Mathematical Methods 3 **2006**

Sheet J Green's functions for PDE's

Constructing and using Green's functions to solve PDE's.

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1. The Green's function for the Poisson equation ($\nabla^2\phi = f$) in three dimensions is given by $g(\mathbf{x}, \boldsymbol{\xi}) = -1/(4\pi|\mathbf{x} - \boldsymbol{\xi}|)$. Hence find $\phi(\mathbf{x})$, when

   (a) $f \equiv f_1(\mathbf{x}) = \delta(r)$;

   (b) $f \equiv f_2(\mathbf{x}) = \dfrac{-a^2\mathrm{e}^{-ar}}{4\pi r}$,

   where $r = |\mathbf{x}|$. Thus show that when $f = f_1 + f_2$,

   $$\phi(\mathbf{x}) = \frac{-\mathrm{e}^{-ar}}{4\pi r}.$$

2. Given that the Green's function for Poisson's equation in the region $-\infty < x, y < \infty$ is given by

   $$g(x, y; \xi, \eta) = \frac{1}{2\pi} \log r$$

   where $r^2 = (y - \eta)^2 + (x - \xi)^2$, use the method of images to determine the Green's function for Laplaces equation in the *half-space* $-\infty < x < \infty$, $y > 0$ satisfying

   (a) $g(x, 0; \xi, \eta) = 0$ - i.e. a Dirichlet condition on $y = 0$

   (b) $\dfrac{\partial}{\partial y} g(x, 0; \xi, \eta) = 0$ - i.e. a Neumann condition on $y = 0$.

   Hence express the solution of

   $$\nabla^2\phi = f(x, y), \qquad -\infty < x < \infty, \quad y > 0$$

   with $\phi(x, 0) = 0$, in terms of a double integral involving your Green's function.

3. The forced heat equation is given by

   $$\frac{\partial\theta}{\partial t} - \frac{\partial^2\theta}{\partial x^2} = f(x, t), \qquad -\infty < x < \infty, \quad t > 0$$

   and satisfies the homogeneous initial condition, $\theta(x, 0) = 0$. Use the Green's function for this general problem in Green's formula to find the solution for a particular forcing $f(x, t) = \delta(x - x_0)\delta(t - t_0)$ where $t_0 > 0$.

4. Consider the forced wave equation, with general initial conditions

   $$\frac{\partial^2 u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x, t), \qquad u(x, 0) = \phi(x), \qquad u_t(x, 0) = \psi(x), \qquad -\infty < x < \infty$$

   Use the fact that the problem is *linear* to write $u = u_1 + u_2$, where you should state boundary value problems for each of u_1 and u_2 which can be solved using methods from the course.