Mathematical Methods 3 **2006**

Sheet G Solving PDE's using Laplace Transforms

In this worksheet we use Laplace Transforms to solve certain PDE problems

---

1. The transverse displacement $u(x,t)$ of a semi-infinite elastic string satisfies
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad x > 0, \quad t > 0$$
with initial conditions
$$u(x,0) = \frac{\partial}{\partial t} u(x,0) = 0, \qquad x > 0.$$
and boundary condition
$$-\beta \frac{\partial}{\partial x} u(0,t) = f(t), \qquad t > 0.$$
Show, using Laplace Transforms that the solution can be written
$$u(x,t) = \frac{c}{\beta} H(t - x/c) \int_0^{t-x/c} f(t')dt'$$
where $H(x)$ is the Heaviside function. Can you interpret this result physically ?

2. Use Laplace Transforms to solve the heat conduction problem for the temperature $\theta(x,t)$:
$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \kappa \frac{\partial^2 \theta}{\partial x^2}, \qquad x > 0, \quad t > 0, \\ \theta(0,t) &= T_0 \mathrm{e}^{-bt}, \qquad t > 0, \\ \theta(x,0) &= 0, \qquad x > 0, \end{aligned}$$
for $b > 0$. You may assume that for the Laplace transform of the function $g(t) = \dfrac{x}{2\sqrt{\pi t^3}} \mathrm{e}^{-x^2/(4t)}$ is $L_g(p) = \mathrm{e}^{-x\sqrt{p}}$. You should attempt to give your answer in terms of the Error function. What happens if $b < 0$?

3. Show that the solution of the heat conduction problem for $\theta(x,t)$ given by
$$\begin{aligned} \frac{\partial \theta}{\partial t} &= \frac{\partial^2 \theta}{\partial x^2} - \cos x, \qquad x > 0, \quad t > 0, \\ \theta(0,t) &= \mathrm{e}^{-t}, \qquad t > 0, \\ \theta(x,0) &= 0, \qquad x > 0, \end{aligned}$$
is
$$\theta(x,t) = \mathrm{erfc}(x/(2\sqrt{t})) - \cos x(1 - \mathrm{e}^{-t})$$
using Laplace Transforms, where $\mathrm{erfc}(x)$ is the complementary error function.