Algebraic tunings

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We propose an approach to tuning systems in which octave doubling ratio is replaced by a suitable algebraic unit $\tau$, and note frequencies are proportional to a subset of the ring $\mathbb{Z}[\tau]$. Then it is possible for many difference tones between notes in the tuning to also appear in the tuning. After outlining more general principles, we consider in detail some natural examples based on the golden ratio $g = (1 + \sqrt{5})/2$, limited by norm or by the number of digits in the greedy $\beta$-expansion. We discuss additive and multiplicative properties, implementation and composition using these tunings. The Online Supplement contains MIDI and websynths files to implement the tuning $S_5^{\beta}(g)$ (based on $\beta$-expansions to $g^{-5}$) on websynths.com and a composition Three Places.

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1. Introduction

Conventional modern Western music is based on a 12 tone equal temperament (12TET) tuning in which consecutive pitches differ by a ratio $2^{1/12} \approx 1.059$, so that 12 of these smallest intervals (semitones) give a ratio of 2 (conventional octave). This choice allows music to be transposed exactly into any key. Rational frequency ratios other than powers of 2 are also available, albeit only approximately, for example 7 semitones gives $2^{7/12} \approx 1.498 \approx \frac{3}{2}$. For music using only a few keys, including much composed prior to 1700, tuning in other temperaments may be preferred, in which these intervals are exact or closer to rational approximations; see Ref. Lindley (2001). This is particularly an issue for keyboard music; performers of many other instruments (including voice) are able to a greater or lesser extent adjust the pitch during performance.

One motivation for temperaments using exactly rational frequency ratios is that of difference tones Greated (2001). Due to nonlinearities in the ear, these tones may be perceived when intervals or chords are played. Their frequency is the difference, or another simple linear combination, of the original frequencies, the latter termed “combination tones.” If the frequencies in the scale are exact rational ratios, difference tones will then often correspond to other frequencies in the scale. A recent discussion of difference tones and their use in electronic music composition may be found in Ref. Chechile (2020).

There are also many tuning systems quite different from 12-TET, either from many...
non-Western musical traditions (though it should be noted that 12TET originated in China Gene (2010)) or of more recent origin, in theory, instruments and composition. These are often termed “microtonal” though the intervals may not be smaller than in 12TET. Intervals are still typically measured in cents. As standard in music, the interval corresponding to frequency ratio $f_2/f_1$ is defined to be

$$1200 \log_2 \left( \frac{f_2}{f_1} \right)$$  \hspace{1cm} (1)

cents, so that a 100 cent interval is a 12TET semitone.

Perception of octave equivalence for a frequency ratio of 2 is not universal, for example the Tsimané people of Bolivia do not appear to have this (or any other) octave equivalence Jacoby et al. (2019). This suggests that the human ear may become accustomed to other interval equivalence.

The ninth century treatises Musica enchiriadis and Scolica enchiriadis Erickson (2001) use “dasian” notation which has equivalence at a perfect fifth (frequency ratio 3/2), and a 3-limit tuning system, that is, frequency ratios that are multiples of powers of 2 and 3. This fits naturally with monophonic chant, and with parallel organum using an interval of a perfect fifth, but not with parallel organum with a fourth (ratio 4/3) or with one or more parts doubled at an octave (ratio 2), also common in this period as described in these treatises.

A notable recent example of a tuning system with a different octave is the Bohlen-Pierce (BP) scale Mathews et al. (1988), where consecutive pitches differ by a ratio $3^{1/13}$. This is periodic with a frequency ratio of 3 (“tritave”) and has intervals approximating ratios involving 3, 5 and 7. This makes it well suited for instruments with strong odd harmonics, such as the clarinet, and BP-tuned instruments are available commercially. Interestingly, one motivation for this scale was that of difference tones Mathews et al. (1988).

The aim of this work is to use the above ideas for generating tuning systems, namely difference tones and non-standard octaves, in the context of algebraic number theory. Namely, we observe that if the octave periodicity is an algebraic unit $\tau$, and frequencies are proportional to elements of the corresponding ring $\mathbb{Z}[\tau]$, then difference tones also lie in the ring. The simplest case is the golden ratio $\tau = g = (1 + \sqrt{5})/2 \approx 1.618$. The frequencies involve two integers, being proportional to $ag + b$ where $a, b \in \mathbb{Z}$, and in this sense have the same complexity as the rational numbers. This will be our main example, but we also motivate and develop this approach in more generality.

First, we discuss some relevant approaches in the existing literature. Walter O’Connell O’Connell (1993) noted that $2^{25} = 33554432$ is close to $g^{36} \approx 33385282$, so that dividing each semitone in thirds (that is, 36TET) yields an interval (25/3 semitones, or close to 833 cents) very close to the golden ratio, motivated as below by sum and difference tones. A very similar scale is obtained by splitting a golden octave (frequency ratio of $g$) into 25 equally spaced intervals. He then described compositions based on pentachords noting the factorisation of 25. Frequency ratios of 2, $\sqrt{5}$ and 3 were highlighted. In the present work the main ratio is $g$, although the above ratios also appear.

Some related history and scales can be found in Smethurst (2016). Of note is the Bohlen 833 scale, which does not appear otherwise published except on websites such as Bohlen (2012). This scale consists of a fundamental and its 2, 3 and 4 harmonics and subharmonics, repeated every golden octave. Unlike the equally tempered approach of O’Connell, this scale now has closest intervals of three different sizes. It also has several difference tones in the scale, though the distinct sums and differences are not much less than the generic value of $n(n + 1)/2$ for $n$ notes; see Fig. 2 below.
In this paper we continue further in this direction, constructing scales with many difference tones equal to each other and/or contained in the scale. This has the effect that almost all intervals (i.e., multiplicative ratios) are different.

In section 2, we begin by generalising the difference tone rationale, and discuss the algebraic number systems and choice of the octave periodicity ratio $\tau$. Section 3 contains relevant properties of $g$ and similar algebraic numbers. Section 4 defines the scales we propose, based on $g$. Section 5 considers their additive properties (i.e., difference tones) and section 6 their multiplicative properties (i.e., intervals). Section 7 gives a relation to an interesting open mathematical problem, the sum-product conjecture. Section 8 gives considerations in implementing and experimenting with these scales, and section 9 some ideas for composition, and discussion of a first composition, *Three Places*.

2. Octave equivalence using algebraic units

Define a scale $\hat{S} \subset \mathbb{R}_{>0}$ as a nonempty set of positive real numbers representing frequencies, measured in Hz. One element $f \in \hat{S}$ denotes the “fundamental” frequency. We consider $S = \{x : fx \in \hat{S}\}$ for now; this is equivalent to setting $f = 1$. In Sec. 4, we will choose another convenient value. We also assume that $S$ is locally finite away from zero.

We require that $S$ be log-periodic, that is, the set of logarithms of elements of $S$ has period $\ln \tau$ for some real number $\tau > 1$. In other words, $\forall x \in S$, $\tau x \in S$ and $\tau^{-1} x \in S$; the pitch class of $x$ is the set $\tau^n x$ for $n \in \mathbb{Z}$. In most traditional scales, $\tau = 2$, corresponding to the octave, but here it will differ. We choose $\tau$ to be the smallest such value, since $S$ is also invariant under multiplication and division by integer powers of $\tau$. Though $S$ is locally finite away from zero by assumption, the log-periodicity implies that $S$ accumulates at zero.

We now introduce some standard definitions in algebraic number theory. We denote $\mathbb{Z}[\tau, \tau^{-1}]$ to be the minimal set containing $\{1, \tau, \tau^{-1}\}$ and closed under addition, subtraction and multiplication. This consists of any integer linear combination of arbitrary positive and negative powers of $\tau$, and so is in general not a finitely generated space over $\mathbb{Z}$. However, if $\tau$ is an algebraic integer of degree $d$ i.e. has minimal polynomial of degree $d$ with integer coefficients and leading coefficient 1, then all powers $\geq d$ in the sum can be written in terms of powers $0 \ldots d - 1$. If in addition, $\tau$ is an algebraic unit, i.e. its minimal polynomial also has constant term $\pm 1$, then all negative powers in the sum can be written in terms of the powers $0 \ldots d - 1$. In this case $\tau^{-1}$ is in $\mathbb{Z}[\tau]$ and we can use the latter notation rather than $\mathbb{Z}[\tau, \tau^{-1}]$. The space now has dimension $d$ over $\mathbb{Z}$ and a good candidate to construct scales with (in some reasonable sense) many difference tones in the scale.

One more standard definition needed here is that of the (field) norm of an element of $\mathbb{Z}[\tau]$. Multiplication by an element $x \in \mathbb{Z}[\tau]$ corresponds to a $d$-dimensional linear operator over $\mathbb{Z}$. The norm $N(x)$ is the determinant of this operator, and hence is multiplicative, i.e. $N(xy) = N(x)N(y)$. Strictly speaking, the norm is defined relative to the original ring $\mathbb{Z}$ and extension $\mathbb{Z}[\tau]$ but these are clear in the context and omitted from the notation.

Later we will need the norm in $\mathbb{Z}[g]$, where $g = (1 + \sqrt{5})/2$ is the golden ratio. The minimal polynomial $x^2 - x - 1$ has highest and lowest terms with coefficient $\pm 1$, so $g$ is an algebraic unit. If we write $x \in \mathbb{Z}[g]$ as $x = ag + b$, then $g^2 = g + 1$ implies that...
$xg = (a + b)g + a$ and the determinant of the multiplication operator is

$$N(ag + b) = \begin{vmatrix} a + b & a \\ a & b \end{vmatrix} = b^2 + ab - a^2$$  \hspace{1cm} (2)$$

In conventional just intonation scales, the octave equivalence is $\tau = 2$ (not an algebraic unit). Frequencies are of the form $p^1_1 / p^k_2 \ldots / 2^q$ where the numerator is typically given as a product of small non-negative powers of primes (for example $2, 3, 5$ for 5-limit tuning). Combining the numerator into a single integer $p$, we can reduce the number of required integer parameters to two, that is, frequencies are dyadic rational numbers of the form $p / 2^q \in \mathbb{Z} \left[ \frac{1}{2} \right]$. The dasian scale discussed in the introduction has $\tau = 3/2$ (not an algebraic integer). Frequencies are of the 3-limit form $2^{k_1} 3^{k_2} = p/6^q \in \mathbb{Z} \left[ \frac{1}{6} \right]$, where now $k_1$ and $k_2$ are integers satisfying $5 \leq k_1 + k_2 \leq 8$. Representing either in terms of $p$ and $q$ or $k_1$ and $k_2$ there are two integer parameters.

The same number of parameters is required for $\tau$ a degree 2 (i.e. quadratic) algebraic unit. These are solutions to $x^2 - ax \pm 1 = 0$ for integer $a$, thus those greater than unity are of the form $(a + \sqrt{a^2 \pm 4})/2$, and easily enumerated. Those less than 3 are the golden ratio $g \approx 1.618$, the silver ratio $s = 1 + \sqrt{2} \approx 2.414$, and also $g^2 \approx 2.618$. The degree 3 (i.e. cubic) units in contrast are dense unless there is an additional condition such as the Pisot property discussed in the next section.

3. The golden ratio, properties and generalisations

In this paper we will focus on the golden ratio $g$ as the ratio for octave equivalence. This section describes some other properties of $g$, that are relevant in selecting algebraic units on which to base alternative scales, and in selecting elements of $\mathbb{Z} \left[ \tau \right]$ to include in the scale.

Algebraic numbers give the asymptotic growth of solutions of linear recurrences with coefficients from their minimal polynomial. For $g$, this is the Fibonacci sequence defined by $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$. It is easy to show (for example by induction) that

$$g^n - (-g)^{-n} = F_n \sqrt{5}$$
$$g^n + (-g)^{-n} = F_{n+1} + F_{n-1}$$  \hspace{1cm} (3) \hspace{1cm} (4)$$

Eq. (3) may be used to calculate $F_n$. Eq. (4) shows that for large $n$, $g^n$ is close to an integer; this is due to the following property.

A Pisot number Bertin et al. (2012) is a real algebraic integer greater than 1 with all Galois conjugates (i.e. roots of the minimal polynomial) of complex magnitude less than 1. The distance between the $n$th power of a Pisot number and the nearest integer converges to zero as $n \to \infty$. A positive power of a Pisot number is Pisot. The golden ratio is Pisot; this can be confirmed from the roots of the minimal polynomial. In general there are criteria using inequalities for the coefficients of the minimal polynomial; see Thm. 2.2 of Ref. Akiyama and Gjini (2005). The quadratic and cubic Pisot units less than 3 are tabulated in Tab. 1. This table also includes information about the $\beta$-expansion (see below) and common names for the golden ratio and some of the others.

A $\beta$-expansion Charlier, Cisternino, and Dajani (2021) is an expression for a real
The table shows Pisot units of degree 2 and 3, of magnitude less than 3, and not powers of smaller Pisot units. The $\beta$-expansion of one (defined below) is $d_\beta(1)$ where the over bar denotes a periodic sequence of symbols. “Symbol” gives the notation in this paper; notation varies in the literature. Cyclotomic Pisot numbers (which also include $g$ and $s$) are discussed in Refs. Dettmann and Frankel (1993); Bell and Hare (2005).

### Table 1. Pisot units of degree 2 and 3, of magnitude less than 3, and not powers of smaller Pisot units.

<table>
<thead>
<tr>
<th>Value</th>
<th>Minimal polynomial</th>
<th>$d_\beta(1)$</th>
<th>Name(Symbol)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.32472</td>
<td>$x^3 - x - 1$</td>
<td>0.10001</td>
<td>Plastic($p$)</td>
</tr>
<tr>
<td>1.46557</td>
<td>$x^3 - x^2 - 1$</td>
<td>0.101</td>
<td>Supergolden</td>
</tr>
<tr>
<td>1.61803</td>
<td>$x^2 - x - 1$</td>
<td>0.11</td>
<td>Golden($g$)</td>
</tr>
<tr>
<td>1.83929</td>
<td>$x^3 - x^2 - x - 1$</td>
<td>0.111</td>
<td>Tribonacci</td>
</tr>
<tr>
<td>2.20557</td>
<td>$x^3 - 2x^2 - 1$</td>
<td>0.201</td>
<td>Cyclotomic-7</td>
</tr>
<tr>
<td>2.24698</td>
<td>$x^3 - 2x^2 - x + 1$</td>
<td>0.2011</td>
<td>Silver($s$)</td>
</tr>
<tr>
<td>2.54682</td>
<td>$x^2 - 2x - 1$</td>
<td>0.211</td>
<td></td>
</tr>
<tr>
<td>2.69292</td>
<td>$x^3 - 2x^2 - x - 1$</td>
<td>0.2201</td>
<td></td>
</tr>
<tr>
<td>2.83118</td>
<td>$x^3 - 2x^2 - 2x - 1$</td>
<td>0.221</td>
<td></td>
</tr>
<tr>
<td>2.87939</td>
<td>$x^3 - 3x^2 + 1$</td>
<td>0.222</td>
<td>Cyclotomic-9</td>
</tr>
</tbody>
</table>

Number in powers of an arbitrary real $\beta > 1$,

$$x = \sum_{j=-\infty}^{j_{\text{max}}} c_j \beta^j$$  \hspace{1cm} (5)

where $c_j \in \mathbb{Z} \cap [0, \beta)$. In general this is non-unique: there are many possible $\{c_j\}$ that satisfy this equation. The greedy $\beta$-expansion is obtained by starting from the largest possible $j_{\text{max}}$ and choosing the largest possible $c_j$ for each $j$ decreasing towards $-\infty$. The $\beta$-expansion of 1, denoted $d_\beta(1)$ is the greedy $\beta$-expansion starting from $j = -1$. For Pisot $\beta$ it is known to be finite or repeating; see Tab. 1. The greedy condition can be written that no consecutive set of $c_j$ is lexicographically greater or equal than $d_\beta(1)$. For $\tau = g$ we have $d_\beta(1) = 0.11$ and leads to the simple condition $c_j c_{j+1} = 0$, that is, that the finite sequences of $c_j$ in the greedy $\beta$-expansion are exactly those without consecutive 1s (also, similar to the infinite trailing sequence of 9s that does not occur in decimal expansions, the $\beta$-expansion excludes an infinite trailing sequence of 01). Here, we use $\beta$-expansions as an approach to deciding what elements of $\mathbb{Z}[\beta]$ to include in the scale.

Finally, we mention the Diophantine properties of the golden ratio and related algebraic units (see for example Rockett and Szusz (1992)), in other words, how close are intervals to rational frequency ratios appearing in conventional (just intonation) music? It turns out that the golden ratio is, in a precise sense, maximally irrational, that is, badly approximable by rationals. Hurwitz’s theorem states that for any irrational $\tau$, the equation

$$|\tau - \frac{p}{q}| < \frac{K}{q^2}$$  \hspace{1cm} (6)

has infinitely many integer solutions for $(p, q)$ if $K \geq \frac{1}{\sqrt{5}}$. If $\tau = \frac{a+bg}{c+dg}$ for integers $a, b, c, d$ with $|ad - bc| = 1$ then this bound is sharp. If $\tau = \frac{a+bs}{c+ds}$ with $|ad - bc| = 1$ with $s = 1 + \sqrt{2}$ (the silver ratio) then the bound $K \geq \frac{1}{\sqrt{8}}$ is sharp. All other irrationals are better approximated by rationals, in that the bound may be replaced by $K \geq \frac{5}{\sqrt{221}} = \frac{1}{\sqrt{221}}$.
Each quadratic irrational has its own bound; these are exactly the numbers with eventually periodic continued fraction expansions. There is a quadratic irrational ratio in 12TET, namely six semitones gives a frequency ratio $2^{6/12} = \sqrt{2}$.

However, for algebraic numbers of higher degree, the continued fraction expansions appear to have similar properties to generic real numbers but less is known; it is conjectured that $K$ may be made arbitrarily small in Eq. (6) and it is known that replacing $q^2$ by $q^2+\epsilon$ leads to only finitely many solutions Roth (1955). In summary, cubic units have more irregular approximation properties than quadratic units. For musical purposes, the ear can distinguish only the first few approximations, for example the plastic number $p$ is 11 cents from $4/3$. The next approximant has an error of less than 1 cent, but at $49/37$ is not a simple ratio. Other cubic units have approximations with larger denominators, for example the supergolden number is 1 cent from $22/15$ and the tribonacci number 6 cents from $11/6$. For comparison, the cubic irrationals in 12TET are four and eight semitones, with frequency ratios $2^{1/3}$ and $2^{2/3}$; these are about 14 cents from $5/4$ and $8/5$ respectively, and less than 1 cent from $63/50$ and $100/63$ respectively.

4. Defining scales

Since scales consist of isolated frequencies, we must keep only a finite number of values per $\tau$-octave. There are at least two natural choices based on the definitions we have considered so far. We can use the norm as a bound, that is, define

$$S^N_B(\tau) = \{ x > 0 \mid |N(x)| \leq B \}$$

where the unit $\tau$ appears implicitly in the norm $N$, and the fact it is a unit implies the log-periodicity of the set. Another approach is to note that the set of $x > 0$ with $N(x) > 0$ forms a cone, and if convex (and it is for $g$) is closed under both addition and multiplication and hence suitable for defining a scale $S^B_N(\tau)$, replacing the inequality in the equation above by $0 < N(x) \leq B$; we will not consider this further here, except to note that $S^{36}_N(g)$ has the same number of notes as $S^{36}_g(g)$ and so fits on a MIDI keyboard (see Eqs. 7,8 below).

Alternatively, we can use the greedy $\beta$-expansion and define

$$S^B_\beta(\tau) = \{ x > 0 \mid \exists c_j \text{ greedy, so that } x = \sum_{j=\max}^{j_{\max}-B} c_j \tau^j \}$$

which is again log-periodic by definition. It is possible (but perhaps less natural) to remove the greedy condition, and impose only $c_j < \tau$. We will not consider this further here, except to note that $S^5_\beta(g)_{\text{greedy}} = S^4_\beta(g)_{\text{non-greedy}}$.

We now focus on the golden ratio, and consider two scales in $\mathbb{Z}[g]$ satisfying the previously defined properties. The scale $S^B_\beta(g)$ consists of notes with greedy $\beta$-expansion of $\leq 6$ digits, leading to 8 notes per $g$-octave, and will be denoted $S_\beta$ for brevity. The scale $S^N_N(g)$ contains notes with norms of magnitude $\leq 20$, leading to 10 notes per golden octave, which turns out to be those of $S_\beta$ together with two others (of 7 digit beta representation). This will be denoted $S_N$ for brevity.

As noted in the previous section, the greedy $\beta$-expansion base $g$ is characterised by sequences with no consecutive 1s. The norm is given in Eq. (2) above. We have $N(g) = -1$, so that multiplying or dividing by $g$, corresponding to going up or down by a golden
Table 2. Notes of the golden scale $S_N$. The scale $S_\beta$ consists of those apart from $\alpha^\flat$ and $\gamma^\sharp$. The $\beta$-expansion and norm are to base $g$. The lattice column gives $(a,b)$ for the expression $ag+b$. Transposing by a golden octave (factor of $g$) flips the sign of the norms.

<table>
<thead>
<tr>
<th>Name</th>
<th>Frequency ratio</th>
<th>$\beta$-expansion</th>
<th>Lattice</th>
<th>Exact</th>
<th>Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>1.000000</td>
<td>1</td>
<td>(0,1)</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\alpha^\sharp$</td>
<td>1.05573</td>
<td>1.00001</td>
<td>(-8,14)</td>
<td>$2\sqrt{5}/g^3$</td>
<td>20</td>
</tr>
<tr>
<td>$\beta^\flat$</td>
<td>1.09017</td>
<td>1.000001</td>
<td>(5,-7)</td>
<td>-11</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.14590</td>
<td>1.0001</td>
<td>(-3,6)</td>
<td>$3/g^2$</td>
<td>9</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>1.23607</td>
<td>1.001</td>
<td>(2,-2)</td>
<td>$2/g$</td>
<td>-4</td>
</tr>
<tr>
<td>$\gamma^\sharp$</td>
<td>1.29180</td>
<td>1.001001</td>
<td>(-6,11)</td>
<td>19</td>
<td></td>
</tr>
<tr>
<td>$\delta^\flat$</td>
<td>1.32624</td>
<td>1.001001</td>
<td>(7,-10)</td>
<td>-19</td>
<td></td>
</tr>
<tr>
<td>$\delta$</td>
<td>1.38197</td>
<td>1.01</td>
<td>(-1,3)</td>
<td>$\sqrt{5}/g$</td>
<td>5</td>
</tr>
<tr>
<td>$\epsilon^\flat$</td>
<td>1.47214</td>
<td>1.01001</td>
<td>(4,-5)</td>
<td>-11</td>
<td></td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>1.52786</td>
<td>1.0101</td>
<td>(-4,8)</td>
<td>$4/g^2$</td>
<td>16</td>
</tr>
</tbody>
</table>

Octave, leads to a change of sign in the norm. Using these definitions, the notes of the scales $S_N$ and $S_\beta$ defined above are those presented in Tab. 2.

The naming of the pitch classes uses $\{\alpha, \beta, \gamma, \delta, \epsilon\}$ for numbers represented by at most 5 digits in the $\beta$-expansion. A $\flat$ or $\sharp$ corresponds to taking or adding $g^{-6}$ to these, respectively. For more general $S^B_\beta(\tau)$ or $S^B_N(\tau)$, it is natural to give notes in scales with smaller $B$ separate letters, and accidentals for the rest, but it seems difficult to make from this a precise and workable standard nomenclature.

The $S_\beta$ scale of eight pitch classes can be deployed by retuning a MIDI keyboard, since three golden octaves (24 notes) are close to the same number of semitones, more precisely

$$1200\frac{\log g^3}{\log 2} \approx 2499.27$$

cents, that is, just under 25 semitones. The $S_N$ scale has too many pitch classes for a MIDI keyboard but the advantage that it contains more arithmetic progressions, chords of fixed difference tone frequency.

For a suitable choice of fundamental frequency $f$, there are sequences of the form $fg^n$ which for large $n$ are all close to integers following from the Pisot property of $g$ above. We choose $f$ to be $f = g^{10}/\sqrt{5} \approx 55.0036$ Hz, which due to Eq. (3) is close to a Fibonacci number. The Fibonacci frequency 55 Hz is A1 on the usual 12TET scale, being three octaves below concert pitch (440 Hz=55 x $2^{3}$ Hz). Here we use Greek letters for notes on the golden scale, and denote the fundamental frequency $\alpha_1$. The frequencies for both scales are given in Tab. 3 and a mapping (of $S_\beta$) to a MIDI keyboard is given in Tab. 4; see also the Online Supplement files described at the end of this paper. Note that we have shifted this correspondence by two semitones, so $\alpha_1$ corresponds to B1, in order to reduce the amount of tuning required in higher octaves, and also to balance the most extreme accidentals (C$\flat$ and E$\sharp$).

5. Additive properties - arithmetic sequences

Arithmetic sequences are one of the main motivations for considering scales based on number fields, since they have common difference tones, which may also be in the scale. Looking at the lattice representations of the notes in Tab. 2 it is clear that there are some arithmetic sequences, for example $\{\alpha, \gamma, \epsilon\}$. However, it is not obvious how to identify
all such sequences, given the repetition of the scale in different octaves. If all can be identified for a given fixed scale, it is still not clear how to choose a finite set of pitch classes to maximise (in some precise sense) the number of such sequences.

Whilst we cannot give a definitive answer to these questions here, we can point to a useful approach for identifying arithmetic sequences. In Fig. 1 we plot the notes of the scale with log magnitude \( \ln(a + b) \) on the \( x \)-axis and norm \( b^2 + ab - a^2 \) on the \( y \)-axis. The plot is invariant under transposition by a \( g \)-octave, which corresponds to translation by \( \ln g \approx 0.4812 \) to the right and reflection across the horizontal axis.
Arithmetic sequences are straight lines in \((a, b)\) space, which become suitably curved in these coordinates. We included the most relevant curved segments in Fig. 1; clearly they are in fact dense. The additional two notes in \(S_N\), \(\alpha\frac{5}{3}\) and \(\gamma\frac{3}{2}\), are thus helpful in providing the long arithmetic sequence \(\{\epsilon_1, \delta\frac{2}{3}, \gamma\frac{3}{2}, \epsilon_3, \beta\frac{4}{3}, \gamma_4, \delta_4, \epsilon_4\}\) and in extending two of the others.

### 6. Multiplicative properties - intervals

Given the emphasis on additive properties, that is, many difference tones are equal, it is not surprising that with regard to multiplicative properties, the reverse is true, that is, almost all ratios (corresponding to musical intervals) are different. The intervals, up to three golden octaves, are given in Tab. 5.

Tab. 2 illustrates this, in that since the norm is multiplicative, the ratios are seen to differ for almost all combinations. Even where two norms have magnitude 11 or 19, the ratios are not algebraic integers, since the only units in \(\mathbb{Z}[g]\) differ for almost all combinations. Even where two norms have magnitude 11 or 19, the ratios are not algebraic integers, since the only units in \(\mathbb{Z}[g]\) are powers of \(g\).

Tab. 2 also illustrates some special intervals using the “exact” column. There are rational intervals with ratios 2 (\(\alpha_1-\gamma_2, \gamma_1-\epsilon_2, \delta_1-\alpha_3\)), 3 (\(\alpha_1-\beta_3\)) and related combinations (4/3, 3/2, 4), as well as maximally irrational intervals \(g\) (all notes up a golden octave) and \(\sqrt{5}\) (\(\alpha_1-\delta_2, \gamma_1-\alpha_3\)). Combining \(g\) with the other intervals also yields combinations that occur more than once, for example a golden third of ratio \(2/g = \sqrt{5} - 1 \approx 1.236\), is found for \(\alpha_1-\gamma_1, \gamma_1-\epsilon_1\) and \(\delta_1-\alpha_2\). This interval lies between that of minor and major thirds, for example the just minor third \(6/5 = 1.2\) and 12TET minor third \(2^{\frac{1}{4}} \approx 1.189\) and just major third \(5/4 = 1.25\) and 12TET major third \(2^{\frac{1}{3}} \approx 1.260\). Similarly for other intervals such as a golden fourth \(g^2/2 \approx 1.309\) which is a little less than just perfect fourth \(4/3 \approx 1.333\) and 12TET perfect fourth \(2^{\frac{5}{12}} \approx 1.335\) and corresponds to \(\gamma_1-\alpha_2, \epsilon_1-\gamma_2\) and \(\alpha_1-\delta_1\). The only geometric sequences are \(\alpha a-b-\epsilon c\) and \(\xi a-\xi b-\xi c\) where \((a, b, c)\) are integers in arithmetic progression (including all equal) and \(\xi\) is any pitch class.

Eq. (3) gives some large “approximately integer” intervals. For \(\delta_1-an\) we have a fre-
<table>
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Table 5. Intervals in the golden scales, in cents.

The frequency ratio $g^n/\sqrt{5}$, and for large $n$ the $g^{-n}$ can be neglected leading to a Fibonacci interval. For example $n = 6$:

$$\frac{g^6}{\sqrt{5}} \approx 8.0249$$  \hspace{1cm} (9)

which is about 4 cents above three conventional octaves.
Figure 2. Sum-product phenomenon for the $S_N$ scale (upper), $S_\beta$ scale (middle) and Bohlen 833 scale (lower). The number of distinct sums, positive differences, and products are shown in blue, orange and green, respectively, plotted against the number of notes used. See Sec. 7.

7. The sum-product conjecture

The observation referred to above, that there are few differences suggests that there are many intervals, is close to a well known problem in the mathematical literature. The sum-product phenomenon asserts that for a finite set $A$ in a suitable ambient space (here $\mathbb{R}$), at least one of the sum set $A + A$ and the product set $AA$ should be large. One form of this is the Erdős-Szemerédi sum-product conjecture which states that for all large $|A|$
we have

$$\max\{|A + A|, |AA|\} \geq |A|^{2 - o(1)}$$

The best rigorous bound has an exponent of 1.335 Rudnev and Stevens (2022).
Our scales have relatively few sums but many products, whilst the closest relative, the Bohlen 833 scale (see the introduction), has fewer products. This is depicted in Fig. 2. Note that our scales have fewer sums and differences as expected, with a transition to quadratically increasing behaviour at around five golden octaves, the precision of the $\beta$-expansion. For seven octaves of $S_N$, we have $|A| = 70, |A + A| = 516, |AA| = 678$, so both $|A + A|$ and $|AA|$ are less than $|A|^{1.54}$. This suggests that a growing sequence of sets along these lines (for example, all elements of a ring with bounds on magnitude and norm) may be a possible example to test the sum-product conjecture.

8. Implementing the scales

It is possible to synthesize sounds of arbitrary waveform and frequency electronically, though many software packages assume octave equivalence for powers of 2, even where notes within each octave may be tuned individually. One option for either computer or MIDI keyboard is websynths.org; Tabs. 3, 4 provide the relevant mappings; see the Online Supplement files.

It may also be possible to retune stringed instruments. Guitars with retuned strings and moved frets are commonly used in microtonal music Nielsen (2003). Similarly for bowed stringed instruments or the piano (using the above keyboard mapping), though we have not yet tried and cannot vouch for the safety. Similarly, it is possible using instruments allowing arbitrary pitch including trombone and voice.

Musical instruments apart from synthesized sine waves generate harmonics other than the fundamental frequency. For those with uniform strings or columns of air, these are at close to integer multiples of the fundamental, which is one of the main explanations for rational frequency ratios in music. These however do not correspond to notes on the golden ratio scales except where the fundamental is $\alpha$ or $\delta$.
It is possible to design strings to have specified overtones by using variable thickness. Unfortunately it does not seem practical in this way to generate all the powers of $g$, which would thus lie in the scale for all the notes; see Sethares and Hobby (2018). As a simpler case, we have considered a string of length $L$ and mass $m_S = \mu SL$ with a small weight at a point $xL$ along it of mass $m_2$ using the formalism in the above paper; see Eqs. (1-4) there. More specifically, we have a string of three segments with linear mass densities $\mu_j$ and lengths $l_j$ where $l_1 = xL, l_3 = (1 - x)L, \mu_3 = \mu_1 = \mu_S$ and $\mu_2 = m_2/l_2$. Taking the limit $l_2 \to 0$ with $m_2$ fixed, the mode equation with appropriate boundary conditions (Eq. 4 in the above paper) reduces to

$$\frac{\sin \tilde{\omega}}{\tilde{\omega}} = \frac{m_2}{m_S} \sin(\tilde{\omega}x) \sin(\tilde{\omega}(1 - x))$$

where $\tilde{\omega} = \omega \sqrt{m_S/T}$ is a dimensionless frequency and $T$ is the string tension. Optimising to find $x = 0.116934$ and $m_2/m_S = 0.519991$, the frequencies of vibration relative to the fundamental become $\{1, 1.618, 2.618, 3.812, 5.037, 6.271, \ldots\}$. In other words, it is possible with only a single additional mass to make the harmonic series of a string start with frequency ratios $1 : g : g^2$. 

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Alternatively, geometric sequences of vibration modes may be created in fractal structures; see Strichatz (1999). Note that irrational harmonics lead to aperiodic wave-forms; for harmonics of sufficiently large amplitude, they may have interesting non-differentiable or fractal properties, along the lines of Weierstrass’s original construction of a nowhere-differentiable function Kaplan, Mallet-Paret, and Yorke (1984).

9. Composition

Some ideas and principles can be found in the previous literature on (especially) tunings with a golden ratio octave (refer to the introduction). The main characteristics of the tunings considered here are the arithmetic progressions and many different intervals.

The arithmetic progressions have common difference tones (refer to Fig. 1), and can be used for scales and chords. The long progression containing $\beta_9$ and $\phi$ is relatively close to the harmonic series (the frequencies are in ratios $n - g^{-4}$ for integer $n$).

The large variety of intervals can also be used to great effect. For example, the dissonant interval $\epsilon_1 - \beta_3$ of just greater than an octave at 1232 cents can resolve to the perfect octave $\alpha_1 - \gamma_2$, where the $\gamma_2$ is a perfect fifth below the $\beta_3$. The golden third and sixth intervals are paradoxical in that thirds and sixths are normally considered consonant, but these are as irrational as possible, in the sense given in Sec. 3.

Ideally, composition using the golden tuning should be on its terms, rather than imitating well known 12TET approaches. The 12TET tuning is homogeneous and cyclic: Each note is equivalent, and keys are arranged in the well known circle of fifths. In contrast, the golden tunings are based on a linear backbone of $\alpha$ notes, decorated by the other notes, each with its own place and character, at increasing distance from the backbone. Distance here can refer to the number of digits needed in the $\beta$-expansion, resulting in the ordering $\alpha$, $\delta$, $\gamma$, $\{\beta, \epsilon\}$, $\{\beta_9, \delta_9, \phi\}$, $\{\alpha_9, \beta_9, \gamma_9, \delta_9, \epsilon_9\}$, $. . .$. Or, it can refer to the norm, which gives the similar ordering $\alpha$, $\gamma$, $\delta$, $\beta$, $\{\beta_9, \phi\}$, $\epsilon$, $\{\gamma_9, \delta_9\}$, $\alpha_9$, $\ldots$.

The parallel organum common in ninth century chant mentioned in the introduction also works here, with the perfect fourth or fifth replaced by the golden ratio octave. We could alternatively (or as well) use two golden ratio octaves, which is 34 cents below a 12TET perfect eleventh.

The composition Three Places (in the Supplemental Online material) is written using the keyboard mapping of $S_\beta$ shown in Table 4. The intention was to approach the tuning system entirely intuitively, allowing the ear to guide the process and pick out interesting chords and pitch combinations. Given the lack of octave periodicity, the piece is formed from three localised musical ideas and their interactions. Each musical idea is described as a ‘place’, and whilst the notes contained in each idea are relatively consonant, the transition from one idea to the next can be abrupt and surprising. With music composed using unfamiliar tuning systems, a ‘settling in’ period of adjustment can be helpful to the listener, provided in Three Places by the oscillating tones at the opening of the piece.

10. Conclusion

Golden ratio scales have led us to wide vistas of mathematics, including many aspects of algebraic number theory, the sum-product conjecture, and non-differentiable curves. There are many open mathematical questions, for example, if we constrain the number of notes per octave, are these scales optimal from the point of view of the number of arithmetic sequences or of sum or difference tones in the scale? From a musical point
of view there remains the challenging task of further developing relevant principles of melody and harmony. The same approach, defining scales by bounding the $\beta$-expansion or norm, can be applied to other algebraic units, such as those in Tab. 1.

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Supplemental Online material

Supplemental Online material for this article can be accessed at TBC. The Online Supplement contains the following files, unless otherwise specified using the $S_\beta$ tuning described in this paper; see Tables. 3 and 4 above:

- **Golden_Ratio(precise).txt**
  MIDI file to implement the $S_\beta$ tuning
- **Golden_Flute_Beta.websynths-patch.json**
  to implement the $S_\beta$ tuning on a MIDI keyboard at https://www.websynths.com/microtonal/
- **Golden_Flute_Norm.websynths-patch.json**
  to implement the $S_N$ tuning on a QWERTY keyboard at https://www.websynths.com/microtonal/
- **Three Places - Full Score.pdf**
  Score for composition *Three Places*
- **Three Places.mp3**
  Audio file for composition *Three Places*

Disclosure statement

No potential conflict of interest was reported by the authors.

References


