Mathematical Methods 3 — 2006

Sheet F — Laplace Transforms

In this worksheet we consider properties of Laplace Transforms and their application to ODE's. The Laplace transform is denoted

$$L_f(p) \equiv \overline{f}(p) = \int_0^\infty f(t)\mathrm{e}^{-pt}\,\mathrm{d}t.$$

1. For functions $f(x)$ absolutely integrable on $(-\infty, \infty)$ and zero for $x < 0$ show that
   (a) $\mathcal{F}\{f\} = L_f(-ik)$
   (b) $\mathcal{F}_s\{f\} = -\mathrm{Im}(L_f(-ik))$
   (c) $\mathcal{F}_c\{f\} = \mathrm{Re}(L_f(-ik))$
2. Assuming that the Laplace transforms of $f(t)$ and $g(t)$ exist verify the following relations:
   (a) $g(t) = \mathrm{e}^{at}f(t) \Rightarrow L_g(p) = L_f(p-a)$
   (b) $g(t) = f(at) \Rightarrow L_g(p) = \frac{1}{a}L_f(p/a)$
   (c) $g(t) = t^n f(t) \Rightarrow L_g(p) = (-1)^n \frac{d^n}{dp^n} L_f(p)$
   (d) $g(t) = \int_0^t f(t')dt' \Rightarrow L_g(p) = \frac{1}{p}L_f(p)$
   (e) $g(t) = \frac{f(t)}{t} \Rightarrow L_g(p) = \int_p^\infty L_f(p')dp'$
   (f) $g(t) = t^{-1/2} \Rightarrow L_g(p) = \sqrt{\frac{\pi}{p}}$
3. Find the inverse Laplace transform of the following:
   (a) $(p^2 - 3p + 2)^{-1}$
   (b) $p^{-2}(p^2+1)^{-1}$
   (c) $p/(p^2 - 2p + 5)$
   (d) $(2p+1)/(p(p+1)(p+2))$
4. The Laplace transform of $t^{-3/2}\mathrm{e}^{-1/t}$ is denoted $F(p)$. Show that

$$\frac{\mathrm{d}F}{\mathrm{d}p} = -\frac{F}{p^{1/2}}.$$

   Hence deduce that $F(p) = \sqrt{\pi}\mathrm{e}^{-2\sqrt{p}}$.

5. Find the Laplace transforms of the functions $f(t) = t^a$ and $g(t) = t^b$ where $a, b \in \mathbb{N}$ and f & g are assumed to vanish for $t < 0$. Construct the convolution of $f(t)$ and $g(t)$, find its Laplace transfrom and deduce that

$$\int_0^1 y^a (1-y)^b \, \mathrm{d}y = \frac{a!b!}{(a+b+1)!}.$$

6. Solve the differential equation

$$\frac{d^2}{dt^2}y(t) + 4y(t) = 3\cos 2t, \qquad y(0) = 1, \qquad \frac{d}{dt}y(0) = 0$$

using Laplace transforms.

7. Use Laplace transforms to find the solution of the following simultaneous differential equations for unknowns $x(t)$, $y(t)$:

$$\begin{aligned} x' + y' + x &= -\mathrm{e}^{-t} \\ x' + 2y' + 2x + 2y &= 0 \end{aligned}, \qquad \text{with} \quad x(0) = -1, y(0) = 1.$$

8. Obtain the solution of

$$y''(x) + (\alpha + \beta)y'(x) + \alpha\beta y(x) = f(x), \qquad \text{with} \quad y(0) = y'(0) = 0$$

where $\alpha \neq \beta$ using Laplace Transforms leaving your answer in the form of an integral.