Recent extensions of periodic orbit theory

C. P. Dettmann

School of Mathematics, University of Bristol

in collaboration with P. Cvitanović, P. Dahlqvist, R. Mainieri, G. P. Morriss, G. Palla, N. Søndergaard, G. Váttay and A. Voros

Classical periodic orbit theory

Classical periodic orbit theory [1] is a formalism for relating long time properties of classical chaotic systems (averages, Lyapunov exponents, correlation decay rates) to unstable periodic orbits. It is exact, given all the periodic orbits, in contrast to semiclassical descriptions such as the Gutzwiller trace formula, which are valid to lowest order in Planck's constant. Periodic orbit theory is optimal for low dimensional hyperbolic systems with finite Markov partition; many examples of transient chaos fall into this category, and are also difficult to simulate directly.

Why periodic orbits?

Periodic orbits give a natural description of chaotic dynamics, since

- They and their linear stability are independent of a choice of coordinates, unlike cutting phase space into cubes.
- Their properties can be computed from a finite trajectory, unlike brute force computation of an average.
- They come naturally from the mathematical formalism (below)

The periodic orbits form a set of zero measure, but this is not a problem as long as they are dense (usually a good assumption). A good analogy is the computation of an integral by evaluating the function only at rational values.

Finding periodic orbits

There are several methods, depending on the type of system considered:

- If all Lyapunov exponents are positive, inverse iteration.
- If approximate orbits known, multi-point Newton method.
- If one orbit already found, look for a homoclinic tangle, ie intersections of stable and unstable manifolds.
- If all orbits are required, modify dynamics to stabilise.
- If all orbits on an attractor are required, run a long trajectory, search for recurrences, and use one of the above methods to refine the result.

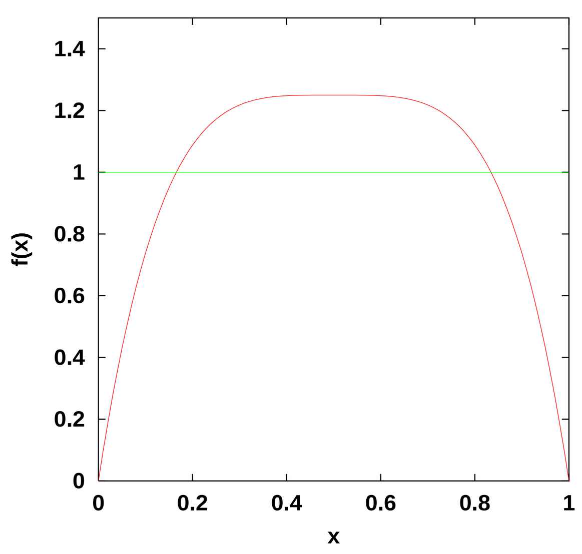

FIGURE 1: Example map $f(x) = 5[1-(1-2x)^4]/4$

Example of the theory

Let us compute the escape rate for some open chaotic 1D map such as $x_{n+1} = f(x_n)$ where $f(x)$ is given in Fig. 1. Only a Cantor set of initial conditions remain in the unit interval for infinite time; all other initial conditions are removed at some finite time. We will see how periodic orbits arise naturally in the elegance of the basic theory. Extra features (averages, Lyapunov exponents, correlation decay, higher dimensions, flows) can all be achieved by modest modifications.

In an open chaotic map, we expect that given a uniform distribution of initial conditions, the number $N(n)$ remaining at discrete time n decreases exponentially, leading to a limit

$$\gamma \equiv -\lim_{n\to\infty} \frac{\ln N(n)}{n}$$

where γ is the escape rate. However, the exponential rate of decay is related to the largest eigenvalue of the Perron-Frobenius operator

$$(\mathcal{L}\rho_n)(y) \equiv \rho_{n+1}(y) = \int \delta(y - f(x))\rho_n(x)dx$$

which describes the evolution of a probability density $\rho(x)$ on an interval containing the repeller but excluding the attractor at $x = -\infty$. In particular $\gamma = -\ln z$ where z is the reciprocal of the leading eigenvalue. We have "reduced" a nonlinear problem in one dimension to a linear problem in the infinite dimensional space of functions $\rho(x)$.

At this point we want to use the characteristic equation $0 = \det(1 - z\mathcal{L})$ and solve for z, but need to define this determinant of an integral operator. We use the matrix formula $\ln\det M = \mathrm{tr}\ln M$, and expand in powers of z, since the leading eigenvalue corresponds to minimum $|z|$:

$$0 = \det(1 - z\mathcal{L}) = \exp(\mathrm{tr}\ln(1 - z\mathcal{L})) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{n}\mathrm{tr}\mathcal{L}^n\right)$$

$$= 1 - z\mathrm{tr}\mathcal{L} - \frac{z^2}{2}\left[\mathrm{tr}\mathcal{L}^2 - (\mathrm{tr}\mathcal{L})^2\right] + \ldots$$

We cut this expansion at a fixed power z^n; rapid convergence in n will be achieved if expressions such as the one in brackets $[\ldots]$ are small due to cancellations.

The trace of $\mathcal{L}^n$ has a simple meaning: iterate the dynamics n times, set the initial and final points equal, and sum. The result is a sum over periodic points of length n. Thus we have

$$\mathrm{tr}\mathcal{L}^n = \int \delta(x - f^n(x))dx = \sum_{x:f^n(x)=x} \frac{1}{|1 - \Lambda(x)|}$$

where we have used the standard formula for integrating the delta function, and the linear stability is $\Lambda(x) = (d/dx)f^n(x)$.

To summarise, we find all periodic orbits up to length n, construct the traces and solve for the reciprocal of the leading eigenvalue z. The rapid convergence in this case is illustrated in the first column of Tab. 1 below. It is a result of "shadowing", that is, long periodic orbits are well approximated by combinations of shorter orbits. In the optimal case, we find that convergence is better than exponential in the orbit length n.

Intermittency

The above example makes it clear that in cases where $\Lambda = 1$ for some periodic orbit (for example intermittency or bifurcations), there is trouble. It turns out that a finite number of such points can be ignored, leading to correct results but slower convergence.

A typical 1D example is a map which, near $x = 0$, has the form

$$f(x) = x(1 + 2|2x|^\alpha) + o(x^{1+\alpha})$$

with $\alpha > 0$, hence $\Lambda(0) = 1$. Simple arguments lead to $\Lambda \sim n^{1+1/\alpha}$ for a sequence of periodic orbits near $x = 0$. Compare this with a hyperbolic system, with $\Lambda \sim e^{n\lambda}$. The terms in the expansion are of order Λ^{-1}. This means that for results accurate to $\epsilon \ll 1$ in an intermittent system, we need to include some long orbits, $n \approx \epsilon^{-\alpha/(\alpha+1)}$. But the number of orbits up to length n grows exponentially with n, so very many orbits need to be included in the sum, as we previously defined it. A number of techniques have been developed for this situation:

Stability ordering

First we reorder the expansion by stability Λ rather than length n, that is, considering all orbits with $\Lambda < \Lambda_{\max}$. It is not clear when this is valid since the expansion is only conditionally convergent, but it seems to give the right results in a number of cases that have been attempted so far. It works, but generally slows convergence in hyperbolic systems.

Stability smoothing

Ordering by stability removes the natural cancellations obtained by grouping orbits with fixed length n. In one sense this is not too significant, since the shadowing properties are not very good for intermittent systems. There is, however a smoothing technique which can mitigate the effects of the cut off at $\Lambda_{\max}$: weighting terms by a factor $1 - (\Lambda/\Lambda_{\max})^2$. This is chosen to vanish at the end point $\Lambda_{\max}$ but lead to a constant error of $\Delta\Lambda/\Lambda_{\max}^2$ irrespective of Λ for a shadowing situation. The results do not depend greatly on this function.

Series matching

The standard periodic orbit expansion gives a power series for the determinant around $z = 0$. For intermittency, we expect that the leading eigenvalue corresponds to a branch point with terms $(z - z_b)^{k+(1/\alpha)}$ for $k = 0, 1, 2, \ldots$. We construct a series of these and integer powers then match it to the series around $z = 0$, solving for the branch point coefficients and hence computing properties of the system.

More details and results of these methods can be found in Ref. [2].

Stochastic perturbations

All real systems have some degree of noise, so we would like to treat problems of the form $x_{n+1} = f(x_n) + \sigma\xi_n$ where $\sigma \ll 1$ is the noise strength and ξ_n are normalised independent identically distributed random variables. The evolution operator associated with this process is a natural generalisation of its deterministic counterpart

$$(\mathcal{L}_\sigma\rho)(y) = \int \delta_\sigma(y - f(x))\rho(x)dx$$

where δ_σ is the noise distribution, for example $\delta_\sigma(z) = \exp[-z^2/(2\sigma^2)]/(\sigma\sqrt{2\pi})$. In a distributional sense we can write

$$\delta_\sigma(z) = \delta(z) + \frac{\sigma^2}{2}\delta''(z) + \frac{\sigma^4}{8}\delta''''(z) + \ldots$$

and evaluate the integrals by changing variable to the argument of the delta function then integration by parts. The result is

$$\mathrm{tr}\mathcal{L}_\sigma^n = \sum_{x:f^n(x)=x}\left[\left|\frac{dx}{dy}\right| + \frac{n\sigma^2}{2}\frac{d^2}{dy^2}\left|\frac{dx}{dy}\right| + \ldots\right]$$

where $y = f^n(x) - x$. Thus we have a perturbation theory for weak noise, expressed in terms of the periodic orbits of the unperturbed deterministic dynamics. We obtained the first term in the previous calculation.

n	ν_0	ν_2	ν_4	ν_6
1	0.308	0.42	2.2	17.5
2	0.37141	1.422	32.97	1573.4
3	0.3711096	1.43555	36.326	2073.0
4	0.371110995255	1.435811262	36.3583777	2076.4793
5	0.3711109952349	1.4358112481974	36.35837123374	2076.47704929
6	0.3711109952349	1.4358112481975	36.35837123384	2076.47704933

TABLE 1: Noise coefficients of the eigenvalue of the map shown in Fig. 1, computed using periodic orbits up to length n.

Numerical results for the map shown in Fig. 1 are given in Tab. 1. Here we expand the eigenvalue $z^{-1}(\sigma) = \nu_0 + \sigma^2\nu_2 + \sigma^4\nu_4 + \sigma^6\nu_6 + \ldots$. We observe rapid convergence in terms of the orbit length n, but divergence in terms of the noise strength σ.

There have been a variety of approaches to this problem, including Feynman diagrams, smooth conjugations, numerical matrix techniques and contour integration. For the latest results please refer to my web page below. The above discussion is based on Ref. [3].

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Carl.Dettmann@bris.ac.uk http://www.maths.bris.ac.uk/~macpd/

References

[1] P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner and G. Váttay, *Classical and quantum chaos* www.nbi.dk/ChaosBook/ Niels Bohr Institute (Copenhagen 2002).

[2] C. P. Dettmann and P. Dahlqvist, Phys. Rev. E **57**, 5303-5310 (1998).

[3] P. Cvitanović, C. P. Dettmann, R. Mainieri and G. Vattay, Nonlinearity **12** 939-953 (1999).