Mathematical Methods 3 — 2006

Sheet H — Generalised Functions

In this worksheet we consider some properties of generalised functions

1. Verify the following properties of the Dirac δ-function:

   (a) $\delta(ax) = \dfrac{1}{|a|}\delta(x)$,

   (b) $\delta(t^2 - a^2) = \dfrac{\delta(t+a) + \delta(t-a)}{2|a|}$,

   (c) $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\delta(t-b)dt = \delta(a-b)$

   (d) $\delta(t) = -t\delta'(t)$

   where a and b are real constants, and the prime denotes differentiation.

2. Prove the Fourier inversion formula

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} f(\xi)\mathrm{e}^{\mathrm{i}k(\xi - x)}d\xi$$

   by interchanging the order of integration and using the definition of the δ-function.

3. Show that $\delta(\sin x) = \displaystyle\sum_{n=-\infty}^{\infty} \delta(x - n\pi)$.

4. The function $f(x)$ is defined by

$$f(x) = \begin{cases} 1+x, & -1 < x < 0 \\ 1-x, & 0 \le x < 1, \end{cases}$$

   and vanishes for $|x| \ge 1$.

   (a) Calculate the Fourier transform of $f(x)$.

   (b) Find the first and second derivatives of $f(x)$ and show that

$$f''(x) = \delta(x+1) - 2\delta(x) + \delta(x-1). \qquad (1)$$

   Hence recalling that $\mathcal{F}(f'') = -k^2\mathcal{F}(f(x))$, calculate the Fourier transform of $f(x)$ from (1).

5. The definition of the δ-function given in the lecture notes is not the only way of defining a δ-function. Show that the following representations of the δ-function also satisfy all the required conditions (equations (4.1) and (4.2) in the lecture notes):

$$\text{(i) } \delta(x) = \lim_{a\to 0}\left(\frac{1}{\pi}\frac{a}{a^2 + x^2}\right) \qquad \text{(ii) } \delta(x) = \lim_{a\to 0}\left(\frac{1}{a\sqrt{\pi}}\mathrm{e}^{-x^2/a^2}\right)$$

   [HINT: The result $f(x) = 1/(a^2 + x^2) \Rightarrow F(k) = (\pi/a)\mathrm{e}^{-a|k|}$ will help with (i)]

6. In §3.6 of the lecture notes, we considered the problem of a string falling under gravity, defined by the equation $u_{tt} = c^2 u_{xx} + g$ with B.C. $u(0,t) = 0$, $t > 0$ and I.C's $u(x,0) = u_t(x,0) = 0$, $x > 0$ and used Laplace Transforms to give

$$L_u(x,p) = \frac{g}{p^3}(1 - \mathrm{e}^{-p(x/c)})$$

Show that

$$\mathcal{L}\{\delta((x/c) - t)\} = \mathrm{e}^{-p(x/c)}$$

and hence perform the inverse Laplace Transform of L_u above using this result, convolutions and properties of the δ-function to give

$$u(x,t) = \frac{gt^2}{2} - \frac{g}{2}H(t - (x/c))(t - (x/c))^2$$

as in the lecture notes.

7. During example 1.8 of the course notes, it was claimed that the solution to a certain problem was *unique* without proving this. This question is designed to demonstrate that this solution was in fact the unique solution. We will need the result from the lectures,

$$\delta(k) = \frac{1}{\pi}\int_0^\infty \cos kx dx.$$

As a reminder, example 1.8 was given as

$$u_{xx} + u_{yy} = 0, \quad y > 0, \quad -\infty < x < \infty$$

with $u(x,0) = 0$ and $u_y(x,0) = \sin(nx)/n$ (and $n > 0$ w.l.o.g.) It was claimed that the *unique* solution was

$$u(x,y) = \frac{\sinh ny \sin nx}{n^2}. \tag{2}$$

To prove this, let us start with the most general representation of the solution to Laplace's equation satisfying $u(x,0) = 0$:

$$u(x,y) = \int_0^\infty \{A(k)\cos kx + B(k)\sin kx\}\sinh ky\, dk. \tag{3}$$

(a) Verify that (3) does indeed satisfy Laplace's equation and $u(x,0) = 0$. Why is it sufficient for the integral in (3) to start from $k = 0$ instead of $k = -\infty$ if it is to be the most general representation of the solution ?

(b) Apply the second BC, namely $u_y(x,0) = \sin(nx)/n$, multiply (in turn) the result by $\sin px$ and $\cos px$ ($p > 0$) and integrate between $-\infty < x < \infty$ to show that $A(k) = 0$, $B(k) = \delta(n - k)/(kn)$. Hence show that the solution is given by (2).

(c) Comment on the analogy between the procedure in part (b) and Fourier series.