Computing the diffusion coefficient for intermittent maps: Resummation of stability ordered cycle expansions

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We compute the diffusion coefficient and the Lyapunov exponent for a diffusive intermittent map by means of cycle expansion of dynamical zeta functions. The asymptotic power law decay of the coefficients of the relevant power series is known analytically. This information is used to resum these power series into generalized power series around the algebraic branch point whose immediate vicinity determines the desired quantities. In particular, we consider a realistic situation where all orbits with instability up to a certain cutoff are known. This implies that only a few of the power series coefficients are known exactly, and many of them are only approximately given. We develop methods to extract information from these *stability ordered* cycle expansions, and compute accurate values for the diffusion coefficient and the Lyapunov exponent. The method works successfully all the way up to a phase transition of the map, beyond which the diffusion coefficient and Lyapunov exponent are both zero. [S1063-651X(98)05405-1]

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I. INTRODUCTION

Given the task of computing an average, such as a Lyapunov exponent or diffusion coefficient of a chaotic system, one can take two different approaches. First, simulation is usually simple, and it provides an answer without bothering to understand the topology of the flow, but it may suffer from severe convergence problems.

Second, these averages can be extracted from dynamical ζ functions and their expansions, known as *cycle expansions*. The basic advantage of expanding the average over cycles is that the asymptotic limit $t\to\infty$ is already taken from the beginning. Longer cycles provide corrections to the results obtained from shorter ones [1].

Real success applying ζ functions has so far only been demonstrated for quite a restricted class of dynamical systems [1–3]. The topology of the flow should be Markovian—symbolic dynamics may be introduced and this symbolic dynamics is of finite subshift type (meaning that there is only a finite number of forbidden substrings). In addition the system needs to be hyperbolic—the stability of cycles is exponentially bounded with length. The class of systems complying with these two properties is called Axiom-A. This class is far too restricted to have any major relevance in applications.

Success in expanding a ζ function depends on its analytic structure. Convergence is hampered by singularities close to the zero being studied. However, if the nature of a disturbing singularity is known, one can utilize this knowledge in a resummation scheme. If the singularity is solely due to intermittency the convergence problem is thus tamed to a large extent [4].

To appreciate the relevance of stability ordering of cycle expansions, we imagine a fairly generic system, given by some set of differential equations. The problem of finding periodic orbits in a systematic way is largely facilitated if one has some symbolic dynamics. For a few potentials this is possible, for example the x^2y^2 model [5], the helium atom [6], the diamagnetic Kepler problem [7] and the anisotropic Kepler problem [8,9].

For generic flows it is often not clear what Poincaré section should be used, and how it should be partitioned to generate a symbolic dynamics. Cycles can be detected numerically by searching a long trajectory for near recurrences. The long trajectory method for finding cycles discussed in Refs. [10,11] preferentially finds the least unstable cycles, regardless of their topological length. If you can find all cycles with stability Λ_p less than a certain cutoff, you can use *stability* ordered cycle expansions. Stability ordering was introduced in Refs. [5,12]. It was later studied more systematically in Refs. [13,14]. It is much easier to implement for a generic dynamical system than the curvature expansions which rely on finite subshift approximations to a given flow.

A general stability ordered cycle expansion looks like $\Sigma_{i=0}^{N_{\max}} a_i \exp(-sl_i)$, where a_i is a monotonically decreasing sequence but l_i is not monotonic. In this paper we will restrict our attention to maps. (It would then be relevant to speak of stability truncation rather than stability ordering.) The expansion looks like $\Sigma_{i=0}^{N_{\max}} a_i z^i$, where a few of the coefficients may be exact whereas the rest are only approximate. In particular, if the system is intermittent, the number of approximate coefficients greatly exceeds the number of exact ones and the main task of this paper is to extract the information they carry. Moreover, we will make use of our *a priori* knowledge of the power law decay of the exact coefficients, and employ the resummation technique of Ref. [4] to improve convergence.

We believe that the idea of stability ordering has its largest potential for systems which cannot be described by a symbolic dynamics of finite subshift type. However, in order

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to identify the problems due only to intermittency, we will study a map with complete symbolic dynamics.

II. THEORY

A. Averages and ζ functions

A nice introduction to chaotic averages is found with proper references in Ref. [15]; we will take a slightly different approach. The reason for this is that the key step in Ref. [15] assumes that the leading zero of a ζ function is isolated. We will try to avoid this assumption by starting from an expression for the invariant density in terms of periodic orbits. The price we pay is that this formula is not rigorously proven for the intermittent systems we will study.

The aim is to compute averages like

$$\langle w(x,n)\rangle = \int \rho(x) w(x,n) dz, \tag{1}$$

where $\rho(x)$ is the invariant density of the ergodic map $x \mapsto f(x)$.

This density can be expressed in terms of periodic orbits via

$$\rho(x) = \lim_{n\to\infty} \sum_p \sum_{r=1}^{\infty} \frac{\delta_{n,rn_p}}{|\Lambda_p|^r} \sum_{x_i \in p} \delta(x - x_i), \tag{2}$$

where r is the number of repetitions of primitive orbit p, having period n_p, and stability $\Lambda_p = (df^{n_p}/dx)|_{x=x_i}$ with x_i being any point along p.

The weight $w(x_0,n)$ is associated with the trajectory starting at x_0 and evolving during n iterations in such a way that it is multiplicative along the flow: $w(x_0,n_1+n_2) = w(x_0,n_1) w(f^{n_1}(x_0),n_2)$. As we are dealing with maps, it is simply

$$w(x_0,n) = w(x_0,1) w(f(x_0),1) w(f^2(x_0),1) \cdots w(f^{n-1},1).$$

The phase space average of $w(x_0,n)$ may now be expanded in terms of periodic orbits as

$$\lim_{n\to\infty} \langle w(x_0,n)\rangle = \lim_{n\to\infty} \sum_p n_p \sum_{r=1}^{\infty} w_p^r \frac{\delta_{n,rn_p}}{|\Lambda_p^r|}, \tag{3}$$

w_p is the weight along with cycle p. ζ functions are introduced by observing that the average (3) may be written as

$$\lim_{n\to\infty} \langle w(x_0,n)\rangle = \lim_{n\to\infty} \frac{1}{2\pi i} \int_C z^{-n} \frac{d}{dz} \ln \zeta_w^{-1}(z) dz, \tag{4}$$

with the ζ function

$$1/\zeta_w(z) = \prod_p \left(1 - w_p \frac{z^{n_p}}{|\Lambda_p|}\right). \tag{5}$$

C is a small contour encircling the origin in clockwise direction. Equation (4) may be verified by inserting the ζ function (5) and let the integral pick up the residues from $z=0$. The result can be recast into a sum over residues outside C, that is, it may be related to the analytic structure of the ζ function.

The *Lyapunov exponent* can be expressed in terms of a generating function

$$\lambda \equiv \lim_{n\to\infty} \frac{1}{n} \langle \ln|\Lambda(x_0,n)|\rangle = \lim_{n\to\infty} \frac{1}{n} \frac{d}{d\beta} \langle \Lambda(x_0,n)^\beta \rangle|_{\beta=0}.$$

One therefore introduces the multiplicative weight

$$w(x_0,n) = \Lambda(x_0,n)^\beta. \tag{6}$$

One can now express the Lyapunov exponent in terms of the associated ζ function

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \frac{1}{2\pi i} \int_C z^{-n} \frac{d}{d\beta} \frac{d}{dz} \ln \zeta_\lambda^{-1}(z)|_{\beta=0} dz. \tag{7}$$

For a diffusive map $\hat{f}$: $\mathbb{R} \mapsto \mathbb{R}$, the diffusion coefficient can also be expressed in terms of a generating function

$$\begin{aligned} D &= \lim_{n\to\infty} \frac{1}{2n} \langle (\hat{f}^n(\hat{x}_0) - \hat{x}_0)^2 \rangle \\ &= \lim_{n\to\infty} \frac{1}{2n} \frac{d^2}{d\beta^2} \langle e^{\beta(\hat{f}^n(\hat{x}_0) - \hat{x}_0)} \rangle|_{\beta=0}, \end{aligned} \tag{8}$$

motivating the introduction of the weight

$$w_D(x_0,t) = e^{\beta[\hat{f}^n(\hat{x}_0) - \hat{x}_0]}. \tag{9}$$

If $\hat{f}(\hat{x}+nL) = \hat{f}(\hat{x}) + nL$ where $\hat{x} \in I$ (I is some interval of length L) then the map can be reduced to a map f: $I \mapsto I$ on the *elementary cell*.

This may be expressed in terms of a ζ function with the weight w_p along cycle p on the elementary cell given by

$$w_p = e^{\beta \sigma_p}, \tag{10}$$

where

$$\sigma_p = \sum_{x_i \in p} [\hat{f}(x_i) - f(x_i)] \tag{11}$$

is the corresponding drift in the full system.

The diffusion coefficient may now be expressed in terms of the associated ζ function

$$D = \lim_{n\to\infty} \frac{1}{2n} \frac{1}{2\pi i} \int_C z^{-n} \frac{d^2}{d\beta^2} \frac{d}{dz} \ln \zeta_D^{-1}(z)|_{\beta=0} dz. \tag{12}$$

In both Eqs. (7) and (12), the asymptotic behavior will be determined by the leading singularity of the integrand. Since the integrand is evaluated at $\beta=0$, the singularity is located at $z=1$. If this singularity is isolated, the asymptotic result is obtained by simply integrating around it. For the intermittent system we are going to consider, there is a complication. The singularity is not isolated. The ζ function has a branch cut

along $\mathrm{Re}(z)\geqslant 1$ and $\mathrm{Im}(z)=0$. To extract the asymptotic behavior of these integrals we need to integrate around this cut.

Let us return to the Lyapunov exponent and assume that

$$1/\zeta_\lambda(z,\beta)=[a_1(1-z)+O((1-z)^\gamma)]+\beta[b_0+O(1-z)]+O(\beta^2), \tag{13}$$

where $\gamma>1$. This particular assumption will be motivated later in this paper. We need to evaluate

$$\frac{1}{2\pi i}\int_{\Gamma_0}(1-s)^{-n}\frac{d}{d\beta}\frac{d}{ds}\ln\zeta_\lambda^{-1}(z(s))|_{\beta=0}ds$$
$$=\frac{1}{2\pi i}\int_{\Gamma_0}(1-s)^{-n}\left(-\frac{b_0}{a_1}\right)\frac{1+O(s^{\gamma-1})}{s^2}ds$$
$$=-\frac{b_0}{a_1}n+O(n^{2-\gamma}), \tag{14}$$

where we have changed variable to $s=1-z$. Γ_0 is a contour encircling the negative real s axis in an anticlockwise direction.

When evaluating these integrals, the following formula is useful:

$$\frac{1}{2\pi i}\int_{\Gamma_0}\frac{1}{s^\rho}e^{st}ds=\frac{t^{\rho-1}}{\Gamma(\rho)}. \tag{15}$$

The Lyapunov exponent is thus found to be

$$\lambda=-\frac{b_0}{a_1}. \tag{16}$$

For the diffusion case, we assume that

$$1/\zeta_D(z,\beta)=[a_1(1-z)+O((1-z)^\gamma)]+\beta^2[c_0+O(1-z)]+O(\beta^4). \tag{17}$$

We now need to evaluate

$$\frac{1}{2\pi i}\int_{\Gamma_0}(1-s)^{-n}\frac{d^2}{d\beta^2}\frac{d}{ds}\ln\zeta_D^{-1}(z(s))|_{\beta=0}ds$$
$$=-2\frac{c_0}{a_1}n+O(n^{2-\gamma}) \tag{18}$$

and

$$D=-\frac{c_0}{a_1}. \tag{19}$$

To obtain the a, b, and c coefficients of this section we need to expand the ζ function (5) in powers of z around $z=0$ which will be discussed in Sec. II B. Then we will resum the series around $z=1$ in Sec. II C.

B. Expanding ζ functions

For the Lyapunov exponent calculation we expand the ζ function

$$1/\zeta_\lambda(z)=\prod_p\left(1-\frac{z^{n_p}}{|\Lambda|^{1-\beta}}\right)$$
$$=\prod_p\left(1-\frac{z^{n_p}}{|\Lambda|}-\beta\frac{z^{n_p}\ln|\Lambda|}{|\Lambda|}+O(\beta^2)\right)$$
$$=\prod_p\left(1-\frac{z^{n_p}}{|\Lambda|}-\beta\frac{z^{n_p}\ln|\Lambda|}{|\Lambda|}\right)+O(\beta^2)$$
$$\equiv\sum_{j=0}^{\infty}\hat{a}_jz^j+\beta\left(\sum_{j=0}^{\infty}\hat{b}_jz^j\right)+O(\beta^2), \tag{20}$$

resulting in two power series. Similarly for the diffusion calculation we expand

$$1/\zeta_D(z)=\prod_p\left(1-\frac{z^{n_p}e^{\beta\sigma_p}}{|\Lambda|}\right)$$
$$1/\zeta_D(z)=\prod_p\left(1-\frac{z^{n_p}}{|\Lambda|}-\beta\frac{z^{n_p}\sigma_p}{|\Lambda|}-\beta^2\frac{z^{n_p}\sigma_p^2}{2|\Lambda|}-\beta^3\frac{z^{n_p}\sigma_p^3}{6|\Lambda|}\right)+O(\beta^4)$$
$$\equiv\sum_{j=0}^{\infty}\hat{a}_jz^j+\beta^2\left(\sum_{j=0}^{\infty}\hat{c}_jz^j\right)+O(\beta^4). \tag{21}$$

We restrict our attention to systems with no net drift, that is,

$$\lim_{n\to\infty}\frac{1}{n}\langle\hat{f}^n(\hat{x}_0)-\hat{x}_0\rangle=0. \tag{22}$$

Therefore only even powers of β appear in Eq. (21).

The set of coefficients we obtain in this way depends on the truncation used in the expansion of the infinite product. For truncation by topological length, we count cycles up to a given length N_{top}. For maps with a few branches, this number is limited to roughly of order $\sim 10^1$, due to the exponential growth in the number of cycles with the topological length. All combinations of cycles with total length less than or equal to N_{top} are also included, as these contribute to the first N_{top} coefficients in each series. Thus we obtain N_{top} exact coefficients in each series by topological length truncation.

For truncation by stability, we count cycles up to a given stability Λ_{max}, and combinations where the product of stabilities is less than Λ_{max}. They have lengths up to N_{max}, and so contribute to all of the first N_{max} coefficients in each series, but are not the only contributions to such coefficients. They give us an approximation to the ζ function which, for the intermittent case, is more accurate than that obtained from the length truncation, but the values of the coefficients themselves are not exact beyond some $N_{\mathrm{exact}}(\Lambda_{\mathrm{max}})$, a quantity growing logarithmically: $N_{\mathrm{exact}}\sim\ln\Lambda_{\mathrm{max}}$. For intermittent maps, as the one we will consider, N_{max} increases as a power of Λ_{max} and $N_{\mathrm{max}}\gg N_{\mathrm{exact}}$.

Often it is found that stability ordered cycle expansions lead to noisy results as a function of Λ_{max}. This is due to the breaking of shadowing pairs. For example, a cycle AB usually gives a contribution roughly equal to and of the opposite sign as the combination of cycles A and B (we will refer to

such a combination as a *pseudocycle*). This means the total contribution is quite small. The phenomenon is called shadowing, and is the main mechanism for the rapid convergence of cycle expansions in hyperbolic systems. It is still present to some degree in intermittent systems. However, if one such term is included but the other is excluded because they lie on opposite sides of $\Lambda_{\max}$, there may be a substantial error generated.

Partial shadowing which may be present can be (partially) restored by smoothing the stability ordered cycle expansions by replacing each term with inverse pseudocycle stability $\Lambda^{-1}=(\Lambda_{p_1}\cdots\Lambda_{p_k})^{-1}$ by $S(\Lambda)\Lambda^{-1}$. Here $S(\Lambda)$ is a monotonically decreasing function, with $S(0)=1$ and $S(\Lambda>\Lambda_{\max})=0$.

A typical "shadowing error" induced by the cutoff is due to two pseudocycles of stability Λ separated by $\Delta\Lambda$, and whose contribution is of opposite signs. Ignoring possible weighting factors the magnitude of the resulting term is of order $\Lambda^{-1}-(\Lambda+\Delta\Lambda)^{-1}\approx\Delta\Lambda/\Lambda^2$. With smoothing there is an extra term of the form $S'(\Lambda)\Delta\Lambda/\Lambda$, which we want to minimize. A reasonable guess might be to keep $S'(\Lambda)/\Lambda$ constant and as small as possible, that is

$$S(\Lambda)=\left[1-\left(\frac{\Lambda}{\Lambda_{\max}}\right)^2\right]\Theta(\Lambda_{\max}-\Lambda).$$

This function still contains a nonanalytic point at $\Lambda=\Lambda_{\max}$, however the discontinuity is now in the derivative, not in the original function, so a smoothing error estimated by $S'(\Lambda)/\Lambda$ ($\Lambda<\Lambda_{\max}$) is finite. We use this smoothing function below when evaluating the ζ coefficients, and demonstrate the improvement numerically.

C. Resumming ζ functions

The result of the *cycle expansions* in Sec. II B is a set of power series of the form $\Sigma_i\hat{a}_i z^i$ around $z=0$. And, according to Sec. II A, what we need are coefficients from some kind of (resummed) series around $z=1$. We now describe a method of obtaining such a series, along the lines of Ref. [4].

Suppose for a moment that the series $\Sigma_{i=0}^{\infty}\hat{a}_i z^i$ has a radius of convergence exceeding unity. In a practical calculation we have only a finite number n (say N_{top} or $N_{\max}$) of coefficients $\hat{a}_i$ at our disposal. We assume them to be exact; the treatment of the approximate coefficients from stability ordered expansions are discussed in Sec. III C. Then we can in principle expand it into another truncated (resummed) Taylor series around $z=1$:

$$\sum_{i=0}^{n}\hat{a}_i z^i=\sum_{i=0}^{n}a_i(z-1)^i. \tag{23}$$

This leads to a linear systems of equations which is trivially invertible:

$$a_i=\sum_{j=i}^{n}\binom{j}{i}\hat{a}_j. \tag{24}$$

In this way one obtains the standard formulae [1]

$$\lambda=\frac{\sum(-1)^k\frac{\ln\Lambda_1+\cdots+\ln\Lambda_k}{|\Lambda_1\cdots\Lambda_k|}}{\sum(-1)^k\frac{n_1+\cdots+n_k}{|\Lambda_1\cdots\Lambda_k|}}, \tag{25}$$

$$D=\frac{1}{2}\frac{\sum(-1)^k\frac{(\sigma_1+\cdots+\sigma_k)^2}{|\Lambda_1\cdots\Lambda_k|}}{\sum(-1)^k\frac{n_1+\cdots+n_k}{|\Lambda_1\cdots\Lambda_k|}}, \tag{26}$$

where the sums run over all distinct pseudocycles.

This approach is particularly cumbersome for intermittent systems, where $\hat{a}_i$ (as well as $\hat{b}_i$ and $\hat{c}_i$) decays according to some power law. Then the coefficients either diverge or converge slowly as $n\to\infty$. So, for intermittent systems, the resummed series cannot be a Taylor series; it has to be some generalized power series.

Assume that the asymptotic behavior of the coefficients is a power law

$$\hat{a}_i\sim n^{-(\gamma+1)}. \tag{27}$$

Then the leading singularity is of the form $(1-z)^\gamma$, and the simplest possible expansion would be

$$\sum_{i=1}^{\infty}a_i(1-z)^i+(1-z)^\gamma\sum_{i=0}^{\infty}\bar{a}_i(1-z)^i=\sum_{i=0}^{\infty}\hat{a}_i z^i. \tag{28}$$

Having only a finite number n of coefficients $\hat{a}_i$, we propose the following resummation [4]

$$\sum_{i=1}^{n_a}a_i(1-z)^i+(1-z)^\gamma\sum_{i=0}^{\bar{n}_a}\bar{a}_i(1-z)^i=\sum_{i=0}^{n}\hat{a}_i z^i+O(z^{n+1}). \tag{29}$$

If $n_a+\bar{n}_a+2=n+1$, we obtain just a linear system of equations to solve in order to determine the coefficients a_i and $\bar{n}_a$ from the coefficients $\hat{a}_i$. It is also natural to require that $|n_a+\gamma-\bar{n}_a|<1$.

The basic philosophy is to build in as much information as possible into the ansatz. If the original power series corresponds to the unweighted ζ function, we know that $a_0=0$. The ansatz is thus accordingly modified. We fix $a_0=0$ and modify n_a or $\bar{n}_a$ so we still obtain a solvable system of equations.

III. NUMERICAL STUDIES OF AN INTERMITTENT DIFFUSIVE MAP

A. Map

In the interval $\hat{x}\in[-\frac{1}{2},\frac{1}{2})$, which we call the elementary cell, our model map, following Ref. [14], takes the form

$$\hat{f}(\hat{x})=\hat{x}(1+2|2\hat{x}|^\alpha). \tag{30}$$

The parameter range we consider here is $\alpha\in(0,1)$, where the Lyapunov exponent and diffusion coefficient are both nonzero. For any value of α, this maps the interval $\hat{x}\in[-\frac{1}{2},\frac{1}{2})$

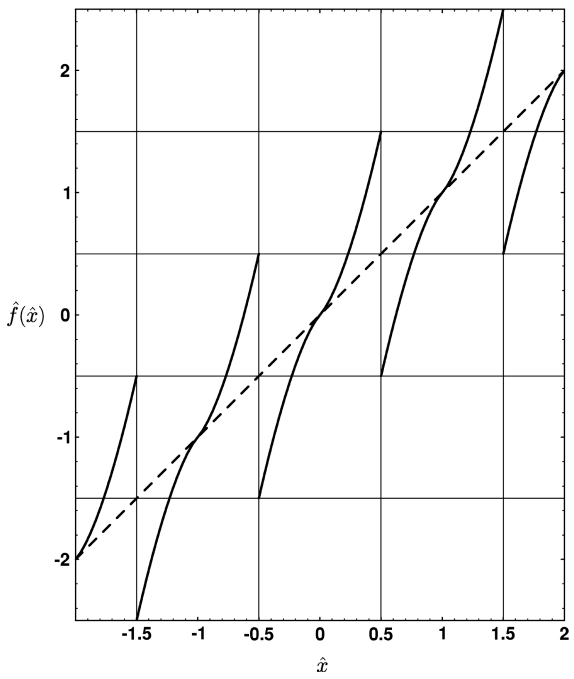

FIG. 1. Map (30) for $\alpha=0.7$.

monotonically to $[-\frac{3}{2},\frac{3}{2})$. Outside the elementary cell, the map is defined to have a discrete translational symmetry,

$$\hat{f}(\hat{x}+n)=\hat{f}(\hat{x})+n, \quad n\in\mathbb{Z}.$$

See Fig. 1. A typical initial $\hat{x}$ in the elementary cell diffuses, wandering over the real line. The map is parity symmetric, $\hat{f}(-\hat{x})=-\hat{f}(\hat{x})$, so the average value of $\hat{x}_{n+1}-\hat{x}_n$ is zero, and there is no mean drift, as expressed in Eq. (22).

We now restrict the dynamics to the elementary cell; that is, we define

$$x=\hat{x}-[\hat{x}+1/2],$$

where $[z]$ is the greatest integer less than or equal to z, so that x is restricted to the range $[-\frac{1}{2},\frac{1}{2})$. The reduced map is

$$f(x)=\hat{f}(x)-[\hat{f}(x)+\tfrac{1}{2}]. \tag{31}$$

As discussed in Ref. [14], the intermittency of this map appears in the form of long cycles near the marginal point with power law stabilities. This is in contrast to Axiom-A systems, for which Λ may be bounded by exponentials of the topological length.

The map has three complete branches in the elementary cell. Symbolic dynamics is introduced by labeling the branches $\{-,0,+\}$, with cycles denoted by an overbar. Due to the completeness of the symbolic dynamics, the ζ functions are approximated by [16]

$$1/\zeta_\lambda(z)\approx 1-\sum_{n=0}^{\infty}\frac{z^{n+1}}{|\Lambda_{\overline{-0^n}}|^{1-\beta}}-\sum_{n=0}^{\infty}\frac{z^{n+1}}{|\Lambda_{\overline{+0^n}}|^{1-\beta}} \tag{32}$$

and

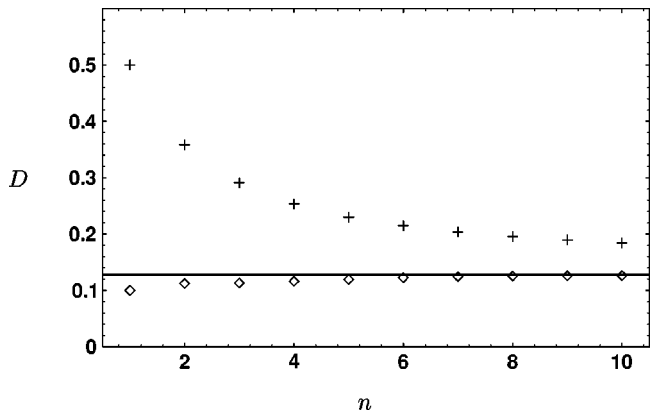

FIG. 2. The diffusion coefficient at $\alpha=0.7$, from direct simulation (solid line), and topological ordered cycle expansions with (diamonds) and without (pluses) resummation.

$$1/\zeta_D(z)\approx 1-e^{-\beta}\sum_{n=0}^{\infty}\frac{z^{n+1}}{|\Lambda_{\overline{-0^n}}|}-e^{+\beta}\sum_{n=0}^{\infty}\frac{z^{n+1}}{|\Lambda_{\overline{+0^n}}|}. \tag{33}$$

This approximation may seem crude. For instance, the ζ functions (32) and (33) fail to preserve flow conservation. However, in Ref. [4] we presented evidence that they capture the leading singularity structure correctly. This was obtained by comparing coefficients of the piecewise linear approximation of the intermittent map (sharing the singularity structure with the approximation above) by the exact cycle expansion. The asymptotic behavior of the fundamental cycles is given by

$$\Lambda_{\overline{-0^n}}=\Lambda_{\overline{+0^n}}\sim n^{1+1/\alpha}, \tag{34}$$

see, e.g., Refs. [4,14] for a derivation. We immediately obtain, from Eqs. (32)–(34),

$$\hat{a}_n\sim n^{-1-1/\alpha}, \tag{35}$$

$$\hat{b}_n\sim n^{-1-1/\alpha}\ln n, \tag{36}$$

$$\hat{c}_n\sim n^{-1-1/\alpha}. \tag{37}$$

This leads to forms (13) and (17), with $\gamma=1/\alpha$, as long as $\alpha<1$.

For a general orbit we can only bound the stability in the range

$$Cn_p^{1+1/\alpha}<|\Lambda_p|\leq(\max|f'|)^{n_p}=(3+2\alpha)^{n_p}, \tag{38}$$

so, when using stability cutoff, for the parameters $N_{\max}$ and N_{exact} discussed in Sec. II B we obtain:

$$N_{\max}\sim\Lambda_{\max}^{\alpha/(1+\alpha)} \tag{39}$$

and

$$N_{\text{exact}}>\frac{\ln\Lambda_{\max}}{\ln(3+2\alpha)}. \tag{40}$$

B. Resumming topologically ordered cycle expansions

We will, most of the time, concentrate on the diffusion coefficient, but a similar analysis holds for the Lyapunov exponent, to which we return at the very end. We calculated the diffusion coefficient from resummed cycle expansions obtained using topological ordering, as described in Sec.

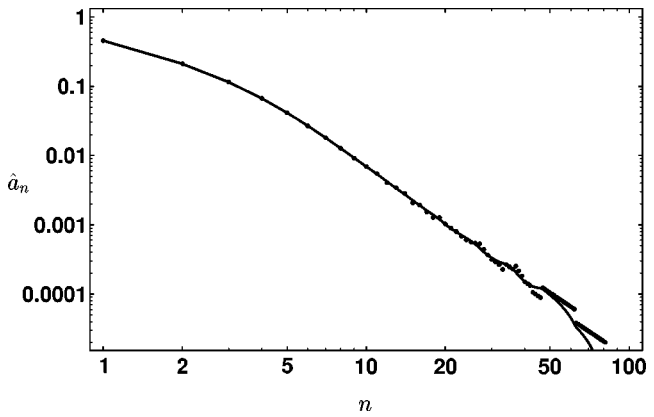

FIG. 3. The magnitude of the ζ coefficients for stability ordered cycle expansions at $\alpha=0.7$ unsmoothed (points) and smoothed (solid line).

II C, with the number of coefficients n determined by the maximum topological length, up to 10. We also used the direct formula (26), and performed direct simulations with roughly the same amount of computer time. The results are shown in Fig. 2, showing that the resummation gives much improvement, and is consistent with direct simulation.

C. Resumming stability ordered cycle expansions

Now we come to the central part of our numerical work: the resummation of stability ordered cycle expansions. First we calculate $\hat{a}_i$, as described in Sec. II B. The coefficients are all negative except $\hat{a}_0=1$, and their magnitudes are plotted in Fig. 3, where we have used $\Lambda_{\max}=10^5$ which corresponds to $N_{\text{exact}}=8$ and $N_{\max}=81$. The unsmoothed coefficients are thus exact for $n\leqslant N_{\text{exact}}$. The smoothed are not exact but are still quite accurate. For $n>N_{\text{exact}}$, we clearly see how the unsmoothed begins to oscillate in an irregular fashion, whereas the smoothed ones are stable for much larger n.

The next issue is how to best make use of the information contained in the $\hat{a}_n$ coefficients. As pointed out in Sec. II B, these coefficients are not exact, but they give a better representation of the ζ function than the limited number of exact coefficients obtained from topological ordering. In order to match the series at $z=1$, we must again solve a linear set of equations, but the number of coefficients ($N_{\max}$) for intermittent systems is much larger than for the topological ordering. We cannot match such a large number of coefficients in both series, because the solution would be unstable to the errors in the coefficients, so we must represent the information contained in $\hat{a}_n$ in the (fewer) number of degrees of freedom that the expansion really contains.

There may be more than one solution to this problem; the solution we use here is to perform *two* resummations, the first from $z=0$ to an intermediate $0<z'<1$, and the second from $z=z'$ to $z=1$:

$$\sum_{i=0}^{N_{\max}} \hat{a}_i z^i = \sum_{i=0}^{N_{\max}} a'_i (z-z')^i, \tag{41}$$

which can be explicitly inverted

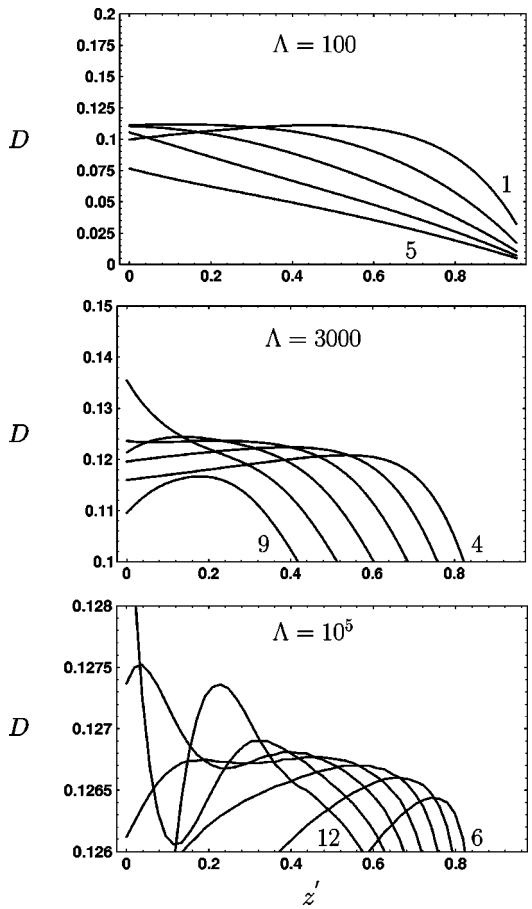

FIG. 4. Diffusion coefficient calculated using resummed stability ordered cycle expansions, showing a dependence on n' (labels on curves) and z' (horizontal axis). Note the scales on the vertical axes.

$$a'_n = \sum_{i=n}^{N_{\max}} \binom{i}{n} \hat{a}_i z'^{(i-n)}. \tag{42}$$

With z' suitably chosen, we have thus used the information available in the $\hat{a}_n$ approximately in proportion to their reliability. That is, the accurate low order coefficients appear with large weights in the first few a'_n, while the less accurate high order coefficients appear with small weights. As we will see, this approach is better than one which simply ignores the higher order coefficients (this corresponds to putting $z'=0$ below).

As for the topological length truncation, the resummation from $z=z'$ to $z=1$ leads to a set of linear equations obtained by equating coefficients in

$$\sum_{i=1}^{n_a} a_i (1-z)^i + (1-z)^\gamma \sum_{i=0}^{\bar{n}_a} \bar{a}_i (1-z)^i = \sum_{i=0}^{n'} a'_i (z-z')^i + O(z^{n+1}). \tag{43}$$

Again, we adjust n_a and $\bar{n}_a$ so as to obtain a consistent series in powers of $z-1$ and a consistent set of linear equations.

We have two parameters in the double resummation scheme, n' and z'. The idea is to calculate λ or D for a range of both parameters and look for the most consistent

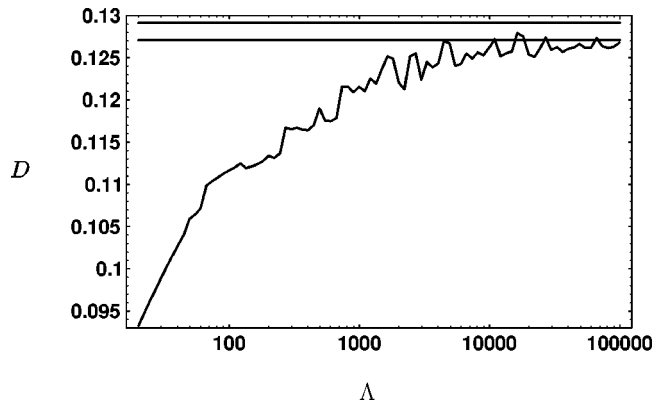

FIG. 5. Diffusion coefficient calculated using resummed stability ordered cycle expansions, showing a dependence on the stability cutoff. The horizontal lines are the range indicated by direct simulation (also see the discussion in the text).

solution. The general behavior is shown in Fig. 4. For each stability cutoff $\Lambda_{\max}$, small values of n' lead to a variation of D with z', which has a single maximum. Larger values of n' lead to functions that are either monotonically decreasing or oscillatory. We estimate the diffusion coefficient by finding the maximum for the largest value of n' before monotonic or oscillatory behavior sets in. The convergence of this method with the stability cutoff is shown in Fig. 5.

This figure also contains the direct simulation results, obtained by estimating the left side of Eq. (8) for 3×10^3 iterations over a sample of 3×10^3 trajectories, similar to the computer time required to find the cycles with $\Lambda<10^5$. The errors were obtained by looking at the scatter in this statistical sample of trajectories; for intermittent maps, the diffusion coefficient always tends to be too high, because long intermittent episodes are not sampled sufficiently. Close to the phase transition at $\alpha=1$, convergence is practically logarithmic in the number of iterations, with exponentially long times required to achieve convergence. For example, with the numerical procedure described above, we find $D=0.0524\pm0.0005$ at $\alpha=1$, where we know $D=0$. Even at $\alpha=0.7$, a reasonable distance from the transition, the resummed cycle expansion result is more accurate than direct simulation.

Our final value for the diffusion coefficient at $\alpha=0.7$ is $D=0.1267\pm0.0003$ with the resummation method. It is then

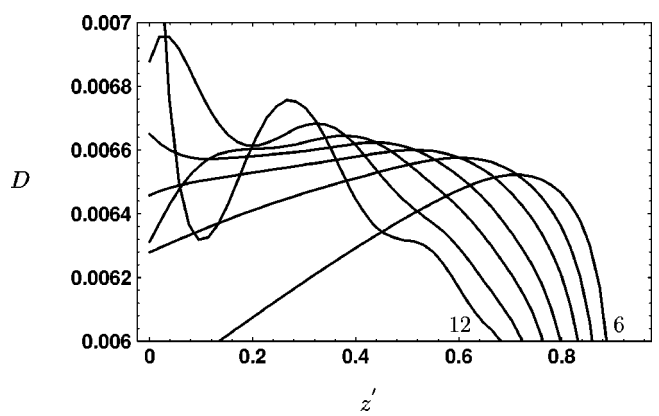

FIG. 6. Diffusion coefficient calculated using resummed stability ordered cycle expansions for $\alpha=0.99$.

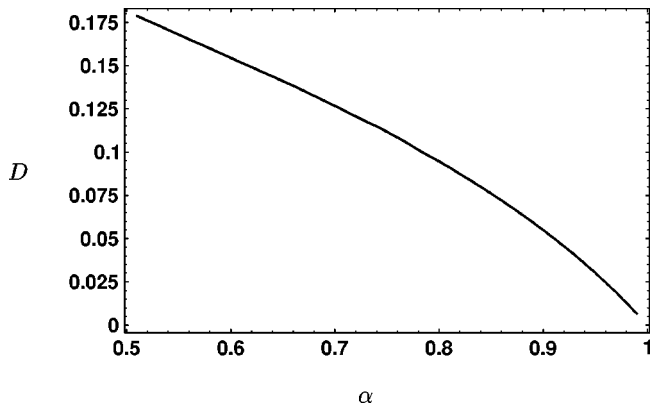

FIG. 7. The diffusion coefficient as a function of the parameter α, showing the approach to the phase transition at $\alpha=1$, beyond which $D=0$.

quite compatible with the topological ordering discussed in Sec. III B, which yielded the result $D=1.262\pm0.0003$. In this example, it is clear that resummed cycle expansions, whether ordered by topological length or stability, provide an accurate method of analyzing intermittent systems. Stability ordering is most important in more complicated systems where topological ordering is not a realistic alternative. There is an additional advantage with the stability ordered expansion, in that it provides a large number of approximate coefficients, thus facilitating a numerical estimate of the power law if it is not known analytically. Recall that this power law is used in the resummation ansatz, and was absolutely essential for the good result in Sec. III B.

D. Phase transition

Having gained confidence in the resummation method for $\alpha=0.7$, far from the phase transition at $\alpha=1$, we now vary α, including values for which direct simulation is totally impractical, due to logarithmically slow convergence. At $\alpha=0.99$ we obtain Fig. 6 for the diffusion coefficient, showing a consistent value of $D=0.0066\pm0.0001$. Plotting D vs α (Fig. 7), we find a linear dependence near the phase transition at $\alpha=1$. Finally, we performed the same analysis for the Lyapunov exponent, which has a similar dependence on α, shown in Fig. 8.

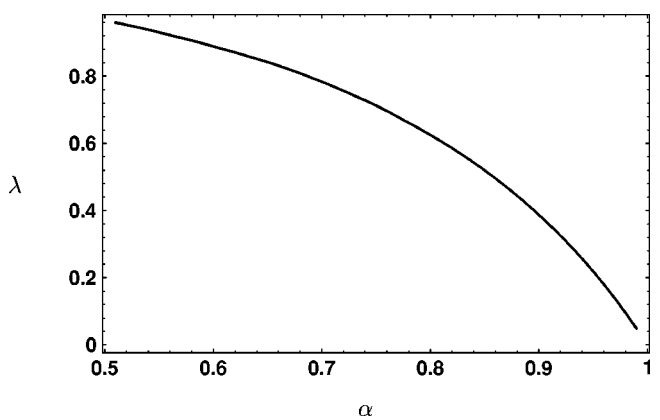

FIG. 8. The Lyapunov exponent as a function of the parameter α, showing the approach to the phase transition at $\alpha=1$, beyond which $\lambda=0$.

IV. CONCLUSION

We have demonstrated that resummed stability ordered cycle expansions can provide accurate estimates of dynamical averages for intermittent maps, even close to a phase transition. This analysis could equally apply to maps with uncontrolled symbolic dynamics, as long as a reliable method exists for locating the cycles.

Our methods can also be applied without much modification to flows. Then the variable z is replaced by $\exp(-s)$, and the cycle expansion is actually a Dirichlet series, $\Sigma_i a_i \exp(-s l_i)$, where the lengths of the pseudo orbits l_i are not restricted to integer values. With an additional resummation step at s' (corresponding to z' in this paper), the ζ function may be represented as a standard power series, thus allowing it to be matched to a generalized power series at $s=0$. For intermittent systems, $s=0$ is again a branch point, and information about it can be obtained from the methods described in Refs. [12,17] or numerically from the stability ordered expansion.

ACKNOWLEDGMENTS

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