Systematization of All Resonance Modes in Circular Dielectric Cavities

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ABSTRACT
Circular dielectric cavities are key components for the construction of optic microresonators and microlasers. They are one of very few cases where the transcendental equations for complex eigenmodes (resonances) of an open system (dielectric cavity) can be found analytically in an exact manner. The behaviour of those eigenvalues in the small opening limit, i.e. when the refractive index of the cavity goes to infinity, is analysed. The analysis allows one to clearly distinguish between internal (Feshbach) and external (shape) resonance modes for both TM and TE polarizations. As a result, unambiguous azimuthal and radial modal indices are assigned to each internal and external resonance mode.
Keywords: circular dielectric cavity, TM/TE modes, internal/external resonances.

1. INTRODUCTION

Among dielectric microcavities of various shapes, thin circular cavities filled with a homogeneous dielectric are one of very few cases where the transcendental equations for complex eigenmodes (resonances), $k = k_r - ik_i$, can be found analytically. These equations are derived from time-independent Maxwell's equations for an infinite cylinder with the aid of the effective refractive index n which takes into account the material as well as the thickness of the cavity, see for example Appendix I of Ref. [1]. Finally, for transverse magnetic polarization of the electromagnetic field (TM; electric field perpendicular to the disk plane) one obtains

$$J_m(knR)H_m'(kR) - nJ_m'(knR)H(kR) = 0\,, \tag{1}$$

and respectively for transverse electric (TE; magnetic field perpendicular to the disk plane)

$$J_m(knR)H_m'(kR) - \frac{1}{n}J_m'(knR)H(kR) = 0\,. \tag{2}$$

The corresponding resonance field Ψ has the form of twofold degenerate (for $m > 0$) whispering gallery (WG) modes

$$\Psi_m = \begin{cases} N_m J_m(knr)\begin{pmatrix}\cos m\varphi \\ \sin m\varphi\end{pmatrix}, & r < R, \\ H_m(kr)\begin{pmatrix}\cos m\varphi \\ \sin m\varphi\end{pmatrix}, & r > R. \end{cases} \tag{3}$$

In the above equations J_m and H_m are Bessel and Hankel functions of the first kind respectively, $m = 0, 1, 2, \ldots$ is the azimuthal modal index, $N_m = H_m(kR)/J_m(knR)$ are constants, and R is the cavity radius. Physically, the azimuthal modal index characterizes the field variation along the disk circumference, with the number of intensity hotspots being equal to $2m$. The second (radial) modal index $q = 1, 2, \ldots$ is used to number the resonances within a set with same azimuthal modal index m in accordance with the increase of their real parts k_r starting from $q = 1$. Physically, for the resonances with relatively small imaginary parts this index characterizes the field variation along the disk radius, with the number of intensity hotspots in that direction being equal to q. However, for TE modes the latter is not always the case.

A more detailed analysis of Eqs. (1, 2) reveals that for each polarization there exist two kinds of resonances, see for example Ref. [2]. Following common terminology we will call them internal (or Feshbach) and external (or shape) resonances. In general, for a fixed refractive index external resonances have significantly larger imaginary parts compared to internal resonances; however, for TE polarization occasionally they can occur in the same range of the complex wavenumber plane. As a result, a set of internal resonances with same m can visually acquire an additional external resonance making the above procedure of assigning the proper radial modal indices within that internal set impossible.

The work reported in this paper was supported by the EPSRC grant EP/C515137/1.

Our purpose here is twofold. First, using the behaviour of the circular cavity (disk) resonances in the small opening limit, i.e. when the refractive index of the cavity diverges, we assign unambiguous azimuthal and radial modal indices to each internal and external resonance mode. To the best of our knowledge this is the first attempt to assign meaningful radial modal indices to external resonances. Second, we study the field patterns of internal and external modes for various cavity refractive indices and various azimuthal and radial modal indices. It is shown that the azimuthal modal index m has clear physical interpretation for all possible modes while the radial modal index q has clear physical interpretation only for TM internal resonances of cavities with refractive indices not very close to 1.

2. TM RESONANCES

We have recently shown in Ref. [3] that the scaled wavenumbers $n\,k_{m,q}R$ of TM internal resonances in the small opening limit, i.e. when $n \to \infty$, contrary to naïve expectations, don't match the corresponding real eigenvalues $j_{m,q}$ of the closed disk with the zero (Dirichlet) boundary conditions. In fact, we have obtained that

$$\begin{aligned} &\lim_{n\to\infty} n\,k_{m,\,q}R = j_{m-1,\,q}, && m \neq 0, \\ &\lim_{n\to\infty} n\,k_{0,\,q}R = j_{1,\,q-1}, && q \neq 1, \\ &\lim_{n\to\infty} n\,k_{0,\,1}R = 0, && \end{aligned} \tag{4}$$

where $j_{m,q}$ are the zeros of Bessel functions. The first limit in Eq. (4) has been also derived in Ref. [4]. For TM external resonance wavenumbers (not scaled with respect to n) we have obtained that

$$\lim_{n\to\infty} k_{m,\,q}R = h_{m,\,q}\,, \tag{5}$$

where $h_{m,q}$ are the complex zeros of Hankel functions. There is only a finite number of such zeros for a given m : none if m is 0 or 1, $m/2$ if m is even, $(m-1)/2$ if m is odd and $m>1$, see Ref. 5. This corresponds to our finding for the number of radial modes in each group of external resonances with fixed m . Fig. 1 illustrates the behaviour of TM resonances for several azimuthal and radial indices under variation of the refractive index.

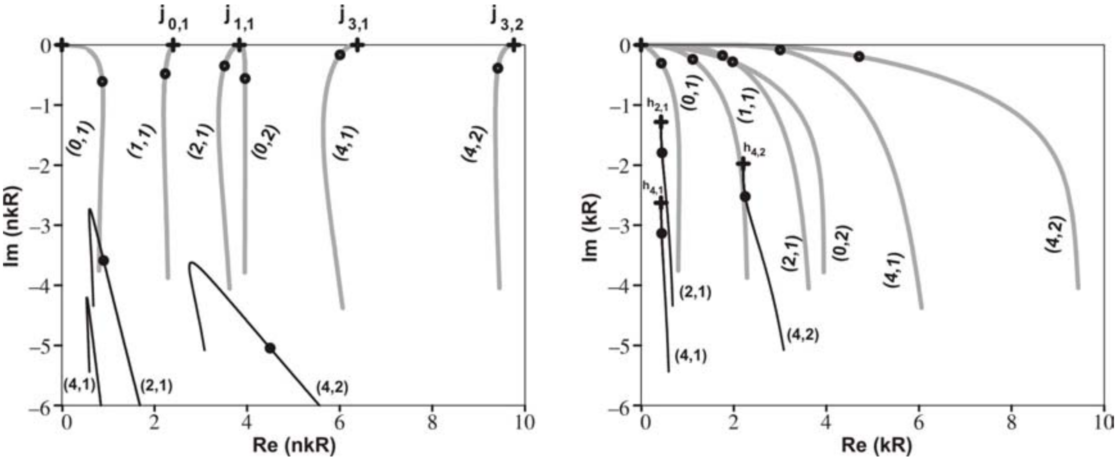

Figure 1. TM internal (thick grey curves) and external (thin black curves) resonances of a dielectric disk of radius R and refractive index n varying from $n = 1.001$ (loose ends) to infinity (crosses) in the complex nkR plane (left panel) and kR plane (right panel). The filled black circles correspond to $n = 2.0$.

As a result, to assign the proper modal indices to TM modes of a circular cavity with the effective refractive index n , one needs to solve Eq. (1) for a fixed azimuthal modal index m , using as initial guesses a fine grid in the complex wavenumber plane. The obtained solutions generate two well-separated (along the imaginary axis) sets of modes: the set with smaller imaginary parts is the set of internal resonances while the set with larger imaginary parts is the set of external ones. Then, one needs to trace one arbitrary resonance from each of those two sets with increasing n till it reaches the limit which unambiguously defines its radial modal index q , see Eqs. (4, 5); the radial modal indices of remaining resonances in each set can be assigned automatically in accordance with the increase (decrease) of their real parts k_r . For TM resonances with relatively small imaginary parts, the physical meaning of the radial modal index q is the number of intensity hotspots in the radial direction inside of the disk. For other TM resonances (all external resonances and internal ones in cavities with the effective refractive index close to 1), the index q has no similar physical interpretation. These resonances are so deep in the complex wavenumber plane that the corresponding Bessel functions, see Eq. (3), have almost no variation inside the disk. This is illustrated in Fig. 2.

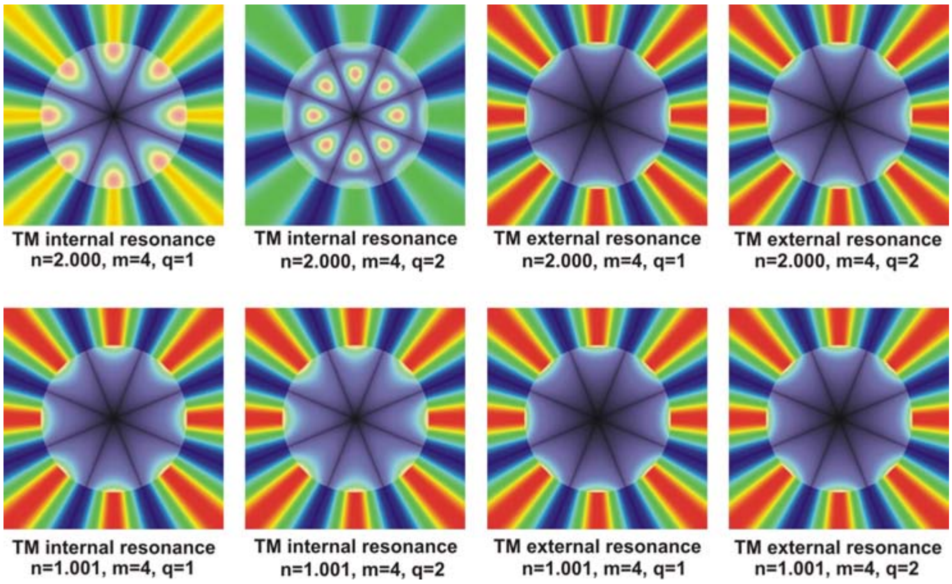

Figure 2. The intensity of TM internal and external resonance modes with the indicated indices in near-field region of the dielectric disk with $R = 1$.

3. TE RESONANCES

In Ref. [3] we have also shown that the scaled wavenumbers $n k_{m,q} R$ of TE internal resonances in the small opening limit satisfy the relation

$$\lim_{n\to\infty} n k_{m,q} R = j_{m,q}, \tag{6}$$

as one intuitively expected. This limit has been derived in Ref. [4] as well. The TE external resonances (not scaled with respect to n) approach the complex zeros $h'_{m,q}$ of the corresponding Hankel functions derivatives

$$\lim_{n\to\infty} k_{m,q} R = h'_{m,q} . \tag{7}$$

Figure 3 illustrates the behaviour of TE resonances for several azimuthal and radial indices under variation of the refractive index.

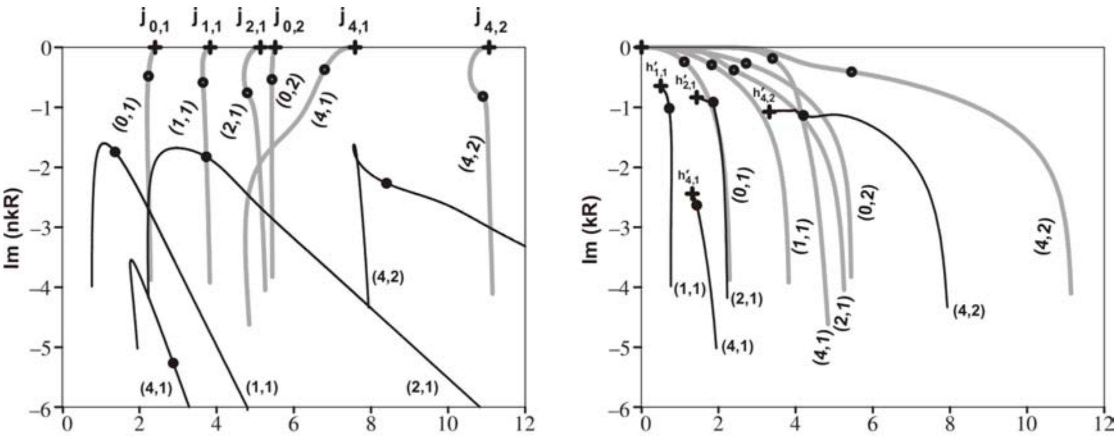

Figure 3. TE internal (thick grey curves) and external (thin black curves) resonances of a dielectric disk of radius R and refractive index n varying from $n = 1.001$ (loose ends) to infinity (crosses) in the complex nkR plane (left panel) and kR plane (right panel). The filled black circles correspond to $n = 2.0$.

As for TM external resonances, there is only a finite number of radial modes in each group of TE external resonances with fixed m : none if m is 0, $m/2$ if m is even, $(m+1)/2$ if m is odd.

There is an additional interesting feature of TE external resonances. For the external resonances $k_{m,(m+1)/2}$ with odd m and $k_{m,m/2}$ with even m, starting from $m=3$ and $m=4$ correspondingly, there is a range of refractive indices where they are mixed with TE internal resonances, see for example the external resonance curve $(4,2)$ in Fig. 3. Their field intensities display some features of the internal resonances as well. We will call them the "special" external resonances. Due to their appearance, to assign the proper radial modal indices to TE resonances one needs to solve Eq. (2) for a fixed azimuthal modal index m and then to trace all obtained solutions with increasing n till they reach their limits, see Eqs. (6, 7).

In Fig. 4 (left panel) we illustrate how the "special" external resonance $(8,4)$ joins the set of internal resonances with same $m=8$. Fig. 4 (right panel) shows field intensities for that "special" external resonance as well as for the internal resonance $(8,2)$ before $(n=2.0)$ and right after $(n=1.4)$ the "special" one joins the internal set. One can clearly see the striking phenomenon of the field pattern change for both of these resonances. In particular, the field intensity of the TE internal resonance $(8,2)$ of the disk with $n=1.4$ has only one intensity peak in the radial direction inside of the disk.

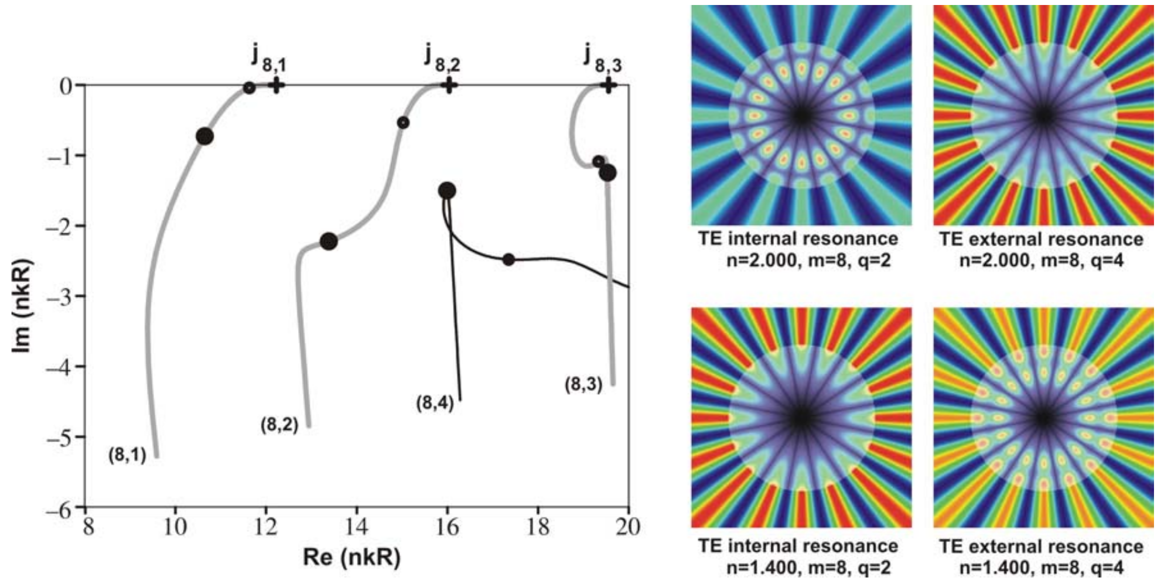

Figure 4. TE internal (grey curves) and the "special" external (a black curve) resonances with $m=8$ of a dielectric disk of radius R and refractive index n varying from $n=1.001$ (loose ends) to infinity (crosses) in the complex nkR plane (left panel). The small filled black circles correspond to $n=2.0$, the large ones to $n=1.4$. The resonance field intensities are shown on the right panel.

4. CONCLUSIONS

The presented analysis assigns unambiguous azimuthal and radial modal indices to each internal and external resonance mode of a circular dielectric cavity. While the double of the azimuthal modal index indicates the number of intensity hotspots along the disk circumference for all possible resonances, the radial modal index has clear physical interpretation (as the number of intensity hotspots along the radial direction) only for internal resonances in cavities with relatively large refractive indices.

REFERENCES

[1] E. I. Smotrova, A. I. Nosich, T. M. Benson, P. Sewell: Cold cavity thresholds of microdisks with uniform and nonuniform gain: quasi-3-D modeling with accurate 2-D Analysis, IEEE Journal of Selected Topics in Quantum Electronics, vol. 11, pp. 1135-1142, Sept./Oct. 2005.

[2] R. Dubertrand, E. Bogomolny, N. Djellali, M. Lebental, and C. Schmit: Circular cavity and its deformations, Phys. Rev. A, vol. 77, pp. 013804(+16), Jan. 2008.

[3] C. P. Dettmann, G. V. Morozov, M. Sieber, H. Waalkens: Internal and External Resonances of Dielectric Disks, arXiv: 0903.4718v1 [physics.optics], March 2009.

[4] J.W. Ryu, S. Kim, Y. J. Park, C. M. Kim, S. Y. Lee: Resonances in a circular dielectric cavity, Physics Letters A, vol. 372, pp. 3531-3536, Feb. 2008.

[5] A. Erdelyi: Higher transcendental functions Vol. II, New York: McGraw-Hill, 1953.