RANDOM GRAPHS AND WIRELESS COMMUNICATION NETWORKS

Part 2: Random Graph Properties

1 hour

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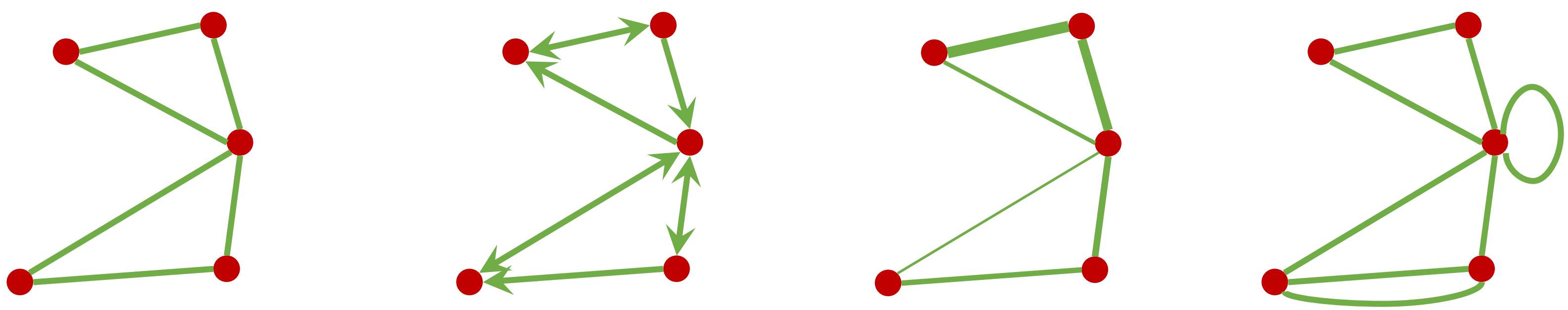
Outline

- Some **Graph examples**
- **Adjacency** and other Matrices
- **Basic** Graph Properties
- **Intermediate** Graph Properties
- **Advanced** Graph Properties
  - **Advanced** Graph Concepts
- **Statistical** Graph Properties

- Basic **Percolation Theory**

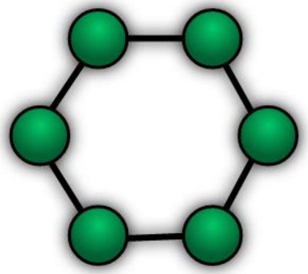
Some Graph examples

- **Mathematical structures used to model pairwise relations between objects**
  - Vertices (Nodes) *V* is the number of nodes
  - Edges (Links) *E* is the number of edges
  - *G(V,E) graph or network*

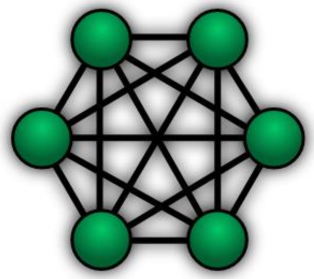


Undirected   Directed   Weighted   Multi-graph Not simple

Some Graph examples

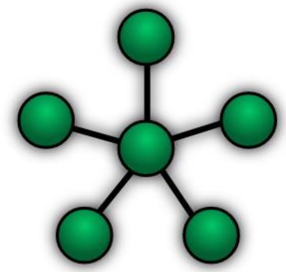


Ring

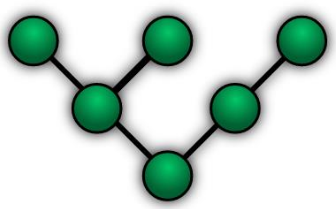


Fully Connected

(Complete)



Star

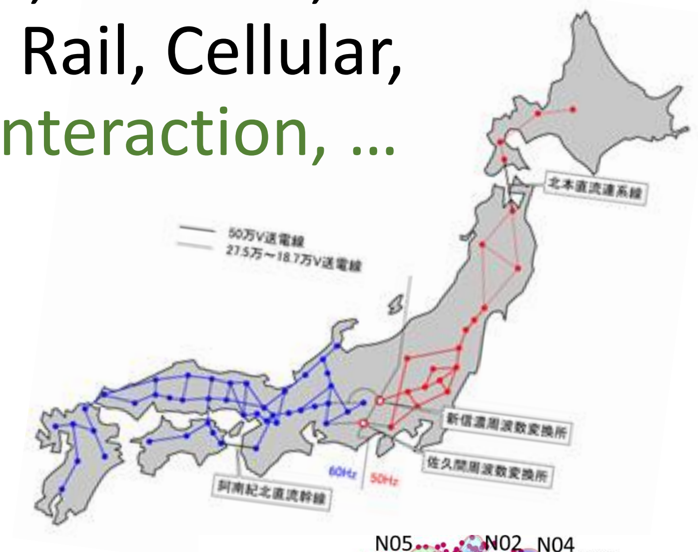


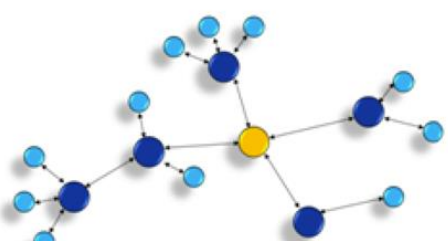
Tree

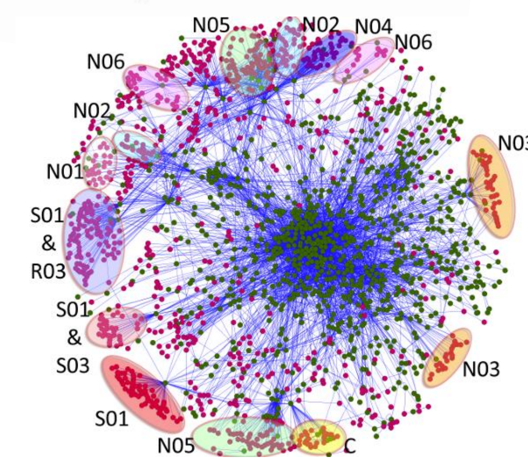
Line

Some Graph examples

Actor movies, Co-authors, Citation, Animal interactions, Social media networks, Internet, Infrastructure, Power grids, Roads, Rail, Cellular, WSNs, Biological, Protein-Protein-Interaction, …







Adjacency and other Matrices

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

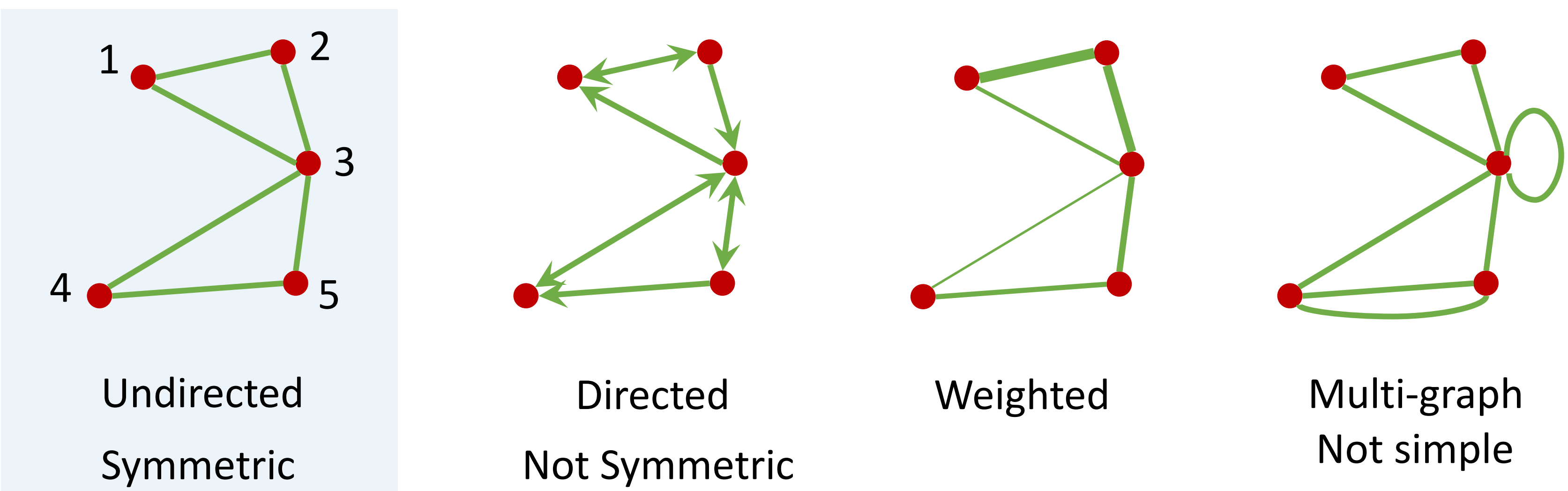
Adjacency Matrix

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Degree Matrix

$$L = [D - A]$$

Laplace Matrix



Basic Graph Properties

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Adjacency Matrix

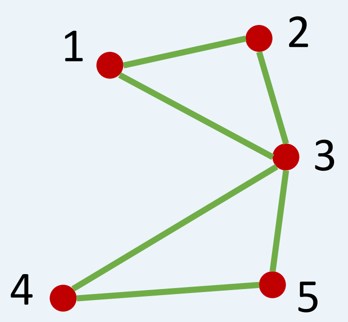
$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Degree Matrix

Node Degree

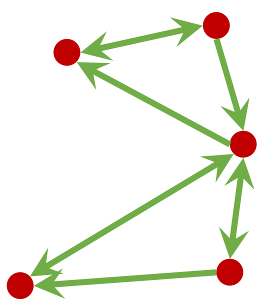
Minimum degree δ(G)=2
Maximum degree Δ(G)=4
Mean degree
μ(G)=(2+2+4+2+2)/5=2.4

Directed degree:
In / Out degrees



Undirected

Symmetric



Directed

Not Symmetric

Basic Graph Properties

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

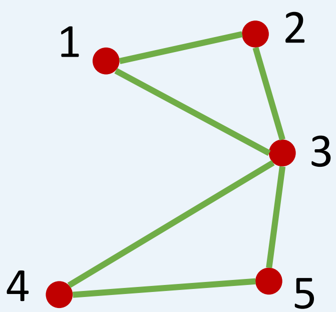
Adjacency Matrix

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Degree Matrix

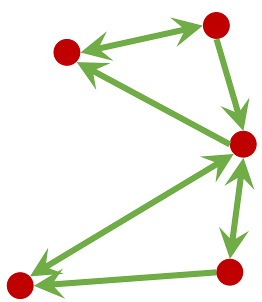
Connectivity
minimum number of elements (nodes or edges) that need to be removed to disconnect the graph.

Edge connectivity *κ=2*
Vertex connectivity e=1



Undirected

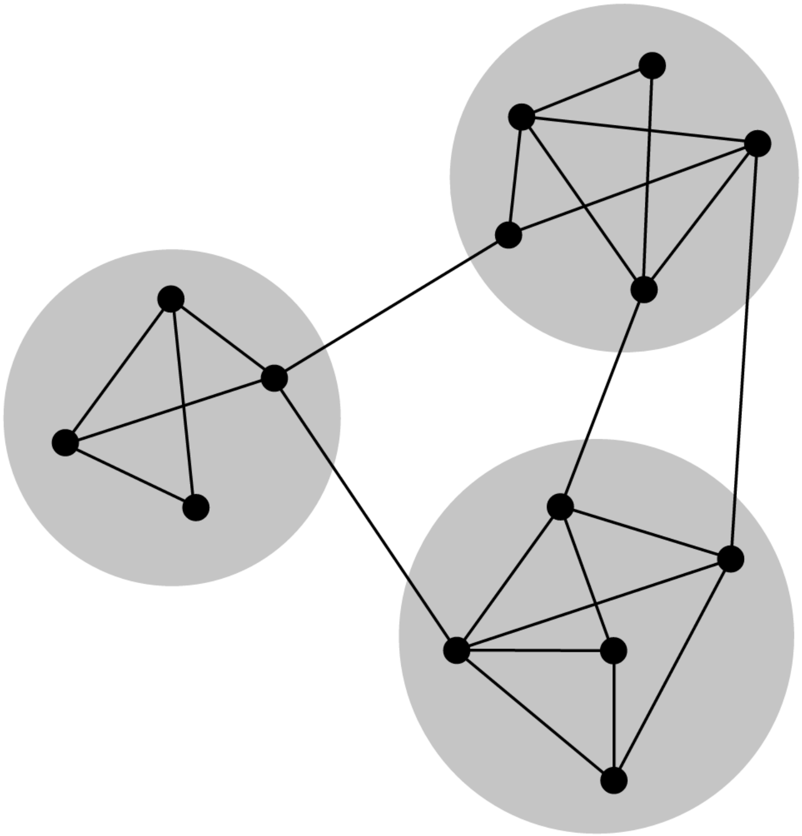
Symmetric



Directed

Not Symmetric

Basic Graph Properties



Adjacency Matrix
Degree Matrix
Laplacian Matrix

Minimum degree δ(G)=2
Maximum degree Δ(G)=4
Mean degree μ(G)= ?
Edge connectivity *κ*=2
Vertex connectivity e=1

Diameter: maximum shortest distance between any 2 nodes
d=5

Distance matrix: NxN matrix with entries node distances (shortest paths)

Intermediate Graph Properties

Reachability (also called Accessibility): ability to get from node *i* to *j*
Available through the **Distance matrix**

Number of walks of length k

$$(A^k)_{i,j} = \sum_{i_1,i_2,\ldots,i_{k-1}} \underbrace{A_{i,i_1} A_{i_1,i_2} \cdots A_{i_{k-2},i_{k-1}} A_{i_{k-1},j}}$$

will be 1 if and only if vertex *i* is adjacent to i_1 which is adjacent to i_2 and so on until we get to *j*

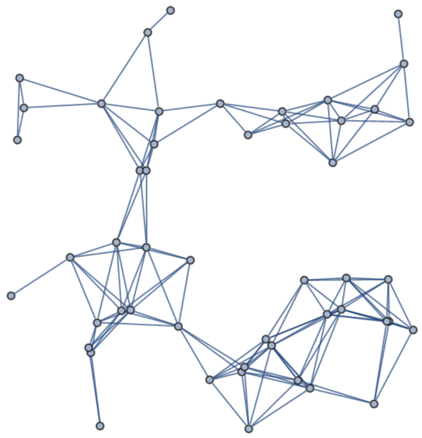
Trace of A^3 gives the number of *closed* paths of length 3. Note that each "triangles" is counted 6 times (3 vertices and 2 directions).

Clustering Coefficient (also called transitivity ratio)

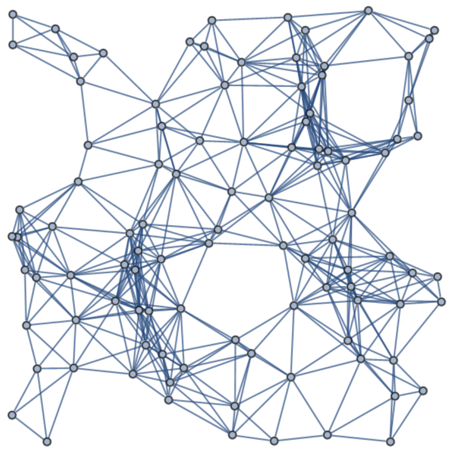
$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triplets of vertices}}$$

Intermediate Graph Properties

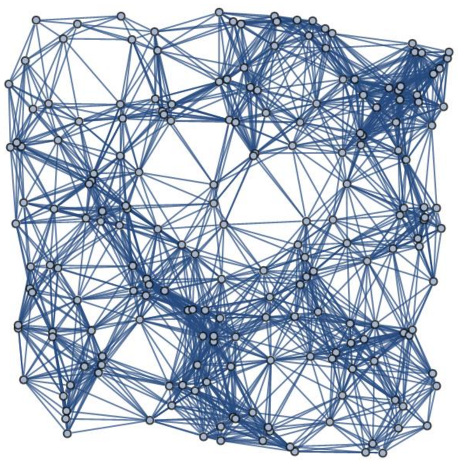
Examples of Random Geometric Graphs (spatially embedded networks)



0.6315



0.5878



0.625

Not a function of density for RGGs

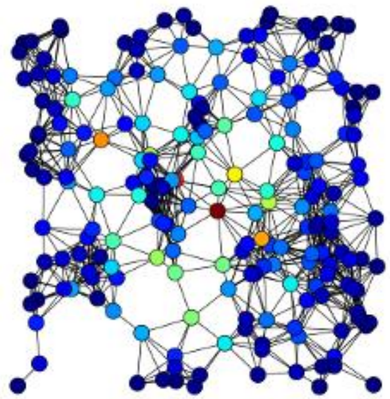
Clustering Coefficient (also called transitivity ratio)

$$C = \frac{3 \times \text{number of triangles}}{\text{number of connected triplets of vertices}}$$

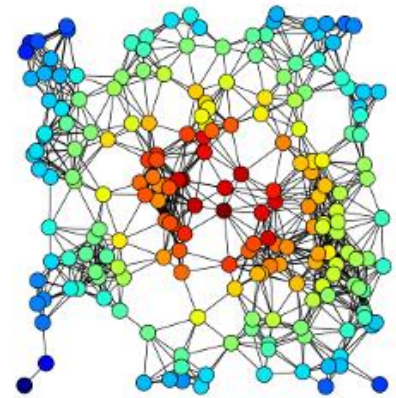
Advanced Graph Properties

Centrality measures: identify the most important vertices within a graph

$$g(v) = \sum_{s \neq v \neq t} \frac{\sigma_{st}(v)}{\sigma_{st}}$$



Betweenness

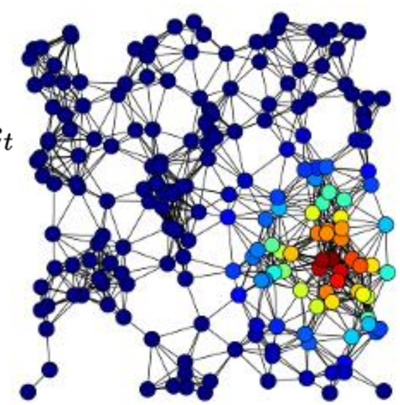


Closeness

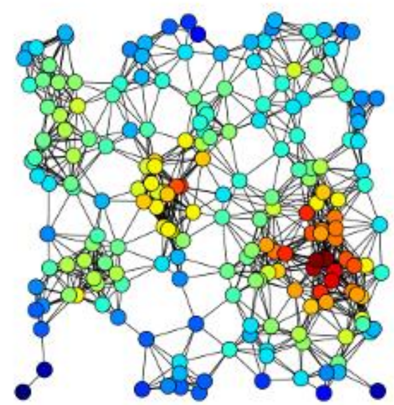
$$C(x) = \frac{1}{\sum_y d(y, x)}$$

$$x_v = \frac{1}{\lambda} \sum_{t \in M(v)} x_t = \frac{1}{\lambda} \sum_{t \in G} a_{v,t} x_t$$

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$



Eigenvector



Degree

Advanced Graph Properties

Number of walks of length k

$$(A^k)_{i,j} = \sum_{i_1, i_2, \ldots, i_{k-1}} A_{i,i_1} A_{i_1,i_2} \cdots A_{i_{k-2},i_{k-1}} A_{i_{k-1},j}$$

Recall the matrix identities:

$$\det(\exp(A)) = \exp(\mathrm{tr}(A))$$

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n$$

Advanced Graph Properties

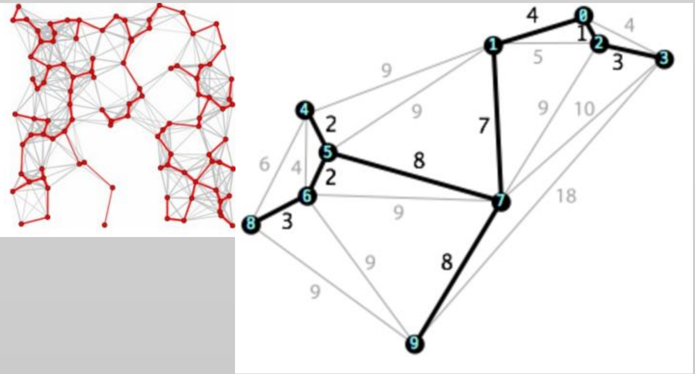
$$L = [D - A]$$

Laplace Matrix spectrum: $0 = \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$

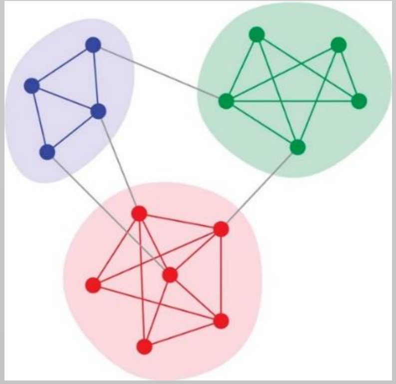
- Contains information about the graph structure
- Number of 0 eigenvalues is the number of connected components
- Second smallest eigenvalue is the *algebraic connectivity*
  - The larger λ_2 is, the better connected the network
- The ratio λ_N/λ_2 is the *network synchronizability*
  - The larger the ratio is, the better the synchronizability

Advanced Graph Concepts

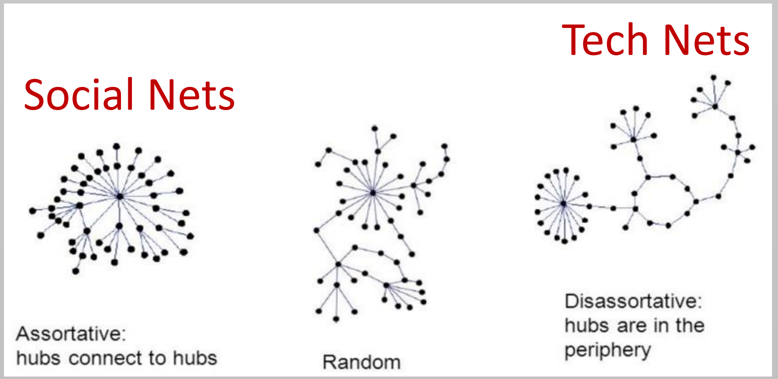
- **Minimum Spanning tree**
  - Shortest Paths
  - Network Flow (Random Walks on graphs)



- **Community Detection**
  - Coverage
  - Centrality
  - Modularity

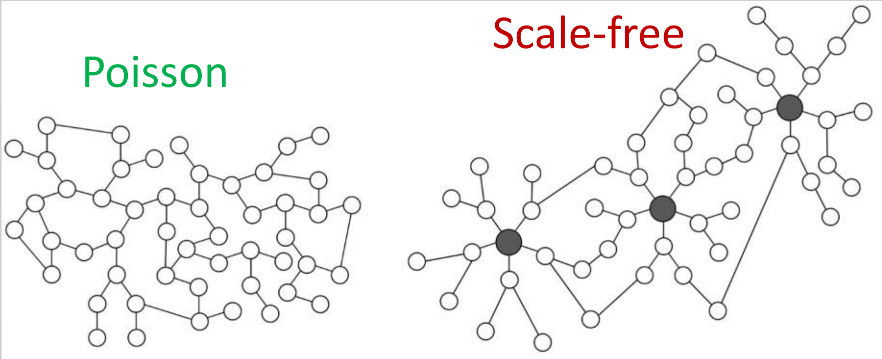


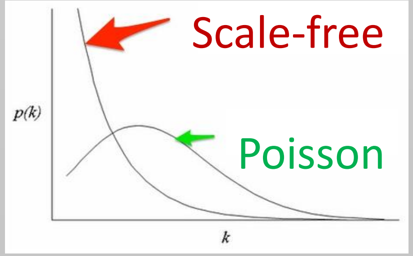
- **Assortativity**
  - preference of nodes to attach to connect with similar (in degree) nodes.



- **Interdependence (Networks of Networks)**

Statistical Graph Properties



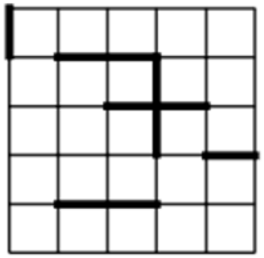


- **Degree Distribution**
  - Poisson for RGGs
  - Power law for scale-free
- **(Pair) Distance Distribution**
  - Diameter, the mean path length
- **Clustering coefficient distribution**
  - some very clustered parts, and some less clustered parts
- **Eigenvalue Distributions**
  - Spectral density
  - Nearest Neighbours spacings (Random Matrix Theory)

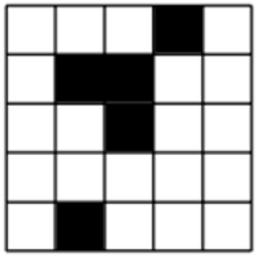
Basic Percolation Theory

Percolation is a phase-transition phenomenon whereby at some critical density ρ_c the largest connected component (cluster) of the system jumps abruptly from being independent of system size (microscopic) to being proportional to it (macroscopic).

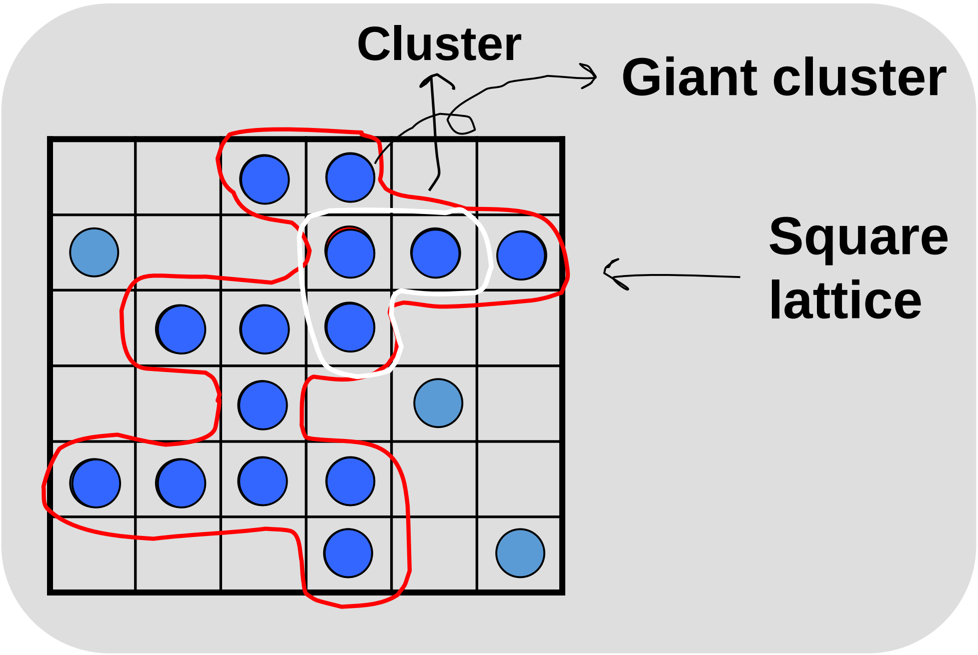
- Nucleation and condensation of gases into liquids (1930s)
- Physical processes such as fluid flow through disordered porous media (1957)
- Mathematical approach – Random Graphs (1959)



bond percolation



site percolation



Basic Percolation Theory

Important Quantities

$C(x)$ = set of all vertices/sites reachable from x

$|C(x)|$ = size of the component

Percolation probability

$$\vartheta(\rho) = P[|C(x)| = \infty] = 1 = \sum_{k=1} P[|C(x)| = k]$$

Mean cluster size

$$\chi_x(\rho) = E[|C(x)|]$$

Basic Percolation Theory

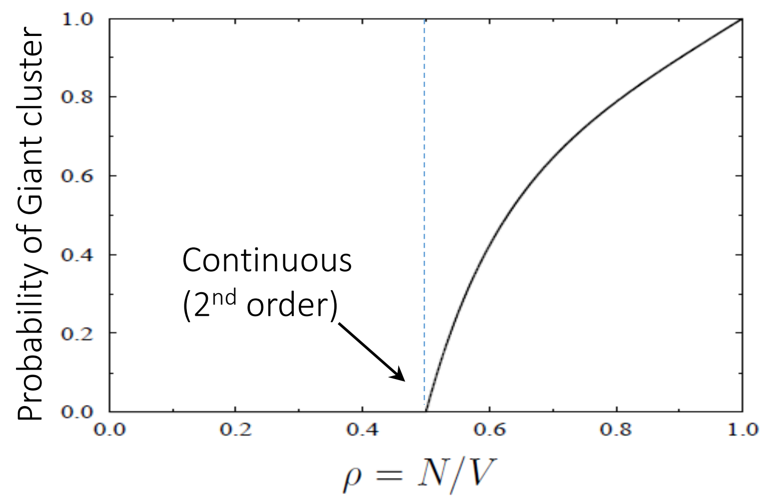
Percolation Transition

No giant cluster --> Giant cluster exists

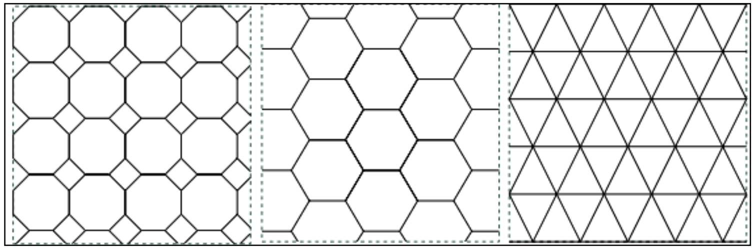
$$\vartheta(\rho) = \begin{cases} 0 & \text{for } \rho < \rho_c \\ > 0 & \text{for } \rho \geq \rho_c \end{cases}$$

$$\chi_x(\rho) = \begin{cases} < \infty & \text{for } \rho < \rho_c \\ \infty & \text{for } \rho \geq \rho_c \end{cases}$$

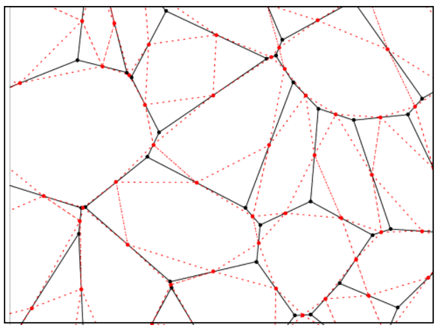
Typical phase transition for an **infinite** system



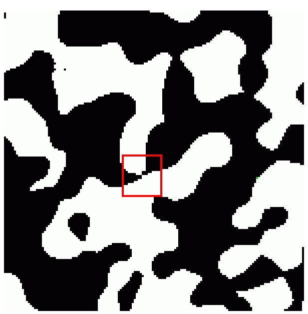
Basic Percolation Theory



regular lattices



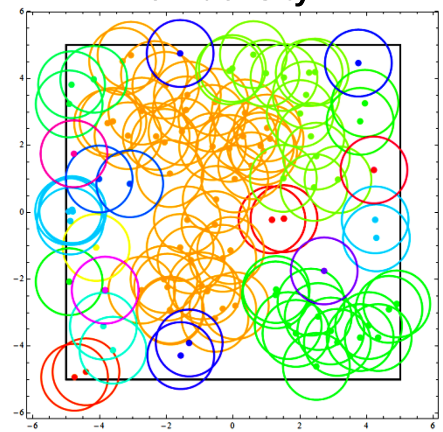
random and quasi-lattices



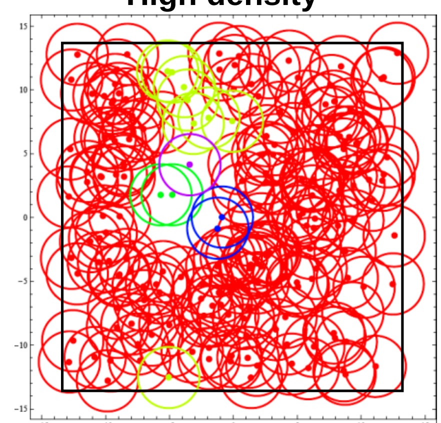
random potential landscape
(continuous)

Unit-Disk model with **periodic BCs** and a step connectivity function (on/off)

Low density

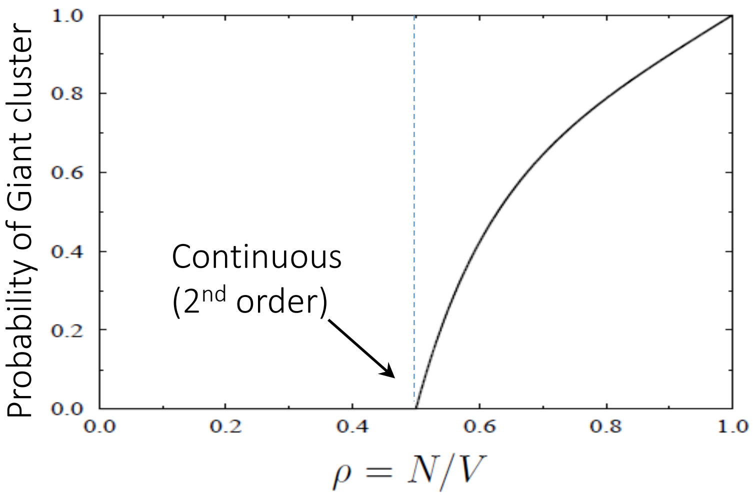


High density



Basic Percolation Theory

Typical phase transition for an **infinite** system



Typical "phase transition" for a **finite** system

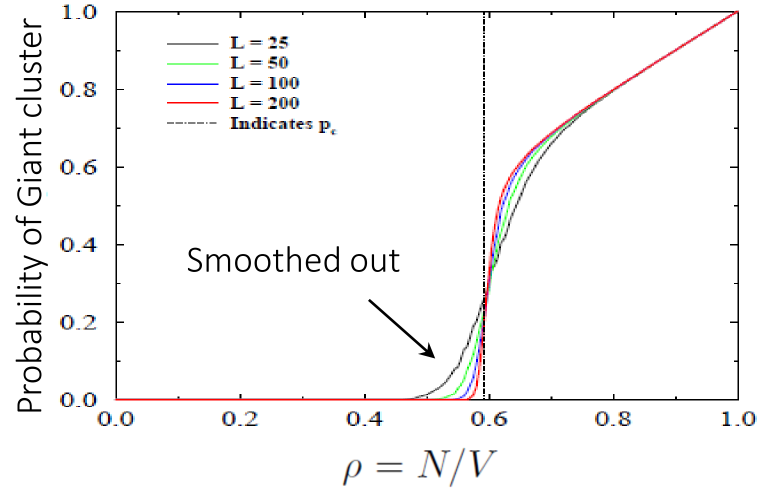
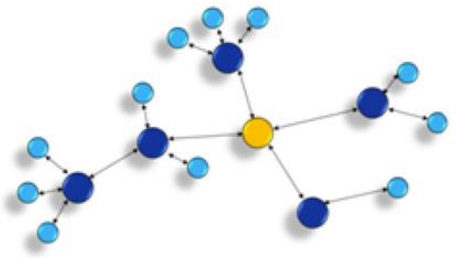
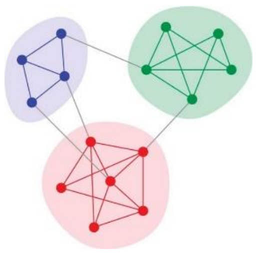


Table of critical densities

Lattice	# nn	Site percolation	Bond percolation
1d	2	1	1
2d Honeycomb	3	0.6962	$1 - 2\sin(\pi/18) \approx 0.65271$
2d Square	4	**0.592746**	**1/2**
2d Triangular	6	**1/2**	$2\sin(\pi/18) \approx 0.34729$
3d Diamond	4	0.43	0.388
3d Simple cubic	6	0.3116	0.2488
3d BCC	8	0.246	0.1803
3d FCC	12	0.198	0.119
4d Hypercubic	8	0.197	0.1601
5d Hypercubic	10	0.141	0.1182
6d Hypercubic	12	0.107	0.0942
7d Hypercubic	14	0.089	0.0787
Bethe lattice	z	**1/(z-1)**	**1/(z-1)**

Summary





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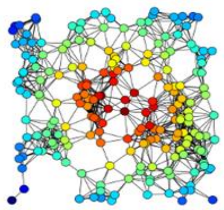
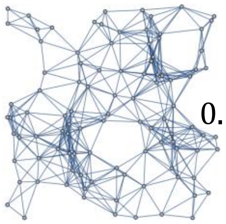
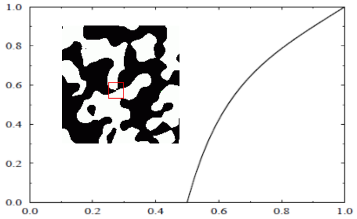
  - **Advanced** Graph Concepts
- **Statistical** Graph Properties

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$





- Basic **Percolation Theory**



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