{cases}$$

3. In the lectures, we derived the solution of the heat equation $\theta_t = \alpha\theta_{xx}$ with I.C. of $\theta(x,0) = g(x)$ for $-\infty < x < \infty$. It was given by

$$\theta(x,t) = \frac{1}{2\sqrt{\pi\alpha t}}\int_{-\infty}^{\infty} g(\xi)\exp\{-(x-\xi)^2/4\alpha t\}d\xi$$

Make the choice $g(x) = A\mathrm{e}^{-\beta x^2}$ and show, by direct substitution into the expression above that

$$\theta(x,t) = \frac{A}{\sqrt{1+4\alpha\beta t}}\exp\{-\beta x^2/(1+4\alpha\beta t)\}.$$

4. The initial-value problem for the temperature $\theta(x,t)$ in a semi-infinite rod being heated uniformly at the end $x=0$ at a variable rate $g(t)$ is

$$\begin{aligned} \frac{\partial}{\partial t}\theta(x,t) &= \frac{\partial^2}{\partial x^2}\theta(x,t), \qquad x>0, \quad t>0, \\ \theta(x,0) &= f(x), \qquad x>0, \\ \frac{\partial}{\partial x}\theta(0,t) &= -g(t), \qquad t>0, \end{aligned}$$

and θ, $\theta_t \to 0$ as $x \to \infty$. Using Fourier cosine transforms, find an expression for $\theta(x,t)$. [HINT: You will need a result from the lectures on how to find the inverse F.C.T. of $\mathrm{e}^{-k^2 t}$ and the result from Sheet D, Q6]

5. The time-dependent transverse displacement $w(x,t)$ in an elastic beam satisfies

$$\frac{\partial^4}{\partial x^4}w(x,t) + \frac{\partial^2}{\partial t^2}w(x,t) = 0, \qquad -\infty < x < \infty, \quad t>0,$$

with initial conditions

$$\begin{aligned} w(x,0) &= f(x), \\ \frac{\partial}{\partial t}w(x,0) &= 0, \end{aligned}$$

for $-\infty < x < \infty$. Use Fourier Transforms to solve this problem. You may use the fact that the inverse Fourier Transform of $H(k) = \cos k^2 t$ is $h(x) = \dfrac{1}{2\sqrt{\pi t}}\cos\left(\dfrac{x^2}{4t^2} - \dfrac{\pi}{4}\right)$