RANDOM GRAPHS AND WIRELESS COMMUNICATION NETWORKS

Part 4: Modelling and Analysis of Ad Hoc Networks

1.5 hours

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Outline

- Applications of ad-hoc networks
- Modelling ad hoc networks
  - Random Geometric Graphs
  - Pairwise Connection function
  - Anisotropic nodes
  - Multiple Antennas
- Local Observables
  - Mean degree
  - Pair Formation
  - Degree distributions
  - Clustering coefficient
- Global Observables
  - Full connectivity
  - Boundary effects
  - K-connectivity

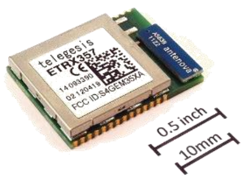
Ad hoc Networks

key ingredients

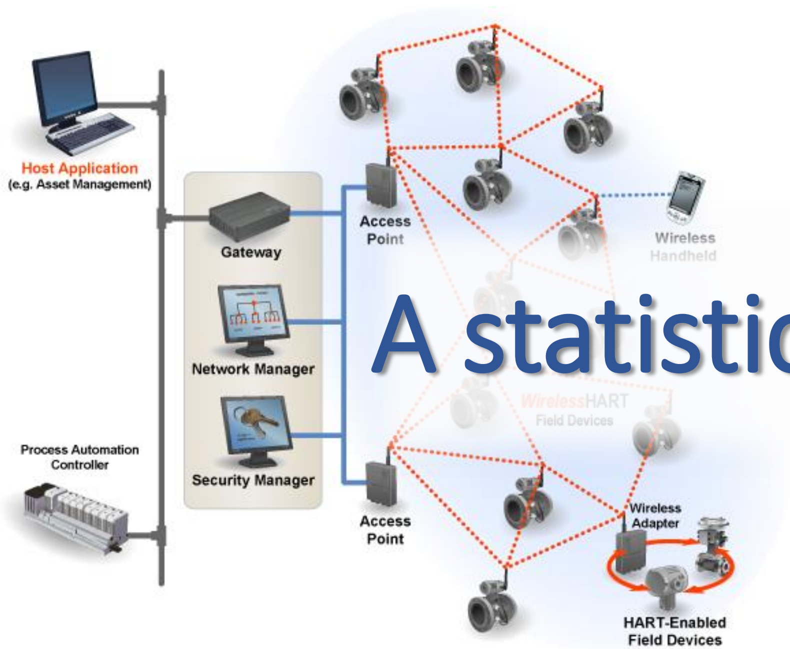
- Decentralized (no central BS but scalable)
- No pre-existing infrastructure “Place and Play”
- Self-configuring “on the fly”
- Multi-hop Routing (dynamic and adaptive)
  - ✓ Table-driven (proactive) routing
  - ✓ On-demand (reactive) routing
  - ✓ Hybrid (both proactive and reactive) routing
  - ✓ Hierarchical routing protocols (tree-based)
- Mobility (MANETS & VANETS)
- SmartPhone (SPANs) D2D, Bluetooth, WiFi-direct, LTE-direct

Applications of ad-hoc networks

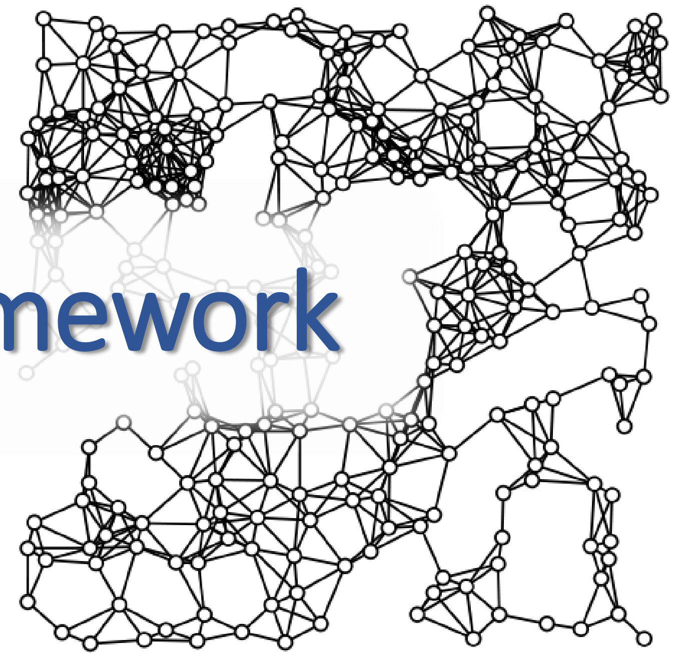
- Standardized under: IEEE 802.15.4
  - ZigBee, WirelessHART, ISA100.11a, and MiWi
- Wireless sensor networks (WSN)
  - Environmental, Agricultural, Industrial, Military
  - Disaster relief solutions
- Building automation
  - Smart metering, Industrial control
- Internet of Things
  - Smart Cities
- Agricultural / Infrastructure / Environmental monitoring



Modelling ad hoc (random) networks



A statistical framework



Modelling ad hoc (random) networks

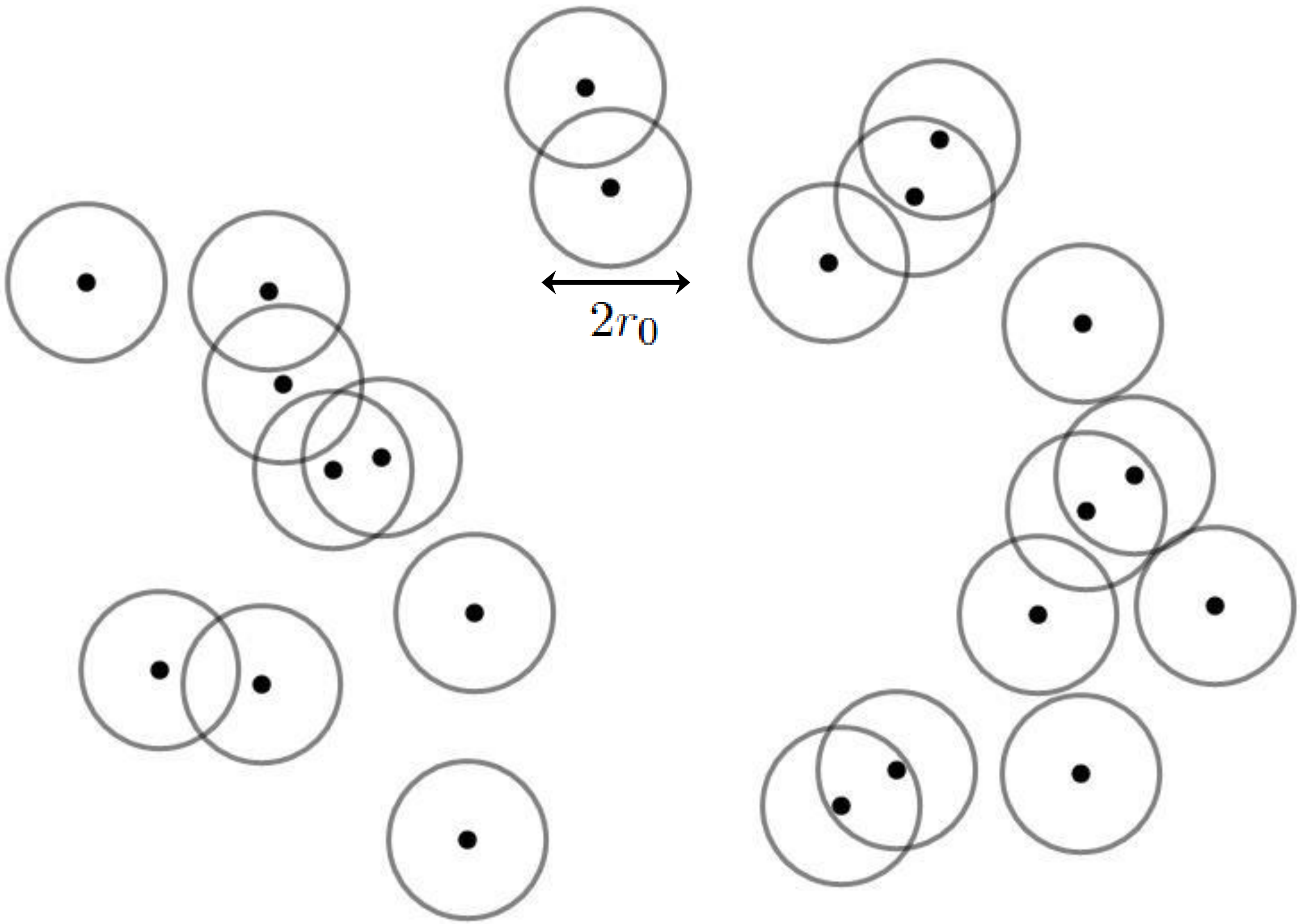
- Number and Location of wireless devices
  - Ad hoc, mobile, physical constraints and costs
- Multipath (fast fading)
- Shadowing (slow fading)
- Power control

  Cooperation - signalling overheads
- MAC protocols

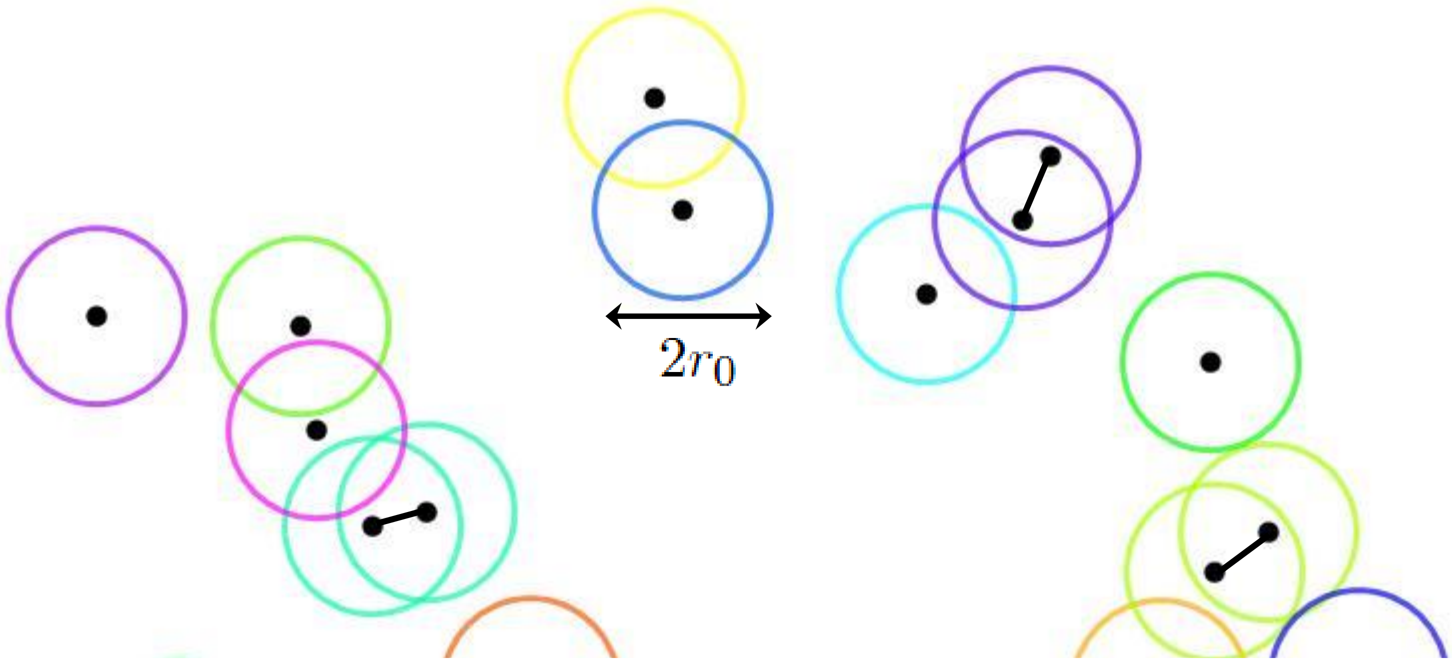
  TDMA / FDMA / CDMA /SDMA ...

  ALOHA / CSMA / CD / CA (802.11)
- Directional antennas
- Multiple antennas
- Transmission scheme (MRC / STBC)

Random geometric network



Random geometric network



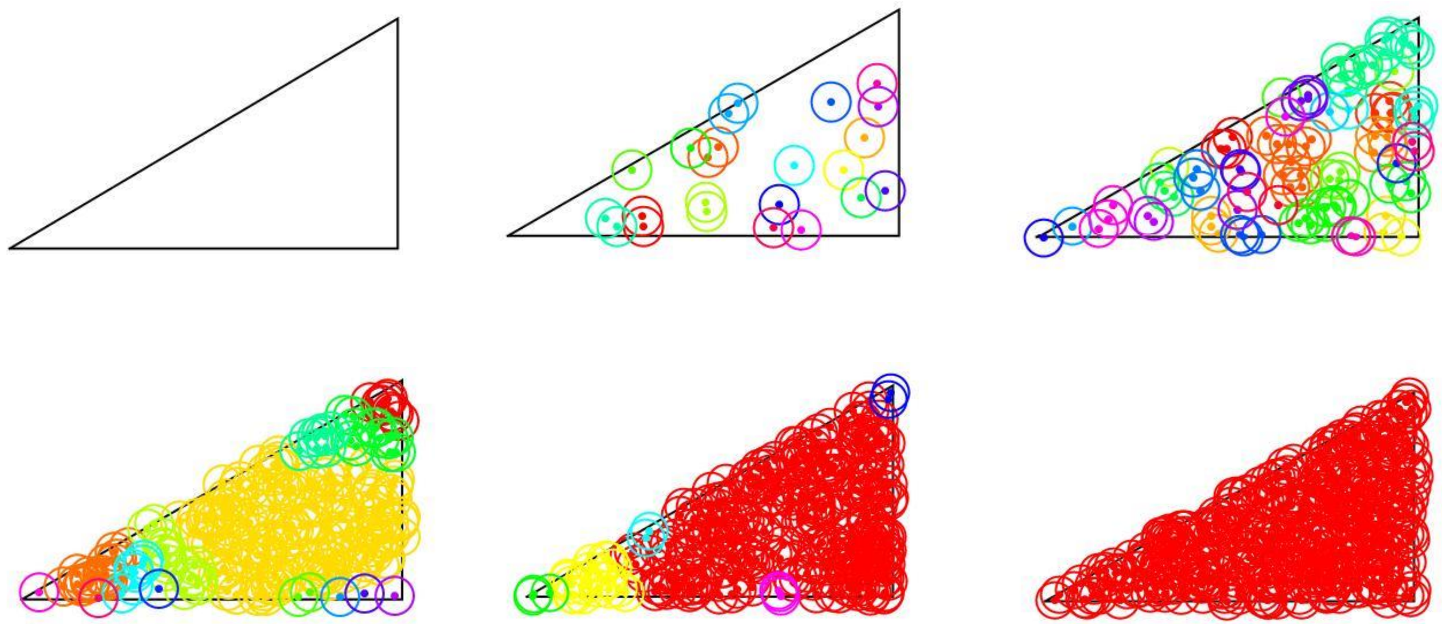
1. G. Gilbert, “Random plane networks,” SIAM J., vol. 9, no. 4, pp. 533–543, 1961.

2. M. D. Penrose, “Random Geometric Graphs”, Oxford University Press, 2003.

3. M. Franceschetti, R. Meester, ”Random networks for communication: from statistical physics to information systems”. Vol. 24. Cambridge University Press, 2008.

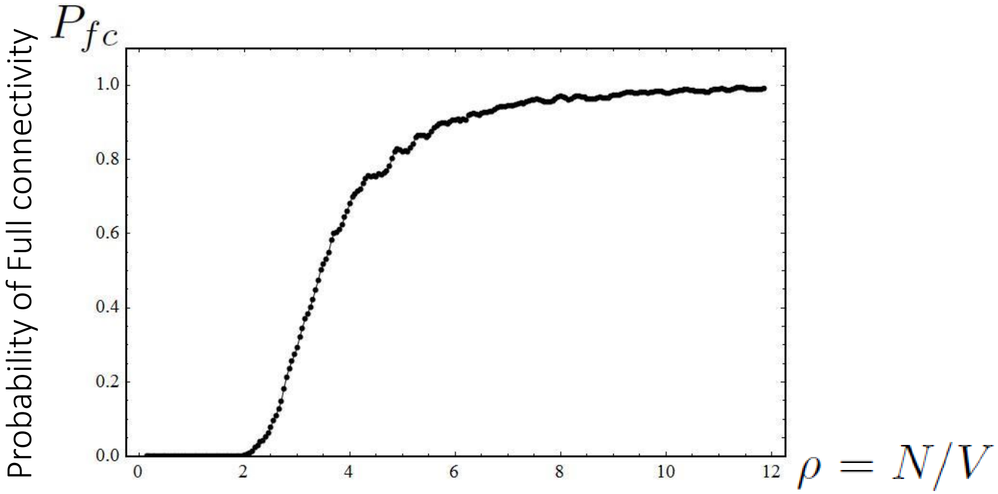
4. M. Walters, “Random Geometric Graphs,” in Surveys in Combinatronics 2011 (Robin Chapman, ed.), Cambridge University Press, 2011

Confined Random geometric networks



What is the probability of achieving a fully connected network at a given density?

Confined Random geometric networks



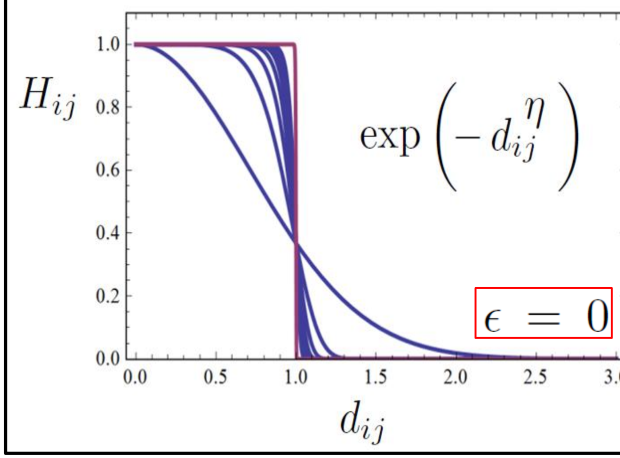
The goal is to develop a theory for P_{fc} that is able to provide useful analytic and physical insight which can support a variety of connectivity models, and can act as a basis for further development and analysis. To this end we will derive a general formula for P_{fc} which is simple, intuitive and practical.

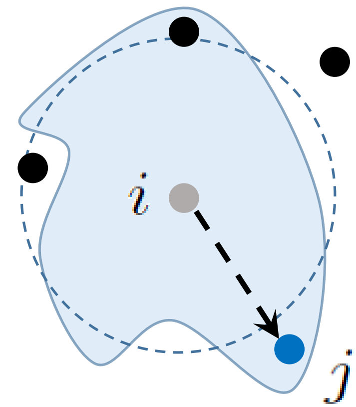
Pairwise Connection function

Complement of outage probability

$$H_{ij} = \mathbb{P}[\mathrm{SINR}_{ij} \geq q]$$

$$\mathrm{SINR}_{ij} = \frac{\mathcal{P}|h_{ij}|^2 g(d_{ij})}{\mathcal{N} + \gamma \mathcal{I}_j}$$





$$g(d_{ij}) = \frac{1}{\epsilon + d_{ij}^{\eta}}$$ Path loss attenuation function

$$\eta > 2$$ Path loss exponent

$$|h_{ij}|^2 \sim \exp(1)$$ Channel gain

$$\boxed{\gamma = 0}$$ Interference factor

F. Baccelli, and B. Blaszczyszyn. *Stochastic geometry and wireless networks: Theory*. Vol. 1. Now Publishers Inc, 2009.

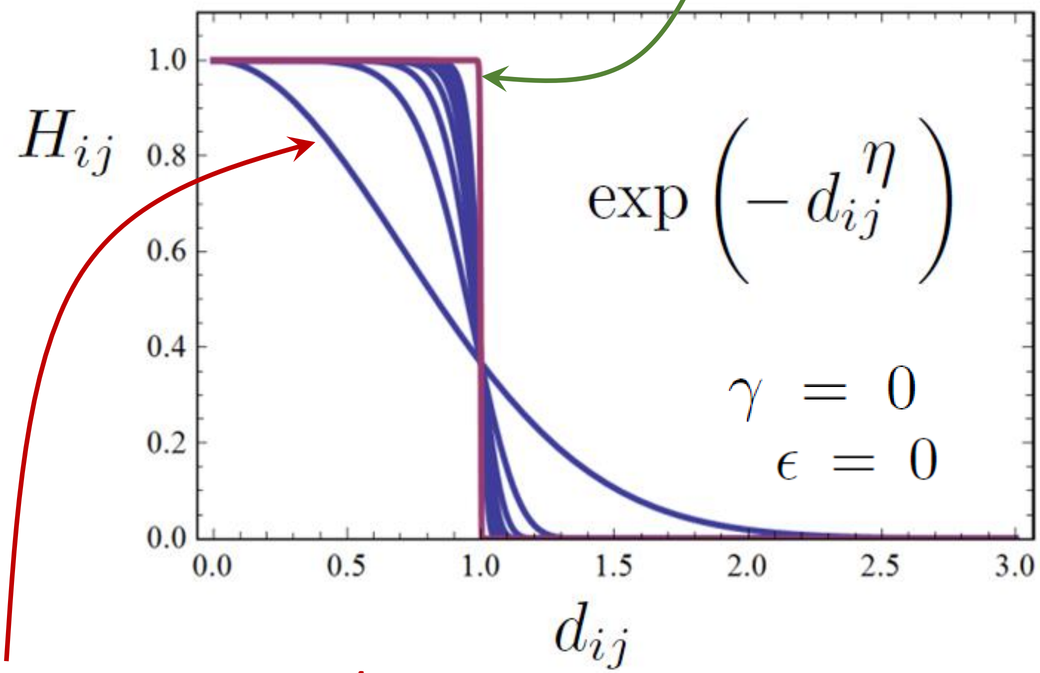
M. Haenggi. *Stochastic geometry for wireless networks*. Cambridge University Press, 2012.

Pairwise Connection function

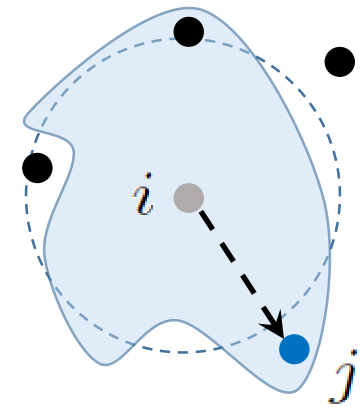
Rayleigh Fading

$$|h_{ij}|^2 \sim \exp(1)$$

Hard = Deterministic



Soft = Stochastic/Probabilistic



Pairwise Connection function

Rayleigh Fading

$$|h_{ij}|^2 \sim \exp(1)$$

$$\boxed{a_0 = 0}$$

Rician Fading

$$E_T(t) = \cos 2\pi f t$$

$$E_R(t) = \sum_{i=0}^{N} a_i \cos(2\pi f t + \phi_i)$$

$$E_R(t) = (a_0 + X)\cos 2\pi f t - Y \sin 2\pi f t$$

inphase and quadrature

$$X = \textstyle\sum_{i=1}^{N} a_i \cos\phi_i$$

$$Y = \textstyle\sum_{i=1}^{N} a_i \sin\phi_i$$

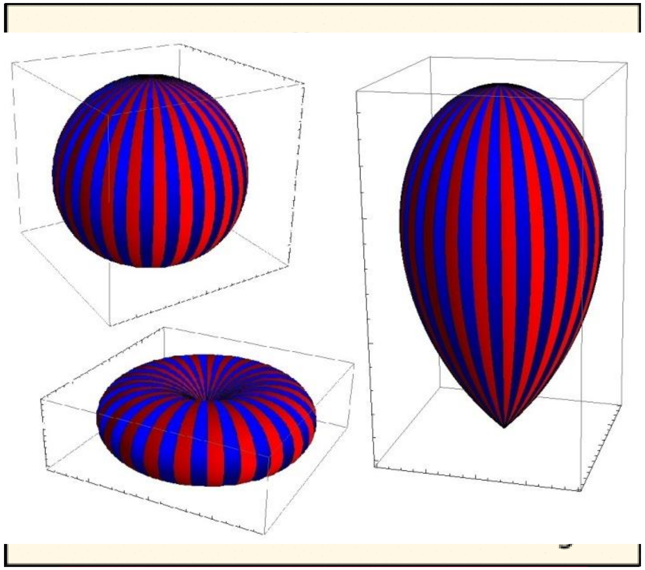
$X, Y \sim N(0, \sigma^2)$ (CLT)

$$P = (a_0 + X)^2 + Y^2$$

$$\boxed{f_P(x) = \frac{1}{2}\exp\left(-\frac{x+\lambda}{2}\right) I_0\left(\sqrt{\lambda x}\right)}$$

$$\lambda = (a_0/\sigma)^2$$

Anisotropically radiating nodes



$$\mathrm{SNR} \propto G_i G_j$$

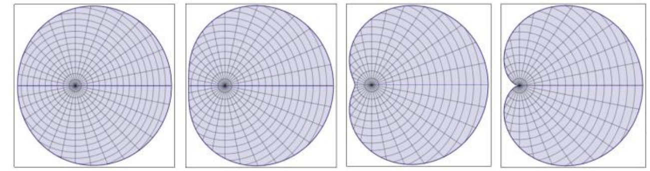
$$H_{ij} = \exp\left(-\frac{\beta r_{ij}^{\eta}}{G_i(\theta_j)G_j(\vartheta_j)}\right)$$

Antenna Tx and Rx gains

Normalization:

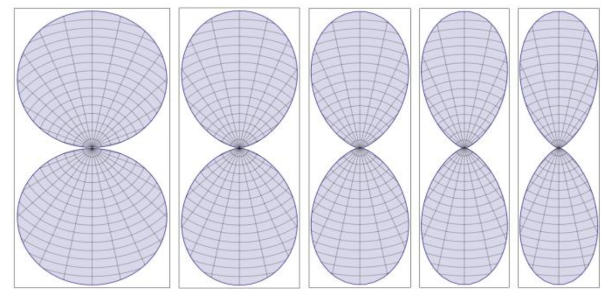
$$\int_0^{2\pi}\int_0^{\pi} G(\theta)\sin\theta \mathrm{d}\theta \mathrm{d}\phi = 4\pi$$

Patch



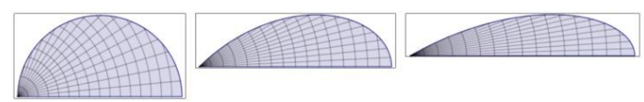
$$G(\theta) = 1 + \epsilon\cos\theta$$

Dipole



$$G(\theta) = \frac{2\Gamma(\frac{3+m}{2})}{\sqrt{\pi}\Gamma(\frac{2+m}{2})}\sin^m\theta$$

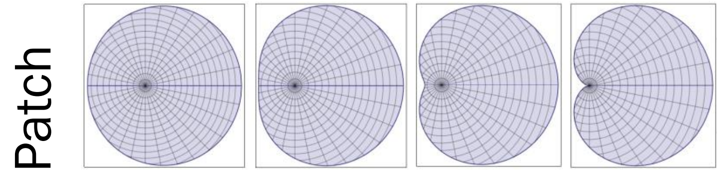
Horn



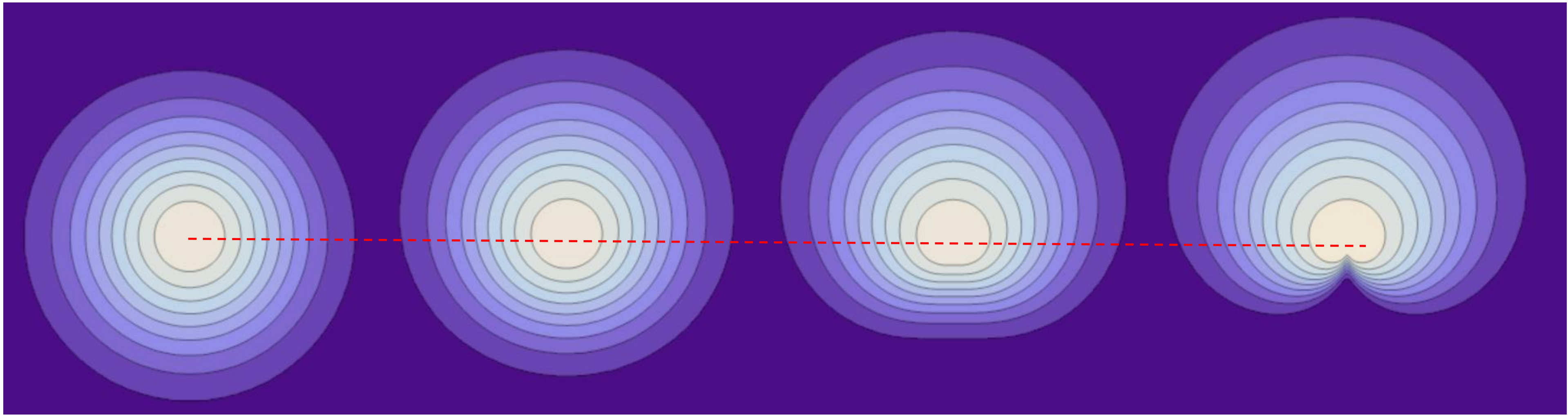
$$G(\theta) = \frac{2(\lambda^2-1)\cos\lambda\theta}{(\lambda\sin\frac{\pi}{2\lambda}-1)}$$

C.A. Balanis, *Advanced engineering electromagnetics*. Vol. 111. John Wiley & Sons, 2012.

Anisotropically radiating nodes



$$G(\theta) = 1 + \epsilon \cos\theta$$



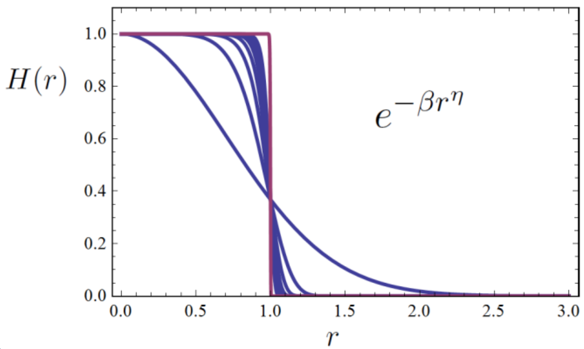
$$H_{ij} = \exp\left(-\frac{\beta r_{ij}^{\eta}}{G_i(\theta_j)G_j(\vartheta_j)}\right)$$

C.A. Balanis, *Advanced engineering electromagnetics*. Vol. 111. John Wiley & Sons, 2012.

Multiple antennas

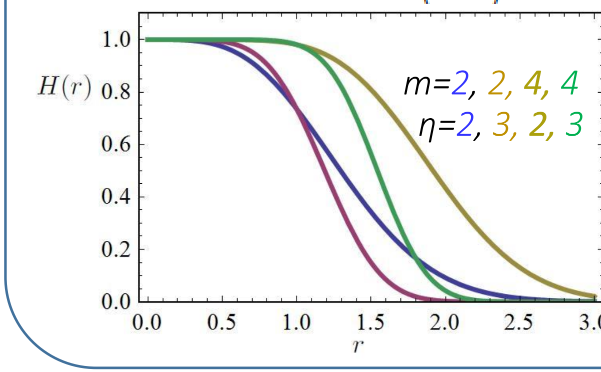
1) Single Input Single Output (**SISO**)

$$H_{ij}(r) = e^{-\beta r^{\eta}}$$



2) Single Input Multiple Output (**SIMO/MISO**)

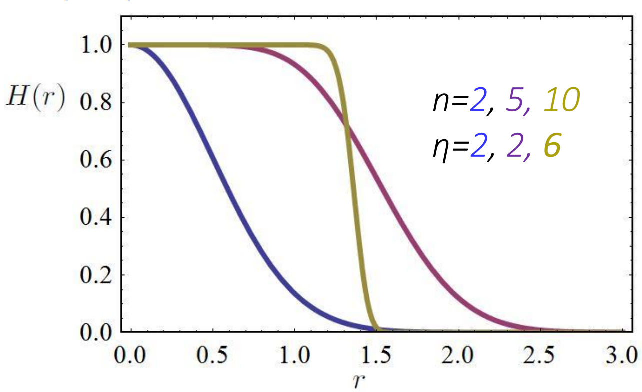
$$H_{ij}(r) = \frac{\Gamma(m, \beta r^{\eta})}{\Gamma(m)}$$



3) Multiple Input Multiple Output (**MIMO-MRC**) with 2 receiving and ***n*** transmitting antennas (or vice versa)

$$H_{ij}(r) = 1 - nP(n-1, \beta r^{\eta}) P(n+1, \beta r^{\eta}) + (n-1) P(n, \beta r^{\eta})^2$$

$P(n, x)$ is the regularised lower incomplete gamma function



Multiple antennas

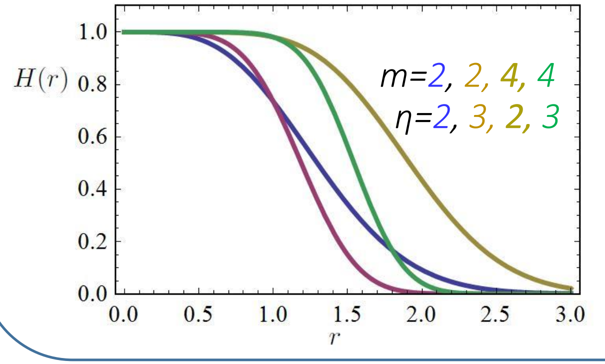
$$H_{ij} = \mathbb{P}[\mathrm{SNR}_{ij} \geq q]$$

SISO: $\mathrm{SNR}_{ij} = \frac{\mathcal{P}|h_{ij}|^2 g(d_{ij})}{\mathcal{N}}$

$$|h_{ij}|^2 \sim \exp(1)$$

2) Single Input Multiple Output (**SIMO/MISO**)

$$H_{ij}(r) = \frac{\Gamma(m, \beta r^\eta)}{\Gamma(m)}$$



MIMO: $n \times m$ channel matrix $\mathbf{H}$
(STBC)

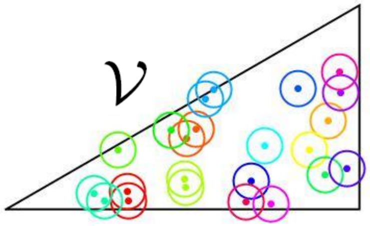
$$\mathrm{SNR}_{ij} = \frac{\mathcal{P}\,|\mathbf{H}|_F^2\;g(d_{ij})}{\mathcal{N}}$$

$$|\mathbf{H}|_F^2 = \sum_{k,l} |h_{kl}|^2$$

χ^2 distributed with *2mn* dof

$$H_{ij}(r) = \frac{\Gamma(mn, \beta r^\eta)}{\Gamma(mn)}$$

Local Network Observables



The probability of some node *i* connecting with some other node

$$H_i(\mathbf{r}_i) = \frac{1}{V}\int_{\mathcal{V}} H(r_{ij})\mathrm{d}\mathbf{r}_j$$

Pair formation probability: The probability that 2 randomly selected nodes connect to form a pair

$$p_2 = \frac{1}{V^2}\int_{\mathcal{V}^2} H(r_{ij})\mathrm{d}\mathbf{r}_i\mathrm{d}\mathbf{r}_j$$

Degree distribution: The probability that node *i* connects with exactly *k* other nodes

$$d_i(k) = \binom{N-1}{k} H_i^k (1-H_i)^{N-1-k}$$

$$d_i(k) \approx \frac{\lambda_i^k}{k!}e^{-\lambda_i} \qquad \lambda_i = (N-1)H_i$$

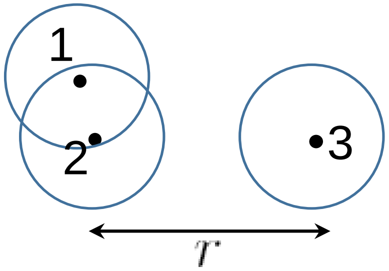
Mean degree

$$\lambda = \int \lambda_i \mathrm{d}\mathbf{r}_i / V = (N-1)p_2$$

$$\lambda \sim (N-1)\pi/(\beta V) \approx \rho\pi/\beta$$

Local Network Observables

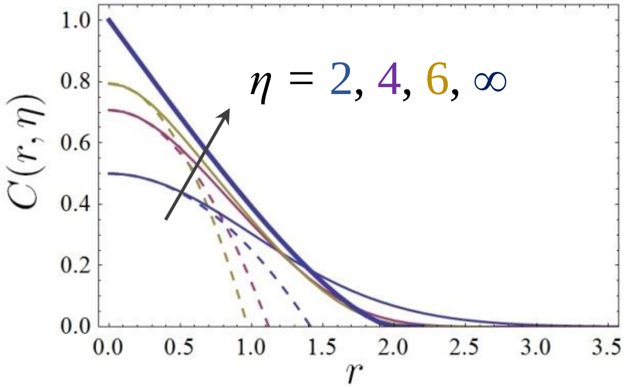
2-node correlation function

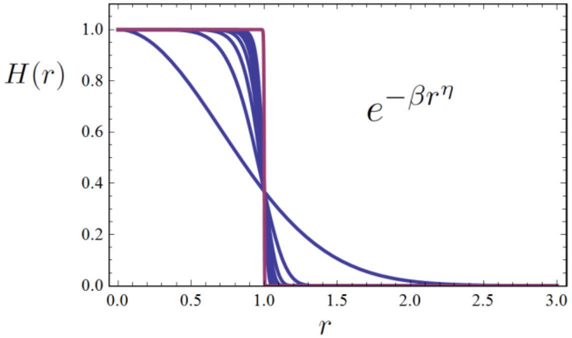


Probability that node 1 connects with node 3, given that node 1 is connected with node 2:

$$C(r,\eta) = \frac{\int_{\mathbb{R}^2} H_{12} H_{13} \mathrm{d}\mathbf{r}_1}{\int_{\mathbb{R}^2} H_{12} \mathrm{d}\mathbf{r}_1}$$

$$C(r,\eta) = 2^{-2/\eta} - \frac{\eta \beta^{2/\eta}}{8\Gamma\left(1 + 2/\eta\right)} r^2 + \mathcal{O}(r^4)$$





Nearby nodes are less correlated for soft connectivity functions

Global Network Observables

Global observables:
Given a graph, what is the probability of achieving full connectivity?
(Erdös 1959)

A graph is fully connected if there exists at least one multi-hop path connecting every two nodes.

A cluster expansion in 3 simple steps

1) Start with the probability of two nodes being connected (or not)

$$1 \equiv H_{ij} + (1 - H_{ij})$$

---

2) Multiply over the complete graph to get the probability of all possible combinations giving $2^{N(N-1)/2}$ terms

$$1 \equiv \prod_{i<j} [H_{ij} + (1 - H_{ij})] = \sum_g \mathcal{H}_g$$

---

3) Group into collections of terms determined by their largest cluster

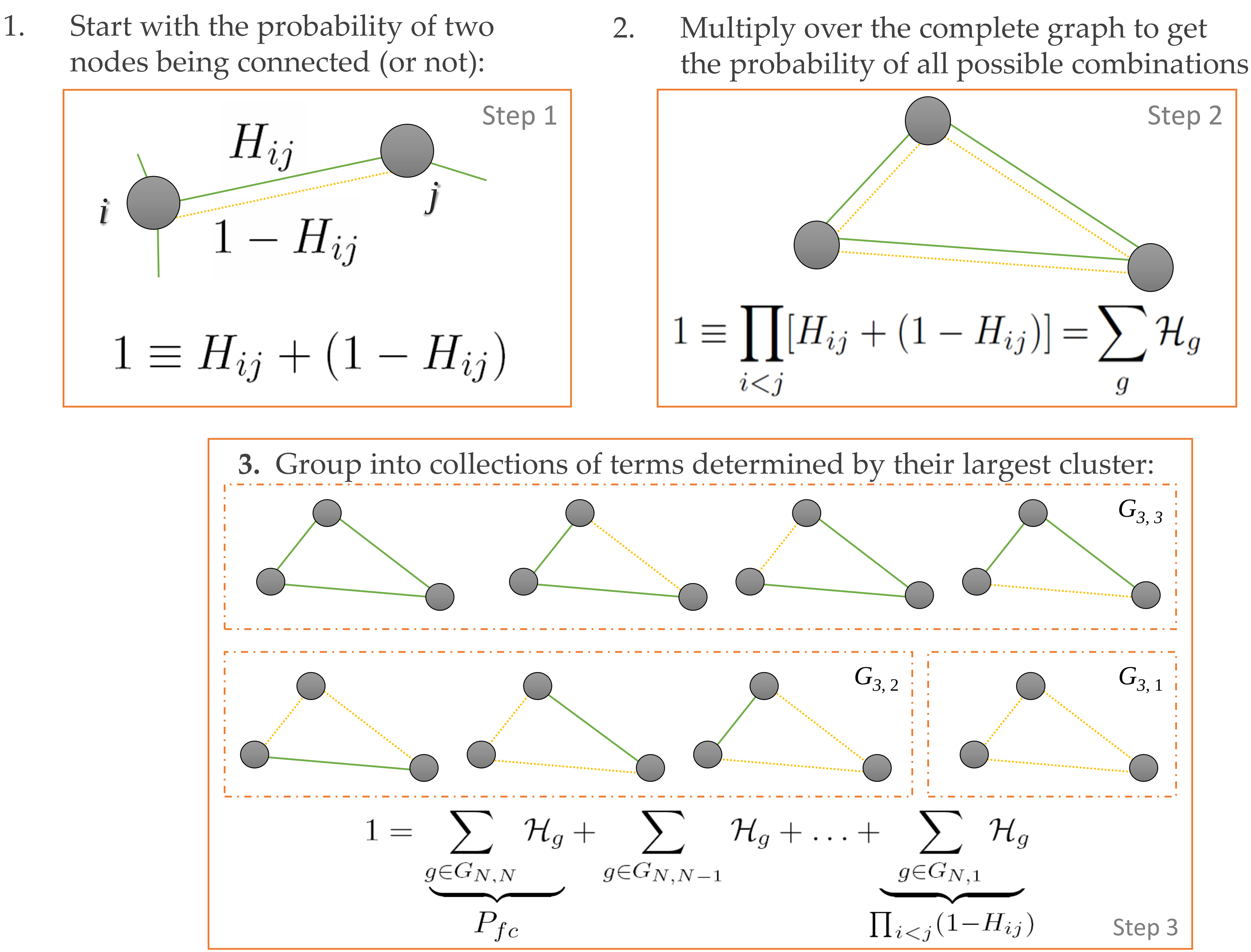
$$1 \equiv \underbrace{\sum_{g \in G_N} \mathcal{H}_g}_{P_{fc}} + \sum_{g \in G_{N-1}} \mathcal{H}_g + \ldots + \sum_{g \in G_1} \mathcal{H}_g$$

---

4) At high densities full connectivity is simply the complement of the probability of an isolated node.

$$P_{fc} = 1 - \sum_{g \in G_{N-1}} \mathcal{H}_g - \ldots$$

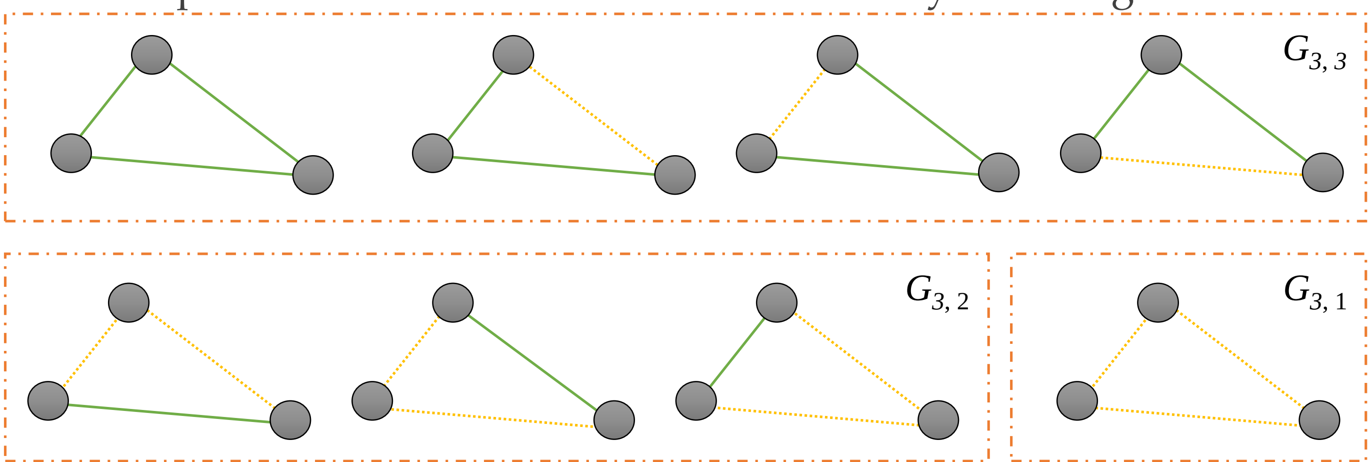
Full Connectivity: Corners, edges and faces, Journal of Statistical Physics, 147 (4), 758-778, (2012)


1. Start with the probability of two nodes being connected (or not):
Step 1
H_{ij}
i
j
$1 - H_{ij}$
$1 \equiv H_{ij} + (1 - H_{ij})$
2. Multiply over the complete graph to get the probability of all possible combinations:
Step 2
$1 \equiv \prod_{i<j} [H_{ij} + (1 - H_{ij})] = \sum_{g} \mathcal{H}_g$
3. Group into collections of terms determined by their largest cluster:
$G_{3,3}$
$G_{3,2}$
$G_{3,1}$
$1 = \underbrace{\sum_{g \in G_{N,N}} \mathcal{H}_g}_{P_{fc}} + \sum_{g \in G_{N,N-1}} \mathcal{H}_g + \ldots + \underbrace{\sum_{g \in G_{N,1}} \mathcal{H}_g}_{\prod_{i<j}(1-H_{ij})}$
Step 3

At high densities, *full connectivity* is the complement of an isolated node:

$$P_{fc} = 1 - \sum_{g \in G_{N-1}} \mathcal{H}_g - \dots$$

3. Group into collections of terms determined by their largest cluster:



$$1 = \underbrace{\sum_{g \in G_{N,N}} \mathcal{H}_g}_{P_{fc}} + \sum_{g \in G_{N,N-1}} \mathcal{H}_g + \dots + \underbrace{\sum_{g \in G_{N,1}} \mathcal{H}_g}_{\prod_{i<j}(1-H_{ij})}$$

Step 3

A cluster expansion in 3 simple steps

Define an average over all possible configurations

$$\boxed{\langle A\rangle = \frac{1}{V^N}\int_{\mathcal{V}^N} A(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\mathrm{d}\mathbf{r}_1\mathrm{d}\mathbf{r}_2\ldots\mathrm{d}\mathbf{r}_N}$$

$$P_{fc} \approx 1 - \langle \sum_{g\in G_{N-1}} \mathcal{H}_g\rangle$$

1) Node *N* is not connected to any of the other *N-1* nodes

2) Multiply by *N* since all nodes are identical

$$= 1 - N\langle \prod_{j=1}^{N-1}(1-H_{jN})\rangle$$

3) Since we are comparing pairs of nodes, *N-2* integrals can be decoupled through a change of variables:

$$\mathbf{x}_j = \mathbf{r}_j - \mathbf{r}_N$$

$$\mathrm{d}\mathbf{x}_j = \mathrm{d}\mathbf{r}_j$$

$$= 1 - \frac{N}{V^N}\int_{\mathcal{V}^N}\prod_{j=1}^{N-1}(1-H(\mathbf{r}_{jN}))\mathrm{d}\mathbf{r}_1\ldots\mathrm{d}\mathbf{r}_N$$

$$= 1 - \frac{N}{V}\int_{\mathcal{V}}\left(1-\frac{1}{V}\int_{\mathcal{V}}H(\mathbf{r}_{1N})\mathrm{d}\mathbf{r}_1\right)^{N-1}\mathrm{d}\mathbf{r}_N$$

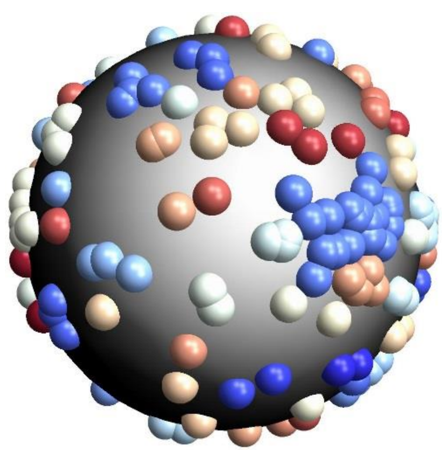
The Homogeneous case

Assuming that the network is homogeneous, implies that there are no boundaries and therefore the system is symmetric under translational transformations. This allows for a final change of variables and we are left with a single integral:

$$P_{fc} \approx 1 - N\left(1 - \frac{1}{V}\int_{\mathcal{V}} H(\mathbf{r})\mathrm{d}\mathbf{r}\right)^{N-1} \longleftarrow \boxed{e^{x} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}}$$

$$= 1 - Ne^{-\rho\int_{\mathcal{V}} H(\mathbf{r})\mathrm{d}\mathbf{r}}\left[1 + \frac{1}{N}\left(\rho\int_{\mathcal{V}} H(\mathbf{r})\mathrm{d}\mathbf{r} - \frac{\left(\rho\int_{\mathcal{V}} H(\mathbf{r})\mathrm{d}\mathbf{r}\right)^{2}}{2}\right) + \mathcal{O}\left(\frac{\rho^{4}}{N^{2}}\right)\right]$$

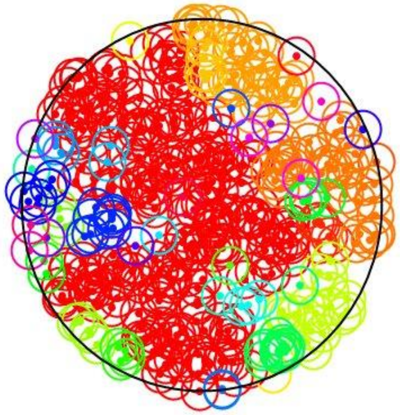
Example of a homogeneous network space: Surface of a Sphere

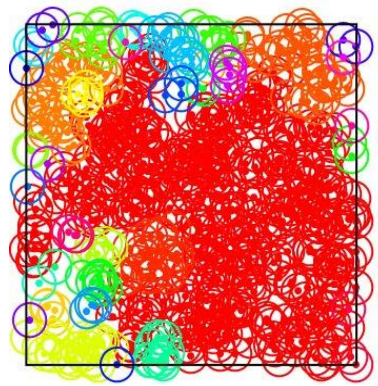


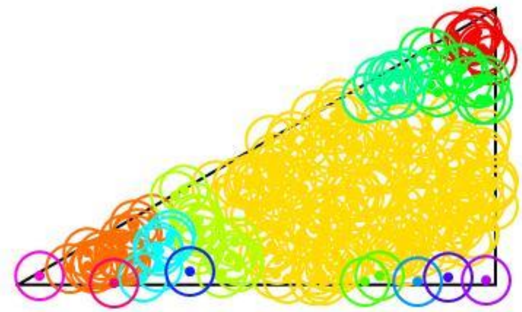
The Inhomogeneous case

System is **not** symmetric under translational transformations and so **border effects** become important.

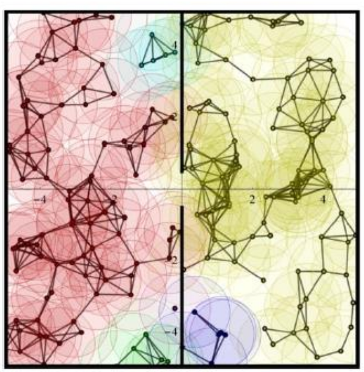
Here are some simple examples:

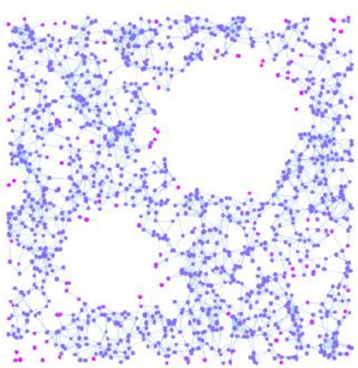


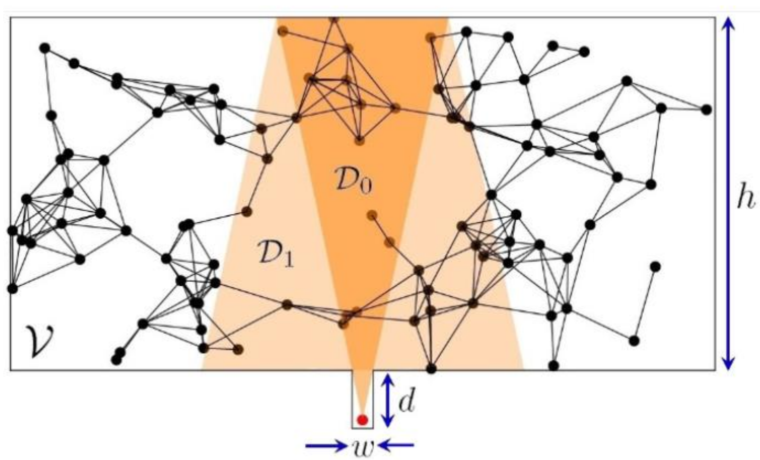


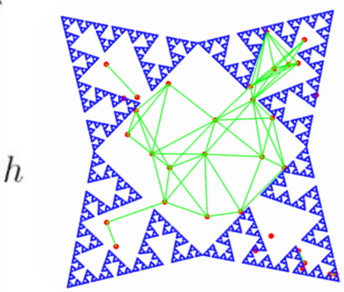


Here are some more interesting (non-convex) examples:









Inhomogeneous problem (Boundary effects)

System is not symmetric under translational transformations

$$P_{fc} \approx 1 - \langle \sum_{g \in G_{N-1}} \mathcal{H}_g \rangle$$

$$= 1 - N \langle \prod_{j=1}^{N-1} (1 - H_{jN}) \rangle$$

1) Node N is not connected to any of the other N-1 nodes

2) Multiply by N since all nodes are identical

3) Since we are comparing pairs of nodes, N-2 integrals can be decoupled through a change of variables:

$$\mathbf{x}_j = \mathbf{r}_j - \mathbf{r}_N$$

$$\mathrm{d}\mathbf{x}_j = \mathrm{d}\mathbf{r}_j$$

For more details see Ref. 2

$$= 1 - \frac{N}{V^N} \int_{\mathcal{V}^N} \prod_{j=1}^{N-1} (1 - H(\mathbf{r}_{jN}))\mathrm{d}\mathbf{r}_1 \ldots \mathrm{d}\mathbf{r}_N$$

$$= 1 - \frac{N}{V} \int_{\mathcal{V}} \left(1 - \frac{1}{V} \int_{\mathcal{V}} H(\mathbf{r}_{1N})\mathrm{d}\mathbf{r}_1\right)^{N-1} \mathrm{d}\mathbf{r}_N$$

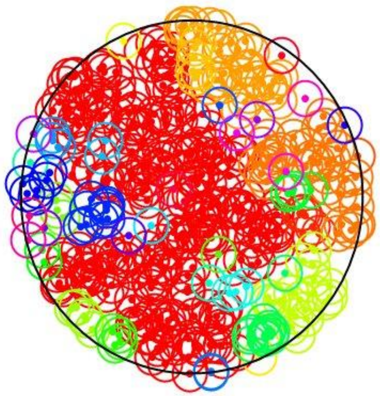
4) Assume that N is large, express the bracket as an exponential, and re-label node N to 2.

$$= 1 - \rho \int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12})\mathrm{d}\mathbf{r}_1} (1 + \mathcal{O}(N^{-1}))\, \mathrm{d}\mathbf{r}_2$$

Observation: The mass of the pair connectedness function is in the exponent.

Conclusion: Exterior integral is **maximum** when interior integral is **minimum**.
Full connectivity is dominated by regions in the network space that are hard to connect to *i.e.* **<u>near the boundaries!</u>**

Example: Ad hoc network in a disk domain



$$P_{fc} = 1 - \rho \int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \, \mathrm{d}\mathbf{r}_2$$

Example 1: **Disk domain of radius *R***

1) Use Euclidean distance between two nodes in polar coordinates:

$$d(\mathbf{r}_1, \mathbf{r}_2) = \sqrt{|\mathbf{r}_1|^2 + |\mathbf{r}_2|^2 - 2|\mathbf{r}_1||\mathbf{r}_2| \cos\theta}$$

2) Set $\eta = 2$ and consider **SISO** link model. Interior integral gives connectivity mass:

$$\int_{d_R} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1 = \int_0^R \int_0^{2\pi} \left(r_1 e^{-\beta(r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta)} \right) \mathrm{d}\theta \mathrm{d}r_1$$

$$= 2\pi \int_0^R \left(r_1 I_0(2r_1 r_2 \beta) e^{-\beta(r_1^2 + r_2^2)} \right) \mathrm{d}r_1,$$

3a) Taylor expand integrand around ***r_2=0*** and integrate to obtain connectivity mass **away** from the boundaries:

$$= \frac{\pi}{\beta} \left(1 - e^{-\beta R^2} \right) + \mathcal{O}(r_2^2) \approx \frac{\pi}{\beta} \qquad \beta R^2 \gg 1$$

3b) Use asymptotic expression of modified Bessel function of he first kind $I_0(x) = \frac{e^x}{\sqrt{2\pi x}} \left(1 + \mathcal{O}(x^{-1})\right)$ to obtain the connectivity mass **near** boundaries:

$$= 2\pi \int_0^R \left(r_1 \frac{e^{2r_1 r_2 \beta}}{\sqrt{4\pi r_1 r_2 \beta}} e^{-\beta(r_1^2 + r_2^2)} \right) \mathrm{d}r_1$$

$$\approx \frac{\sqrt{\pi}}{\sqrt{\beta}} \int_0^R e^{-\beta(r_1 - r_2)^2} \mathrm{d}r_1$$

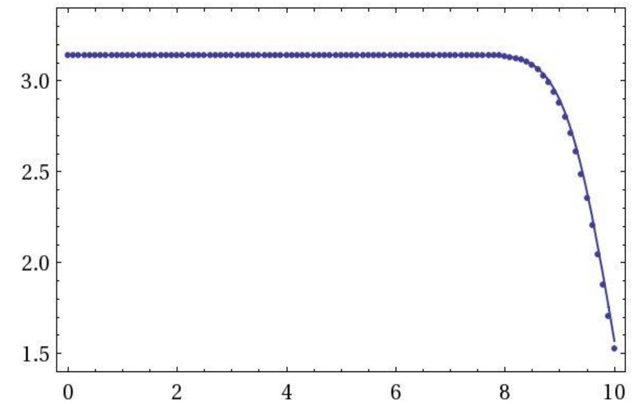
$$= \frac{\pi}{2\beta} \left(\mathrm{erf}\left[\sqrt{\beta}(R - r_2)\right] + \mathrm{erf}\left[\sqrt{\beta} r_2\right] \right)$$

4) Matching the two solutions we obtain an approximation for the connectivity mass:

$$\approx \frac{\pi}{2\beta} \left(\mathrm{erf}\left[\sqrt{\beta}(R - r_2)\right] + 1 \right)$$

Example: Ad hoc network in a disk domain

$$
\begin{aligned}
\int_{d_R} H(\mathbf{r}_{12})\mathrm{d}\mathbf{r}_1 &= \int_0^R \int_0^{2\pi} \left(r_1 e^{-\beta\left(r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta\right)} \right) \mathrm{d}\theta \mathrm{d}r_1 \\
&\approx \frac{\pi}{2\beta} \left(\mathrm{erf}\left[\sqrt{\beta}(R - r_2) \right] + 1 \right) \\
&= \frac{\pi}{2\beta} f(r_2)
\end{aligned}
$$

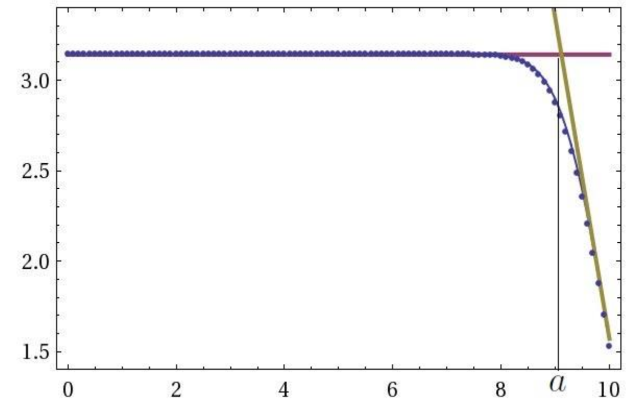


Check approximation obtained using ***β=1*** and ***R=10***. Dots are obtained from numerical integration

5) Further approximate ***f(r)*** by a piecewise linear function:

$$
\tilde{f}(r) = \begin{cases} c_1, & \text{for} \quad 0 < r < a, \\ c_2 - m(r - R), & \text{for} \quad a \leq r < R \end{cases}
$$

$$
\boxed{
\begin{aligned}
c_1 &= 2\mathrm{erf}\left[\sqrt{\beta}\tfrac{R}{2}\right] \approx 2 \qquad & m &= \frac{2\sqrt{\beta}}{\sqrt{\pi}}\left(1 - e^{-\beta R^2}\right) \approx \frac{2\sqrt{\beta}}{\sqrt{\pi}} \\
c_2 &= \mathrm{erf}\left[\sqrt{\beta}R\right] \approx 1 & & \beta R^2 \gg 1
\end{aligned}
}
$$

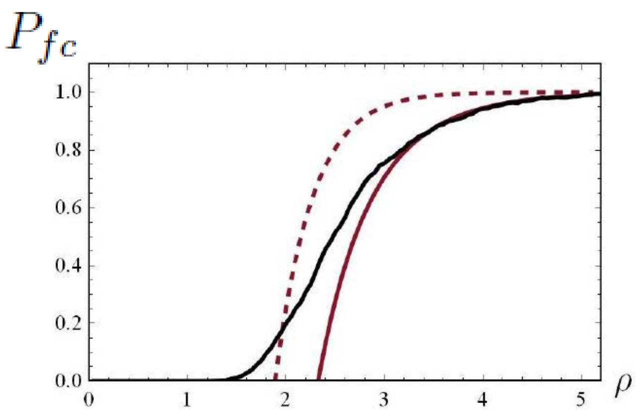


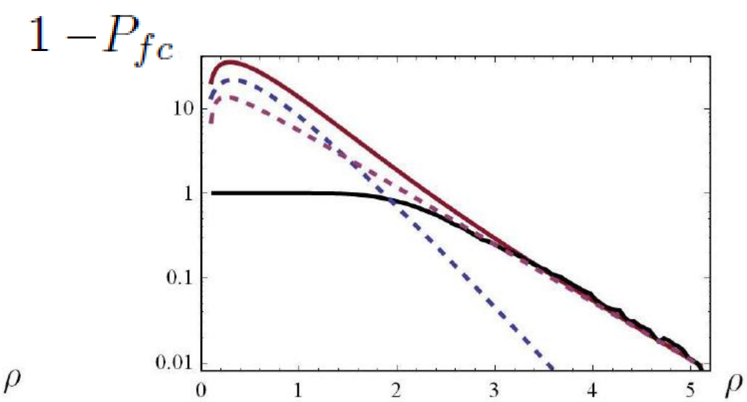
Example: Ad hoc network in a disk domain

6) Calculate probability of full connectivity using the piecewise linear approximation of connectivity mass:

$$\tilde{f}(r) = \begin{cases} c_1, & \text{for} \quad 0 < r < a, \\ c_2 - m(r - R), & \text{for} \quad a \le r < R \end{cases}$$

$$\begin{aligned} P_{fc} &= 1 - \rho \int_{d_R} e^{-\rho \int_{c_R} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \mathrm{d}\mathbf{r}_2 \\ &\approx 1 - 2\pi\rho \int_0^R r e^{-\rho \frac{\pi}{2\beta} \tilde{f}(r)} \mathrm{d}r \\ &= 1 - \underbrace{\pi R^2 \rho e^{-\rho \frac{\pi}{\beta}}}_{\text{Area term}} \left(1 - \frac{\sqrt{\pi}}{\sqrt{\beta} R} - \frac{2\sqrt{\beta}}{\rho\sqrt{\pi} R} + \mathcal{O}(R^{-2})\right) - \underbrace{2\pi R \sqrt{\frac{\beta}{\pi}} e^{-\frac{\pi}{2\beta}\rho}}_{\text{Perimeter term}} \left(1 - \frac{\sqrt{\beta}}{\rho\sqrt{\pi} R} + \mathcal{O}(R^{-2})\right) \end{aligned}$$





Figures: Black curves are numerical simulations. Dashed curve only includes "Area" term. Full curve is the new analytic prediction. Notice that at high densities there is excellent agreement.

A general formula for the probability of full connectivity

$$P_{fc} = 1 - \rho \int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \, \mathrm{d}\mathbf{r}_2$$

1) Boundary components separate and can be summed individually
2) Thus we can postulate the following general formula:

$$\boxed{\mathrm{P}_{fc} \approx 1 - \sum_{i=0}^{d} \sum_{j_i} \rho^{1-i} G_{j_i} V_{j_i} e^{-\rho \omega_{j_i} \int_0^\infty r^{d-1} H(r) \mathrm{d}r}}$$

3) The first sum runs over objects of different co-dimension with ***i=0*** being the volume term, and ***i=d*** being the corner terms.
4) The second sum runs over objects of equal co-dimension e.g. for a cube in ***d=3***, we have ***1*** volume term, ***8*** faces, ***12*** edges, and ***8*** corners.
5) $\mathbf{G}_{j_i}$ is a geometric factor which is ***H(r)*** dependent and can be calculated independently for each distinct boundary component.
6) $\mathbf{V}_{j_i}$ is the volume of each object with respect to the appropriate dimension
7) ω_{j_i} is the solid angle available from the corresponding object e.g. for a cube in ***d=3*** it is simply **4π** for the volume term, **2π** for faces, **π** for edges, and **π/2** for corners
8) The remaining radial integral is **d**-dimensional Homogeneous connectivity mass.

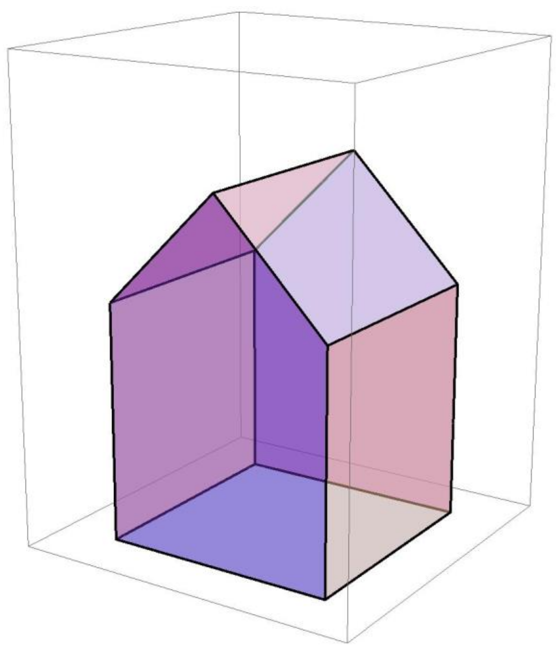
The simple format of this general formula emphasizes the logical decomposition of the domain into objects of different full connectivity importance. Reusable once the terms have been found for particular boundary components – a type of Universality.

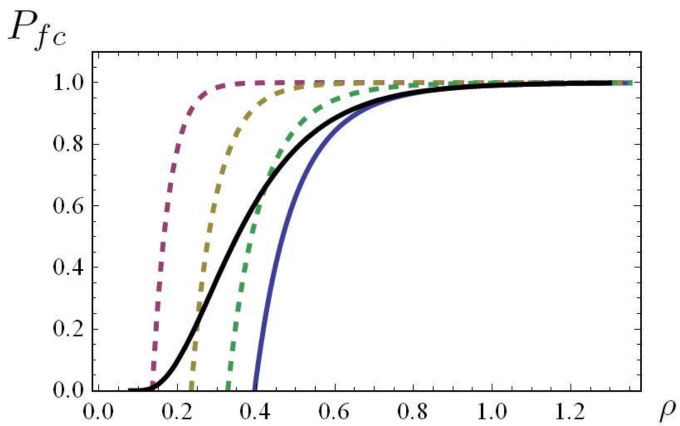
What kinds of geometries can this theory analyse?

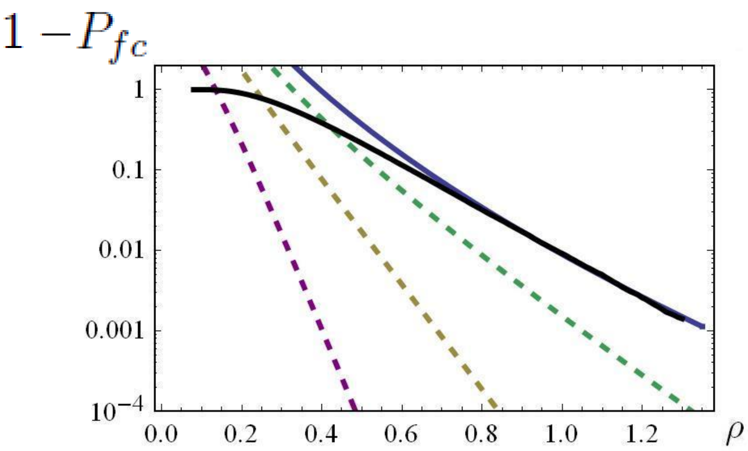
$$\mathrm{P}_{fc} \approx 1 - \sum_{i=0}^{d} \sum_{j_i} \rho^{1-i} G_{j_i} V_{j_i} e^{-\rho \omega_{j_i} \int_0^\infty r^{d-1} H(r) \mathrm{d}r}$$

- Answer: *LOTS!*
  - Limited to convex geometries
  - Complicated polyhedrons, like prisms
  - Example: a right prism in the shape of...

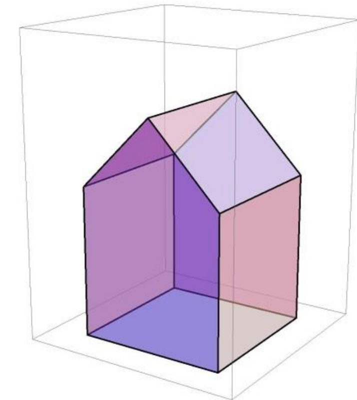
...a house!







Example: "house" domain (using MIMO model)



$$\int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \mathrm{d}\mathbf{r}_2$$

1) We consider a **2x2 MIMO** pair connectedness function with $\eta = 2$: $H(r) = e^{-\beta r^2}\left(\beta^2 r^4 + 2 - e^{-\beta r^2}\right)$

2) We expect that P_{fc} is a sum of the different boundary contributions: $P_{fc} \approx 1 - \rho\left(C_1 + C_2 + E_1 + E_2 + F + U\right)$

3) Start by considering the **corner** terms ***C₁*** and ***C₂*** using cylindrical coordinates: $d(\mathbf{r}_1, \mathbf{r}_2) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2}$

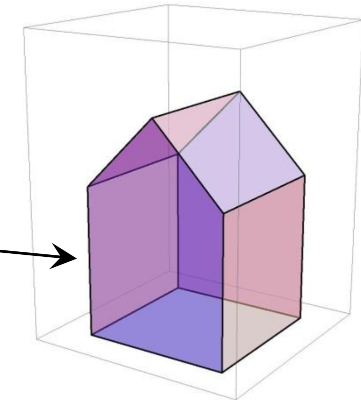
4) Substituting this into ***H(r)*** and Taylor expanding around $r_2=0$ and $z_2=0$ (i.e. near the corner) and keeping only linear terms, we can calculate the connectivity mass:

$$\begin{aligned} M_H(\mathbf{r}_2) &= \int_0^\infty \int_0^\vartheta \int_0^\infty r_1 H(\mathbf{r}_{12})\, \mathrm{d}r_1 \mathrm{d}\theta_1 \mathrm{d}z_1 \\ &= \frac{1}{8\beta}\left(14 z_2 \vartheta + \frac{23-\sqrt{2}}{2}\sqrt{\frac{\pi}{\beta}}\vartheta + 7\pi r_2\left(\sin\theta_2 - \sin(\theta_2 - \vartheta)\right)\right) \end{aligned}$$

5) We can now also calculate the exterior integral to obtain a general expression for the corners

$$\begin{aligned} \int_{\mathcal{V}} e^{-\rho M_H(\mathbf{r}_2)} \mathrm{d}\mathbf{r}_2 &= \iiint r_2 e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \mathrm{d}r_2 \mathrm{d}z_2 \mathrm{d}\theta_2 \\ &= \frac{256\beta^3 \csc\vartheta}{343\pi^2\rho^3\vartheta} e^{-\frac{(23-\sqrt{2})\sqrt{\pi}\rho\vartheta}{16\beta^{3/2}}}. \end{aligned}$$

Example: “house” domain (using MIMO model)



6) We now considering the **Edge** terms E_1 and E_2 using cylindrical coordinates such that $r=z=0$ corresponds to the midpoint of the edge:

$$d(\mathbf{r}_1, \mathbf{r}_2) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2) + (z_1 - z_2)^2}$$

$$\int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \mathrm{d}\mathbf{r}_2$$

7) Taylor expanding around $r_2=0$ and $z_2=0$ (i.e. near the midpoint of the edge) and keeping only linear terms, we can calculate the connectivity mass:

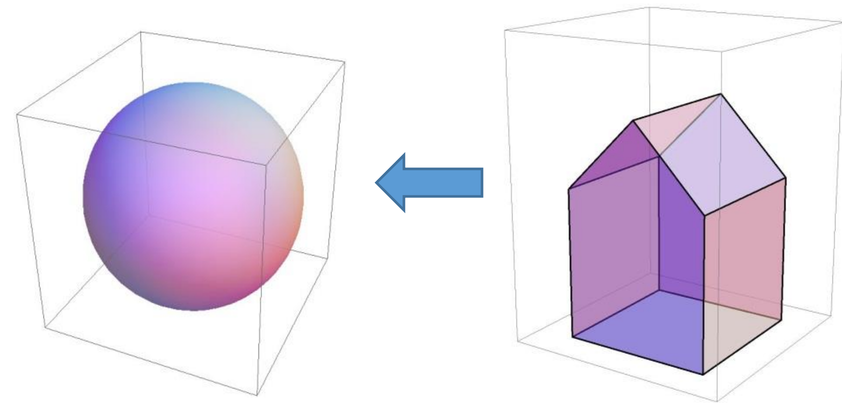
$$M_H(\mathbf{r}_2) = \frac{1}{4\beta}\left(\frac{23-\sqrt{2}}{2}\sqrt{\frac{\pi}{\beta}}\vartheta + 7\pi r_2\left(\sin\theta_2 - \sin(\theta_2 - \vartheta)\right)\right)$$

8) Note that we have assumed that $\sqrt{\beta}L \gg 1$ so that we can make the following approximations: $\exp(-\beta L^2/4) \approx 0$ $\quad \mathrm{erf}\left(L\sqrt{\beta/2}\right) \approx 1$

9) We can now also calculate the exterior integral to obtain a general expression for the edges

$$\int_{-\frac{L}{2}}^{\frac{L}{2}} \int_0^{\vartheta} \int_0^{\infty} r_2 e^{-\rho M_H(\mathbf{r}_2)} \mathrm{d}r_2 \mathrm{d}\theta_2 \mathrm{d}z_2 = \frac{16L\beta^2 \csc\vartheta}{49\pi^2\rho^2} e^{-\frac{(23-\sqrt{2})\sqrt{\pi}\rho\vartheta}{8\beta^{3/2}}}$$

Example: “house” domain (using MIMO model)



$$\int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12})\mathrm{d}\mathbf{r}_1} \mathrm{d}\mathbf{r}_2$$

10) Having already considered the corners and edges, we can treat the **Surface** term independently and so the house domain with surface area: $S = 2B + ph$ is topologically equivalent to a sphere with surface area: $S = 4\pi R^2$.

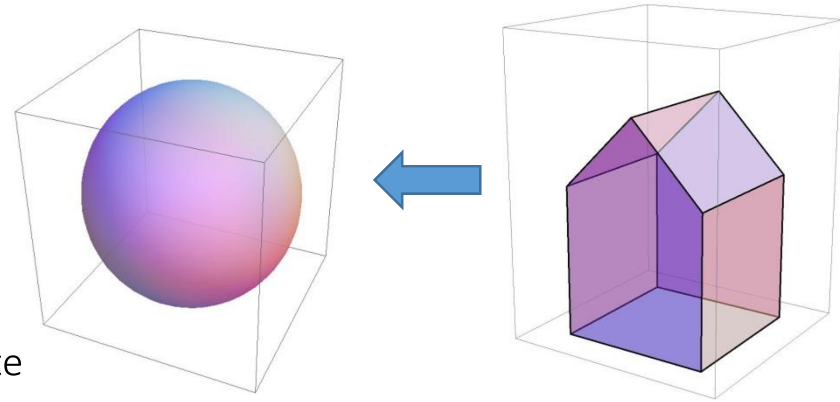
11) We use spherical coordinates $d\left(\mathbf{r}_1, \mathbf{r}_2\right) = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}$ and calculate the connectivity mass by Taylor expanding around ***r_2=R*** (i.e. near the surface):

$$\begin{aligned} M_H\left(\mathbf{r}_2\right) &= 2\pi \int_0^R \int_0^\pi r_1^2 \sin\theta \, H\left(\mathbf{r}_{12}\right) \mathrm{d}\theta \mathrm{d}r_1 \\ &= \frac{\pi}{4\beta}\left(\frac{23-\sqrt{2}}{2}\sqrt{\frac{\pi}{\beta}} + 14\left(R - r_2\right)\right) \end{aligned}$$

12) We can now also calculate the exterior integral:

$$2\pi \int_0^R \int_0^\pi r_2^2 \sin\theta \; e^{-\rho M_H\left(\mathbf{r}_2\right)} \mathrm{d}\theta \mathrm{d}r_2 = \frac{8\beta R^2}{7\rho} e^{-\frac{(23-\sqrt{2})\pi^{3/2}\rho}{8\beta^{3/2}}}$$

Example: “house” domain (using MIMO model)



$$\int_{\mathcal{V}} e^{-\rho \int_{\mathcal{V}} H(\mathbf{r}_{12}) \mathrm{d}\mathbf{r}_1} \mathrm{d}\mathbf{r}_2$$

13) Finally we consider the **Volume** term of a sphere with equal volume as the house domain:

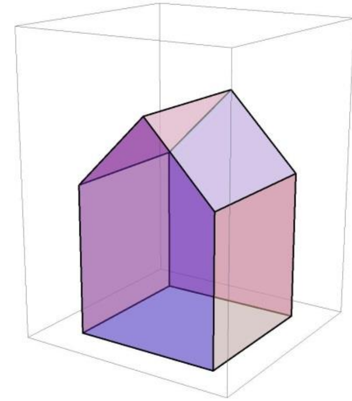
14) We use spherical coordinates and calculate the connectivity mass by Taylor expanding around $\mathbf{r}_2$**=0** (i.e. away from the surface):

$$M_H(\mathbf{r}_2) = 2\pi \int_0^{\infty} \int_0^{\pi} r_1^2 \sin\theta \, H(\mathbf{r}_{12}) \, \mathrm{d}\theta \mathrm{d}r_1$$
$$= \frac{(23-\sqrt{2})}{4} \left(\frac{\pi}{\beta}\right)^{\frac{3}{2}}.$$

15) The exterior integral gives:

$$2\pi \int_0^{R} \int_0^{\pi} r_2^2 \sin\theta \; e^{-\rho M_H(\mathbf{r}_2)} \mathrm{d}\theta \mathrm{d}r_2 = V e^{-\frac{(23-\sqrt{2})\pi^{3/2}\rho}{4\beta^{3/2}}}$$

Example: “house” domain (using MIMO model)



For a **2x2 MIMO** pair connectedness function with $\eta = 2$:

$$H(r) = e^{-\beta r^2}\left(\beta^2 r^4 + 2 - e^{-\beta r^2}\right)$$

we have that P_{fc} is a sum of the different boundary contributions:

$$P_{fc} \approx 1 - \rho\left(C_1 + C_2 + E_1 + E_2 + F + U\right)$$

$$C_1 = 6\frac{512\beta^3}{343\pi^3\rho^3} e^{-\frac{(23-\sqrt{2})\rho}{32}\left(\frac{\pi}{\beta}\right)^{3/2}}$$

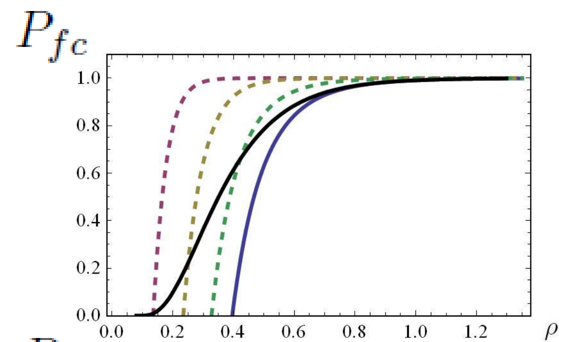
$$C_2 = 4\frac{1024\sqrt{2}\beta^3}{1029\pi^3\rho^3} e^{-\frac{(23-\sqrt{2})3\rho}{64}\left(\frac{\pi}{\beta}\right)^{3/2}}$$

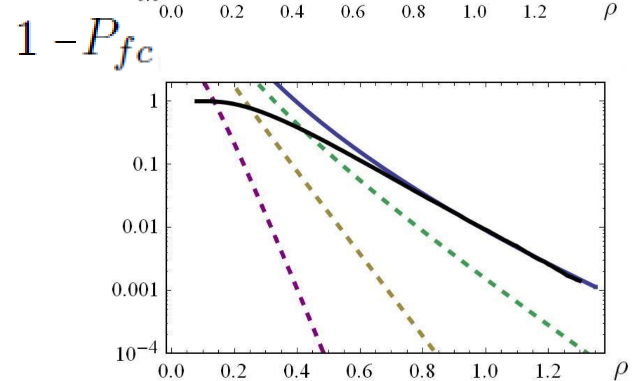
$$E_1 = L\left(9 + 2\sqrt{2}\right)\frac{16\beta^2}{49\pi^2\rho^2} e^{-\frac{(23-\sqrt{2})\rho}{16}\left(\frac{\pi}{\beta}\right)^{3/2}}$$

$$E_2 = 2L\frac{16\sqrt{2}\beta^2}{49\pi^2\rho^2} e^{-\frac{(23-\sqrt{2})3\rho}{32}\left(\frac{\pi}{\beta}\right)^{3/2}}$$

$$F = \frac{2\beta S}{7\pi\rho} e^{-\frac{(23-\sqrt{2})\pi^{3/2}\rho}{8\beta^{3/2}}}$$

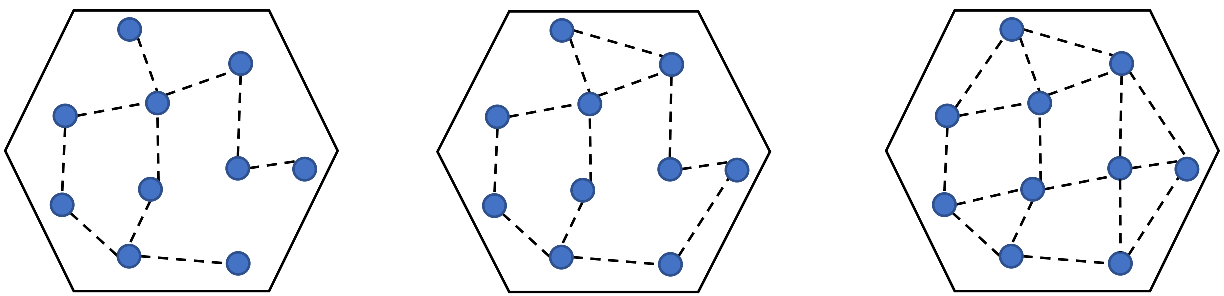
$$U = V e^{-\frac{(23-\sqrt{2})\pi^{3/2}\rho}{4\beta^{3/2}}}$$



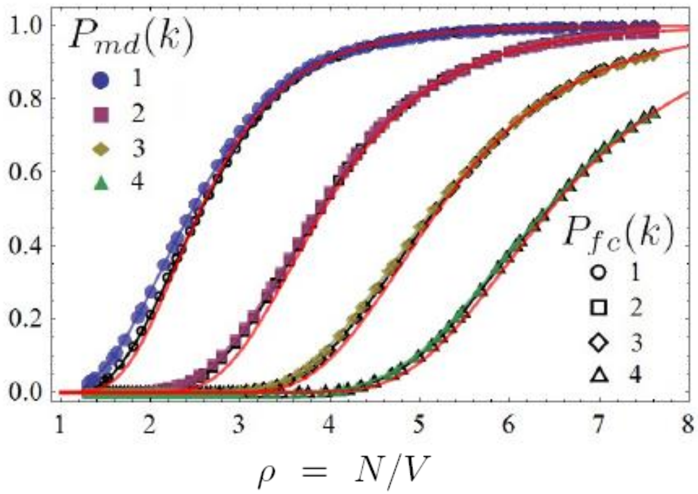


K-connectivity - Reliability

$P_{fc}(k)$ ***k-connectivity***: *network remains fully connected if any k-1 nodes are randomly removed*



1-connected 2-connected 3-connected



K-connectivity - Reliability

The probability of network having minimum degree k

$$P_{md}(k) = \langle \prod_{i=1}^{N} P(\text{degree}(\mathbf{r}_i) \geq k) \rangle = \langle \prod_{i=1}^{N} (1 - D_i(k-1)) \rangle$$
$$\approx [1 - \langle D_i(k-1) \rangle]^N .$$

(1 - probability node has degree at most k-1)^N

$$P_{md}(k) \approx \left[1 - \sum_{m=0}^{k-1} \frac{\rho^m}{m!} \frac{1}{V} \int_{\mathcal{V}} M_H^m(\mathbf{r}_i) e^{-\rho M_H(\mathbf{r}_i)} \mathrm{d}\mathbf{r}_i \right]^N$$

$$D_i(k) = \sum_{m=0}^{k} d_i(m)$$

$$d_i(k) \approx \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

$$\lambda_i = (N-1) H_i$$

$$H_i(\mathbf{r}_i) = \frac{1}{V} \int_{\mathcal{V}} H(r_{ij}) \mathrm{d}\mathbf{r}_j$$

$$M_H(\mathbf{r}_i) = V H_i$$

k-connectivity for confined random networks, Europhysics Letters, 103, 28006, (2013)

K-connectivity - Reliability

$$D_i(k) = \sum_{m=0}^{k} d_i(m)$$

$$d_i(k) \approx \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

$$\lambda_i = (N-1)H_i$$

$$H_i(\mathbf{r}_i) = \frac{1}{V}\int_{\mathcal{V}} H(r_{ij})\mathrm{d}\mathbf{r}_j$$

$$M_H(\mathbf{r}_i) = V H_i$$

Q. What about $P_{fc}(k)$ **?**

A. $P_{md}(k)$ **and** $P_{fc}(k)$ **have the same asymptotic distribution.**

Since 2-connectivity implies 1-connectivity

$$P_{fc}(2) = P_{fc}(1) - X(1)$$

$X(1)$ = probability of obtaining a fully connected network which is not 2-connected.

At high densities, a fully connected network which is not 2-connected will typically contain a single node which is of degree 1.

$$X(1) \approx \langle \sum_{i=1}^{N} \sum_{j \neq i} H_{ij} \prod_{k \neq j \neq i} (1 - H_{ik}) \rangle$$
$$= \rho^2 \int_{\mathcal{V}} M_H(\mathbf{r}_1) e^{-\rho H_M(\mathbf{r}_1)} \mathrm{d}\mathbf{r}_1,$$

<u>Repeating this argument *k* times:</u>

$$P_{fc}(k) = P_{fc}(1) - \sum_{m=1}^{k-1} X(m),$$

$$X(m) = \frac{\rho^{m+1}}{m!} \int_{\mathcal{V}} M_H^m(\mathbf{r}_1) e^{-\rho M_H(\mathbf{r}_1)} \mathrm{d}\mathbf{r}_1$$

K-connectivity - Reliability

The probability of network having minimum degree *k*

$$P_{md}(k) = \langle \prod_{i=1}^{N} P(\text{degree}(\mathbf{r}_i) \geq k) \rangle = \langle \prod_{i=1}^{N} (1 - D_i(k-1)) \rangle$$
$$\approx [1 - \langle D_i(k-1) \rangle]^N .$$

$$D_i(k) = \sum_{m=0}^{k} d_i(m)$$

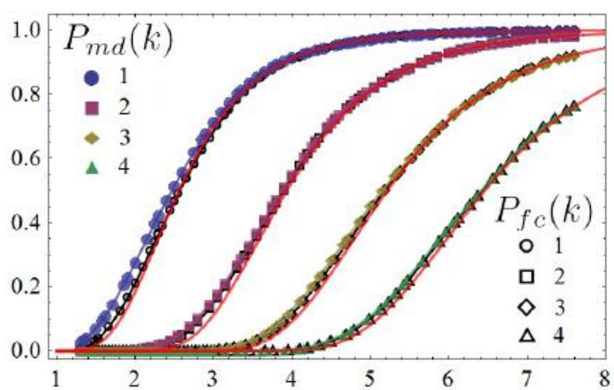
$$d_i(k) \approx \frac{\lambda_i^k}{k!} e^{-\lambda_i}$$

$$\lambda_i = (N-1) H_i$$

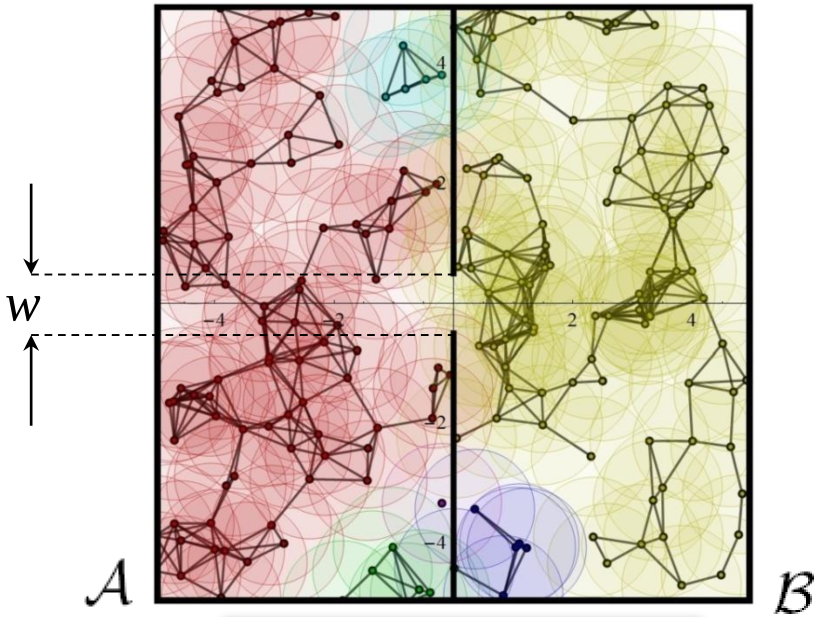
$$H_i(\mathbf{r}_i) = \frac{1}{V} \int_{\mathcal{V}} H(r_{ij}) \mathrm{d}\mathbf{r}_j$$

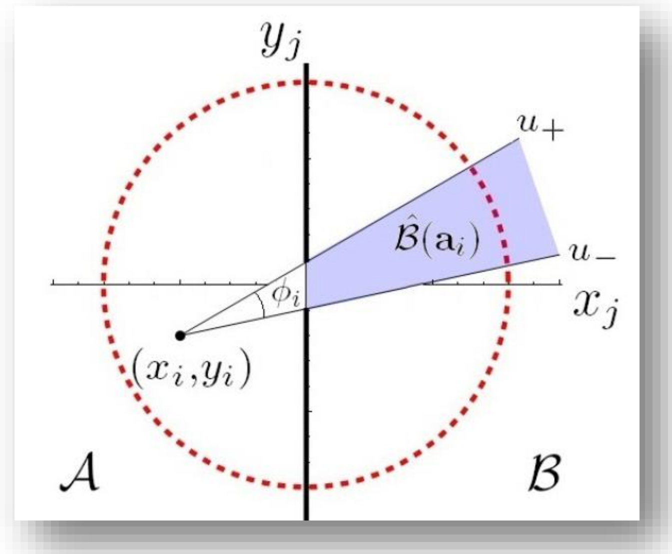
$$M_H(\mathbf{r}_i) = V H_i$$

$$\boxed{P_{fc}(k) \approx \left[1 - \sum_{m=0}^{k-1} \frac{\rho^m}{m!} \frac{1}{V} \int_{\mathcal{V}} M_H^m(\mathbf{r}_i) e^{-\rho M_H(\mathbf{r}_i)} \mathrm{d}\mathbf{r}_i \right]^N}$$



Example: The keyhole setup (non-convex)





$$P_{fc}^{(\mathcal{V})} = P_{fc}^{(\mathcal{A})} P_{fc}^{(\mathcal{B})} X$$

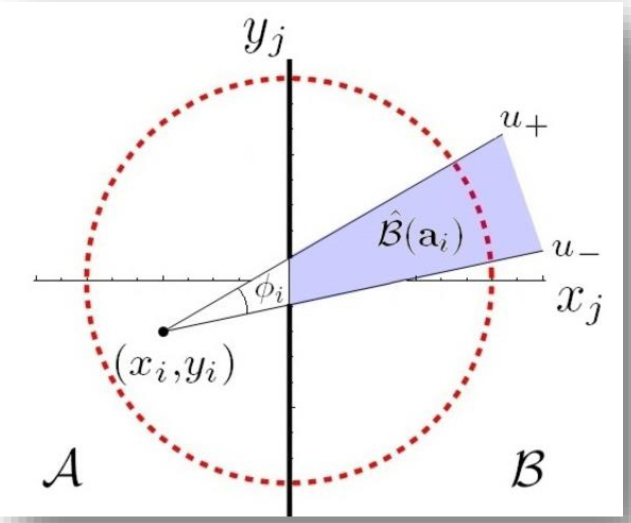
X = the probability of a bridging link between the two sub-domains

= **the complement of** *the probability of* **no** *bridging link between the two sub-domains*

$$X = 1 - \langle \langle \prod_{i=1}^{N_{\mathcal{A}}} \prod_{j=1}^{N_{\mathcal{B}}} (1 - \chi_{ij} H_{ij}) \rangle_{\mathcal{B}} \rangle_{\mathcal{A}}$$

Example: The keyhole setup (non-convex)

$$X = 1 - \langle\langle \prod_{i=1}^{N_{\mathcal{A}}} \prod_{j=1}^{N_{\mathcal{B}}} (1 - \chi_{ij} H_{ij}) \rangle_{\mathcal{B}} \rangle_{\mathcal{A}}$$

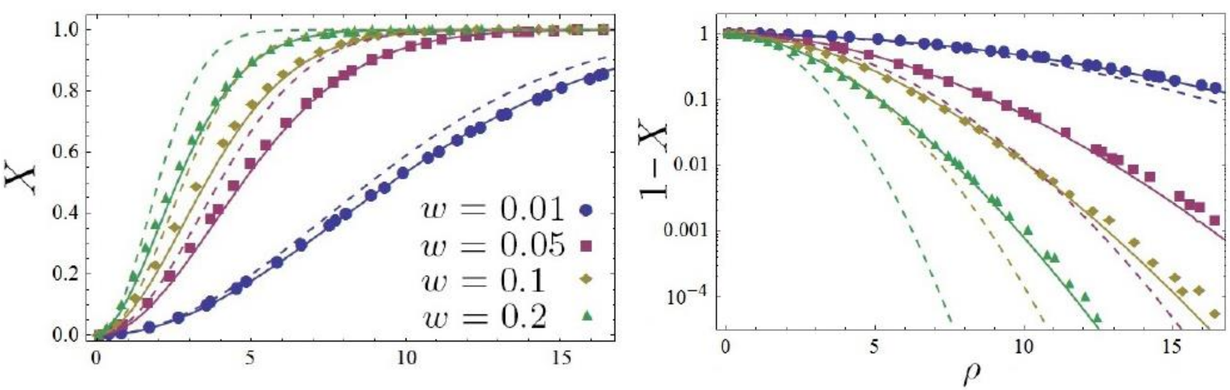


Assumption: all integrals separate out (independence)

$$X \approx 1 - (1 - \langle\langle \chi_{ij} H_{ij} \rangle_{\mathcal{B}} \rangle_{\mathcal{A}})^{N_{\mathcal{A}} N_{\mathcal{B}}}$$
$$= 1 - \exp\left(-\rho_{\mathcal{A}} \rho_{\mathcal{B}} \int_{\mathcal{A}} \int_{\mathcal{B}} \chi_{ij} H_{ij} \mathrm{d}\mathbf{b}_j \mathrm{d}\mathbf{a}_i \right)$$

$$X \approx 1 - \exp\left(-\rho_{\mathcal{A}} \rho_{\mathcal{B}} \frac{\sqrt{\pi} w}{2 \beta^{3/2}} \right)$$

Plotted below using dashed curves

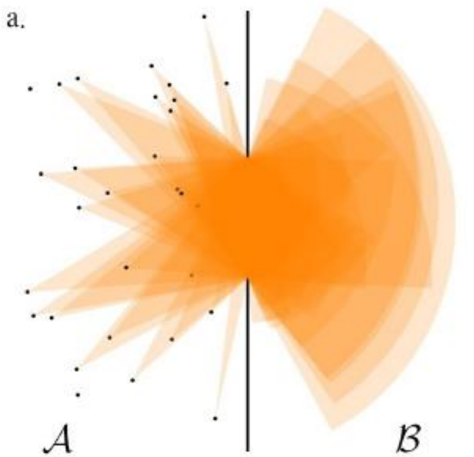


Clearly this was a bad assumption since connections through the keyhole are far from independent.

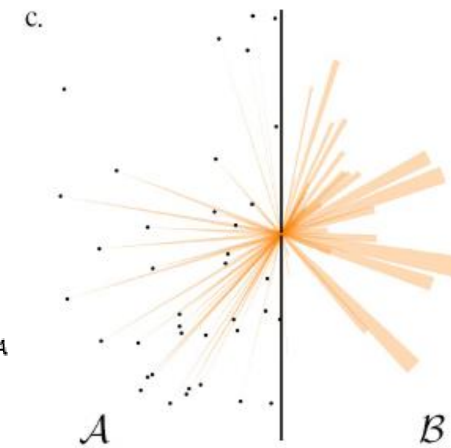
Example: The keyhole setup (non-convex)

*"A system is said to present **quenched disorder** when some parameters are random variables which do **not** evolve with time - they are quenched or frozen. It is opposite to **annealed disorder**, where the random variables are allowed to evolve themselves"*

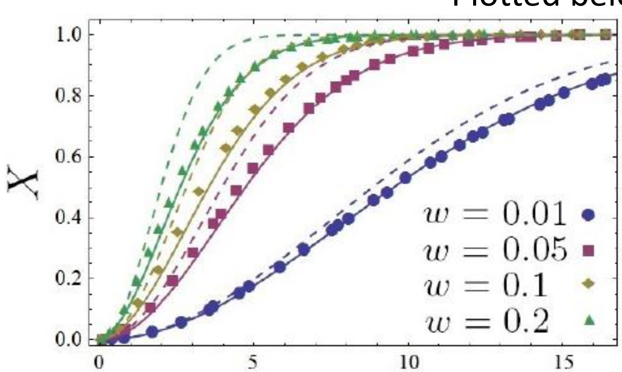
Notice how the LoS connectivity 'cones' overlap (correlated). We must average over each region separately

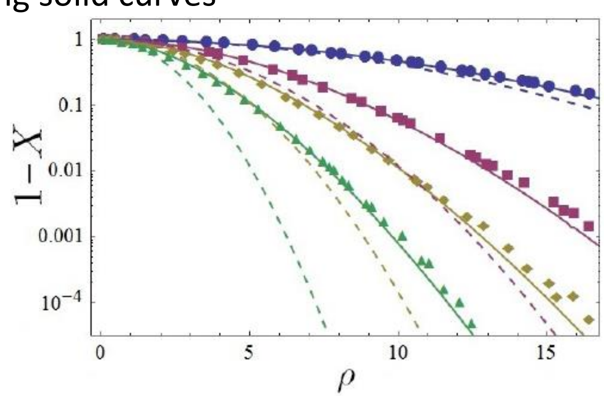


$$
\begin{aligned}
X &= 1 - \langle \prod_{i=1}^{N_{\mathcal{A}}} \langle \prod_{j=1}^{N_{\mathcal{B}}} (1 - \chi_{ij} H_{ij}) \rangle_{\mathcal{B}} \rangle_{\mathcal{A}} \\
&= 1 - \left(\langle \left(1 - \langle \chi_{ij} H_{ij} \rangle_{\mathcal{B}}\right)^{N_{\mathcal{B}}} \rangle_{\mathcal{A}} \right)^{N_{\mathcal{A}}} \\
&= 1 - \left(\langle e^{-N_{\mathcal{B}} \langle \chi_{ij} H_{ij} \rangle_{\mathcal{B}}} \rangle_{\mathcal{A}} \right)^{N_{\mathcal{A}}} \\
&= 1 - \left(\frac{1}{V_{\mathcal{A}}} \int_{\mathcal{A}} e^{-\rho_{\mathcal{B}} \int_{\mathcal{B}} \chi_{ij} H_{ij} \mathrm{d}\mathbf{b}_j} \mathrm{d}\mathbf{a}_i \right)^{N_{\mathcal{A}}}
\end{aligned}
$$



Plotted below using solid curves





Summary

- Applications of ad-hoc networks
- Modelling ad hoc networks
    - Random Geometric Graphs
    - Pairwise Connection function
    - Anisotropic nodes
    - Multiple Antennas
- Local Observables
    - Mean degree
    - Pair Formation
    - Degree distributions
    - Clustering coefficient
- Global Observables
    - Full connectivity
    - Boundary effects
    - K-connectivity

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