Intro to Stochastic Geometry & Point Processes

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5Gwireless

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This Lecture: Tools for System-Level Modeling & Analysis

- What is stochastic geometry ?
- What are point processes ?
- Why are they useful in communications ?
- Basic definitions
- Poisson point processes
- How to compute sums over point processes
- How to compute products over point processes
- Transformations of point processes (displacement, marking, thinning)
- Palm theory, Palm distribution, conditioning
- Packages for analyzing spatial point processes (spatstat in R)
- Some books

- Part II: Motivating, validating and applying all this to cellular networks

What is Stochastic Geometry ?

- Stochastic geometry is the area of mathematical research that is aimed to provide suitable mathematical models and appropriate statistical methods to study and analyze random spatial patterns.

- Random point patterns or point processes are the most basic and important of such objects, hence point process theory is often considered to be the main sub-field of stochastic geometry.

- Random spatial patterns are more general than random point patterns. For example, one can model shapes in multiple dimensions (random shape theory).

What is a Random Point Process ?

- A random (spatial) point process is a set of locations, distributed within a designated region and presumed to have been generated by some form of stochastic mechanism.

- A realization of a spatial point process is termed spatial point pattern, which is a countable collection of points or dataset giving the observed locations of things or events (in a given dimensional space, e.g., in 2-D).

- The easiest way to visualize a 2-D point pattern is a map of the locations, which is simply a scatterplot but with the provision that the axes are equally scaled.

Examples…

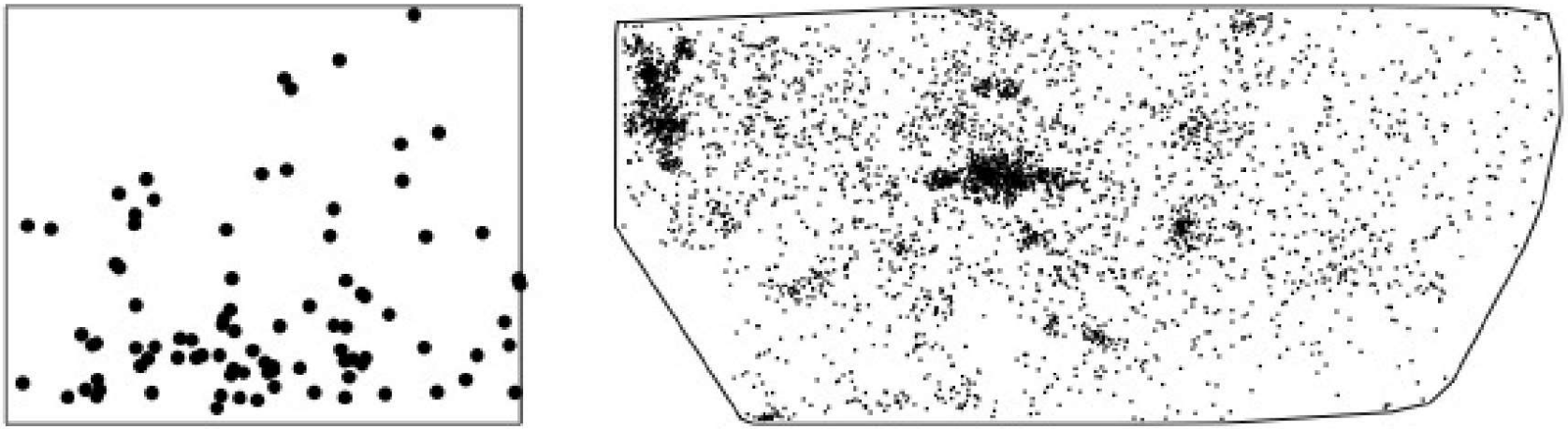


Figure 1.1. *Point pattern datasets with spatially varying density of points.* Left: *enterochromaffin-like cells in histological section of gastric mucosa (T. Bendtsen; see [484, pp. 2, 169]), interior of stomach towards top of picture.* Right: *sky positions of 4215 galaxies in the Shapley Supercluster (M. Drinkwater); survey region about 25 degrees across.*

Examples…

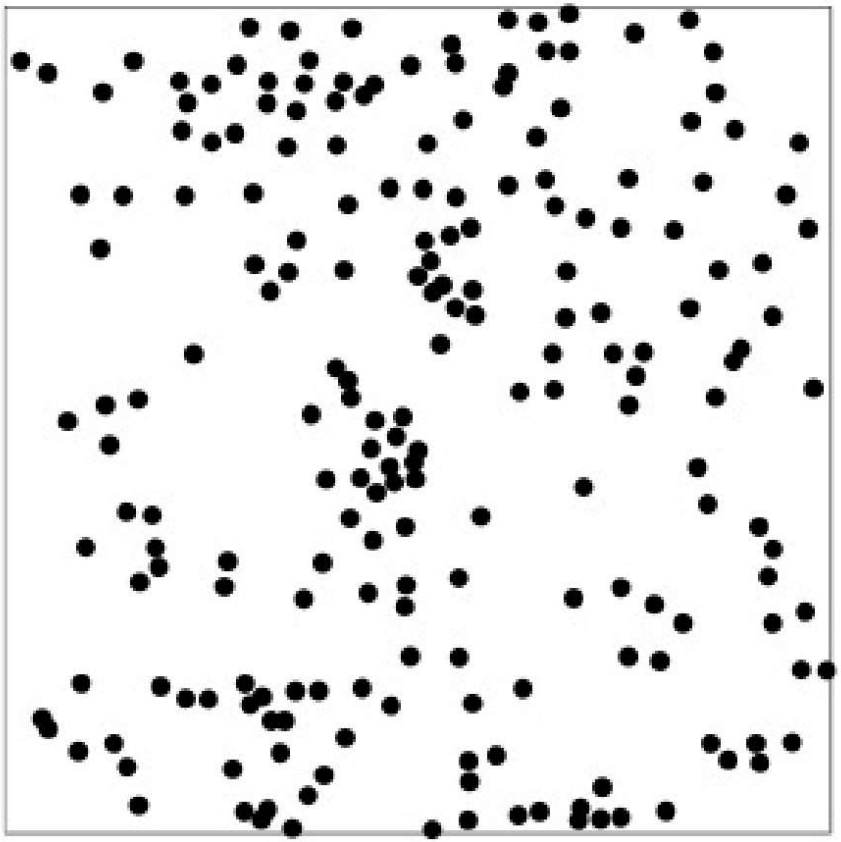


Figure 1.3. *Japanese black pine seedlings and saplings in a 10 metre square sampling region. Data recorded by M. Numata [503], and kindly supplied by Y. Ogata and M. Tanemura.*

Figure 1.3 depicts the locations of 204 Japanese black pine (*Pinus Thunbergii*) seedlings and saplings recorded in a 10×10 metre sampling region within a natural forest stand [503]. In mapping a snapshot of the forest, we hope to understand ecological processes, such as competition for resources (soil nutrients, light, water, growing space), and spatial variation in the landscape, such as variation in soil type or soil fertility.

A detailed analysis of these data [513, 516, 55] concluded that *both* these phenomena are present: there is spatial variation in the density of the forest, and also a tendency for trees to avoid growing close together, suggesting competition between neighbouring plants.

Examples…

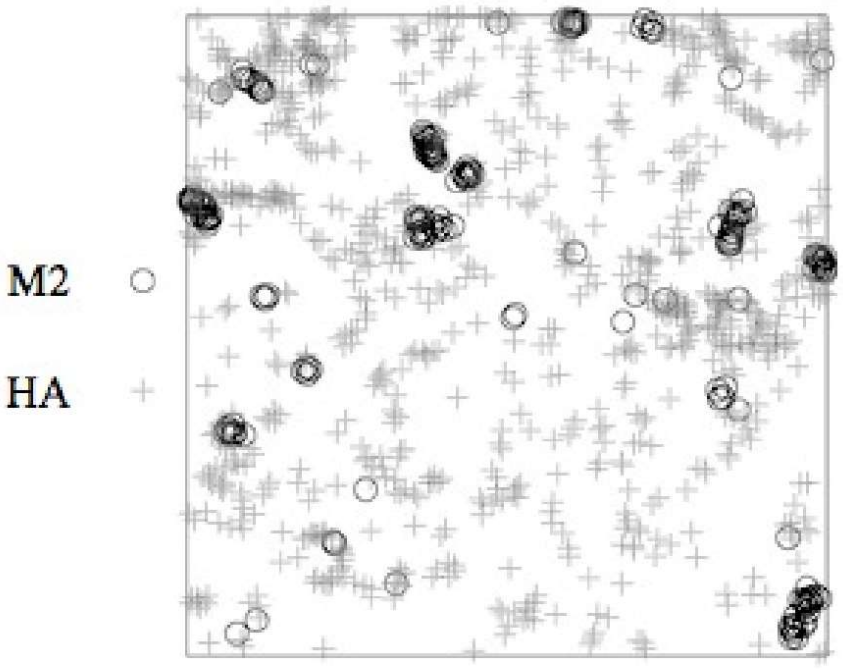
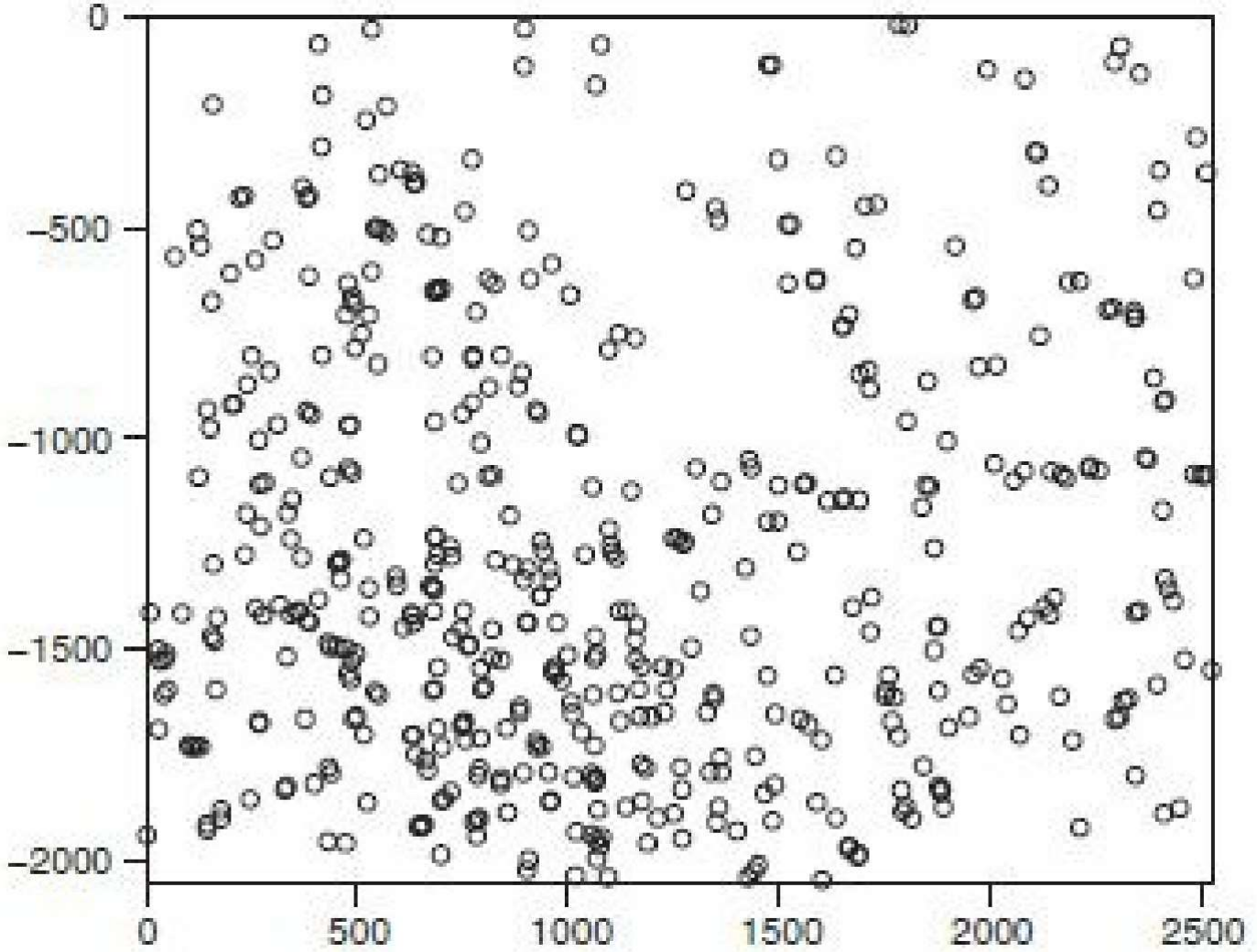


Figure 1.6. *Locations of influenza virus proteins M2 and HA, mapped by immunogold labelling, on the surface of an infected cell. Field width is 3331 nanometres. Data kindly supplied by G.P. Leser and R.A. Lamb.*

Figure 1.6 shows the locations of influenza virus proteins on the surface of an infected cell [146]. The protein locations were mapped by immunogold labelling, that is, by growing antibodies to the specific proteins, attaching a gold particle to each antibody, and subsequently imaging the gold particles in electron microscopy. The research problem is to decide whether there is spatial association between the two proteins: this is important for the study of viral replication. While these data are superficially similar in structure to the amacrine cells data, the required analysis is completely different. Whereas the amacrine cells belong to two highly organised layers, it is appropriate to treat the individual influenza proteins (the individual circles and crosses in Figure 1.6) as individual, mobile, biochemical entities, each responding to its local environment.

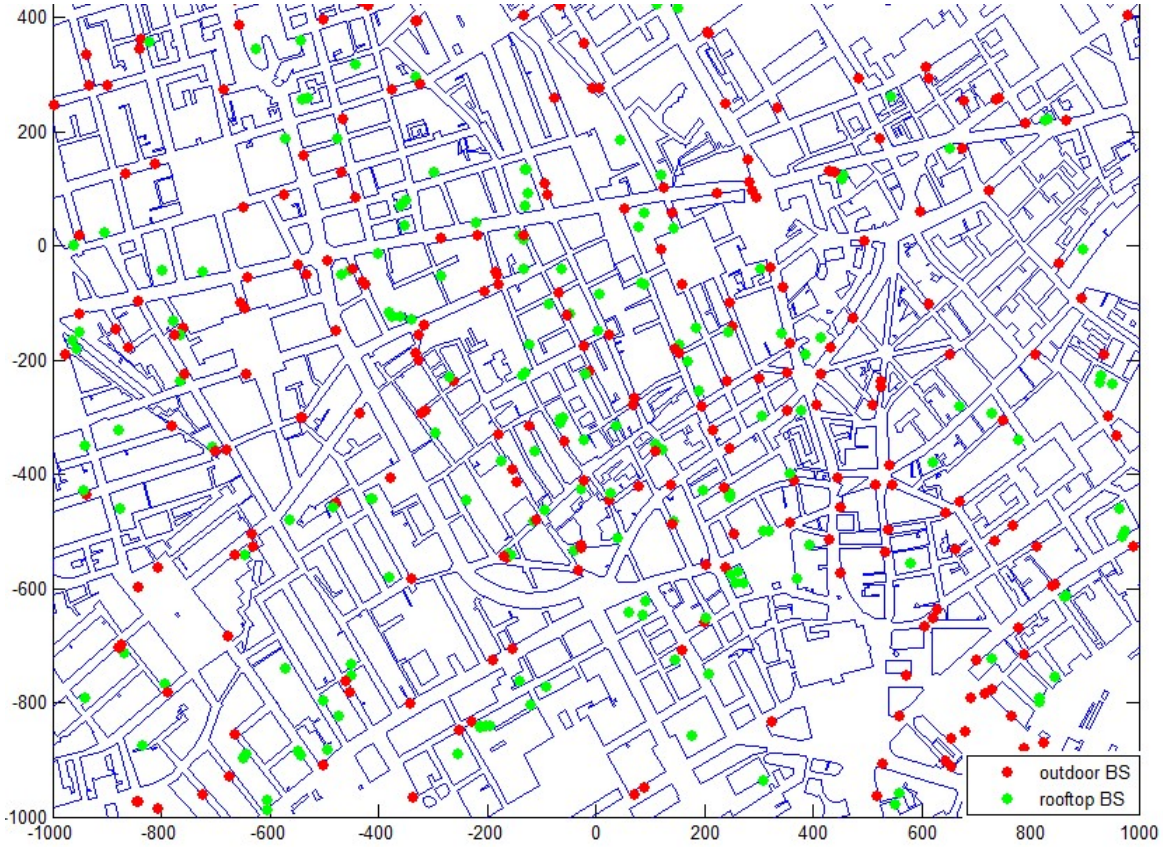
Examples…



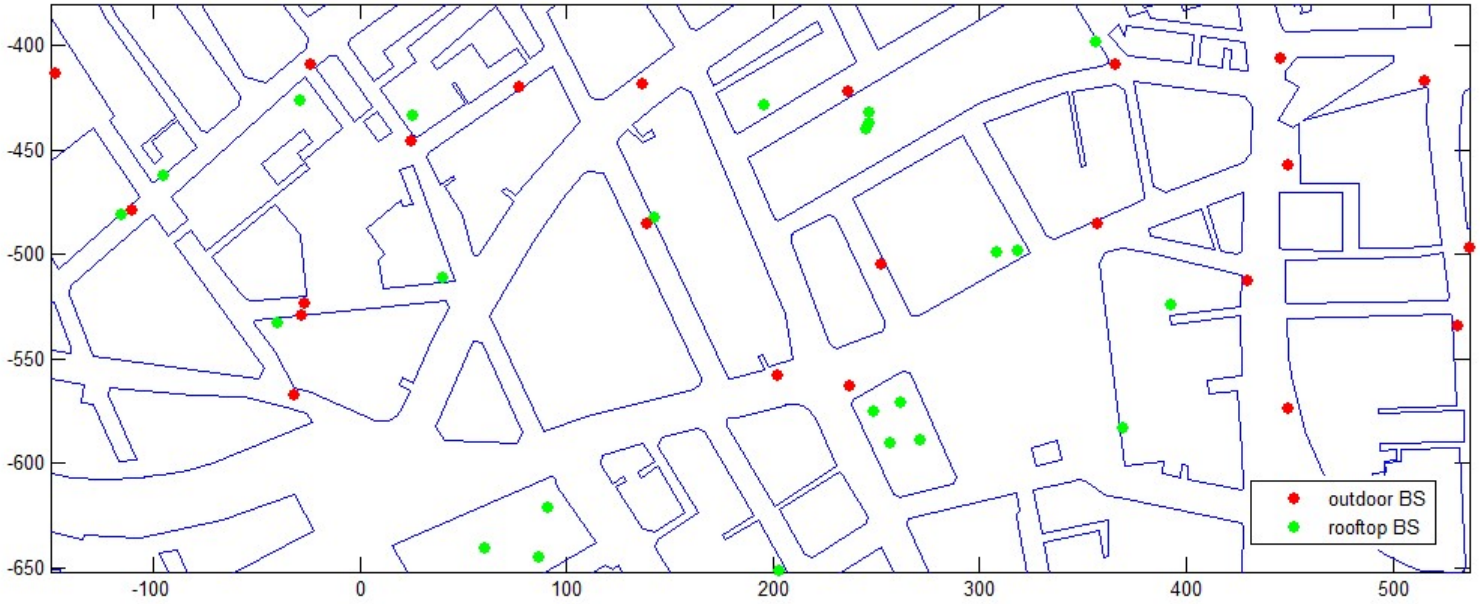
Locations of 493 cellular base stations (5 km square area in central London)

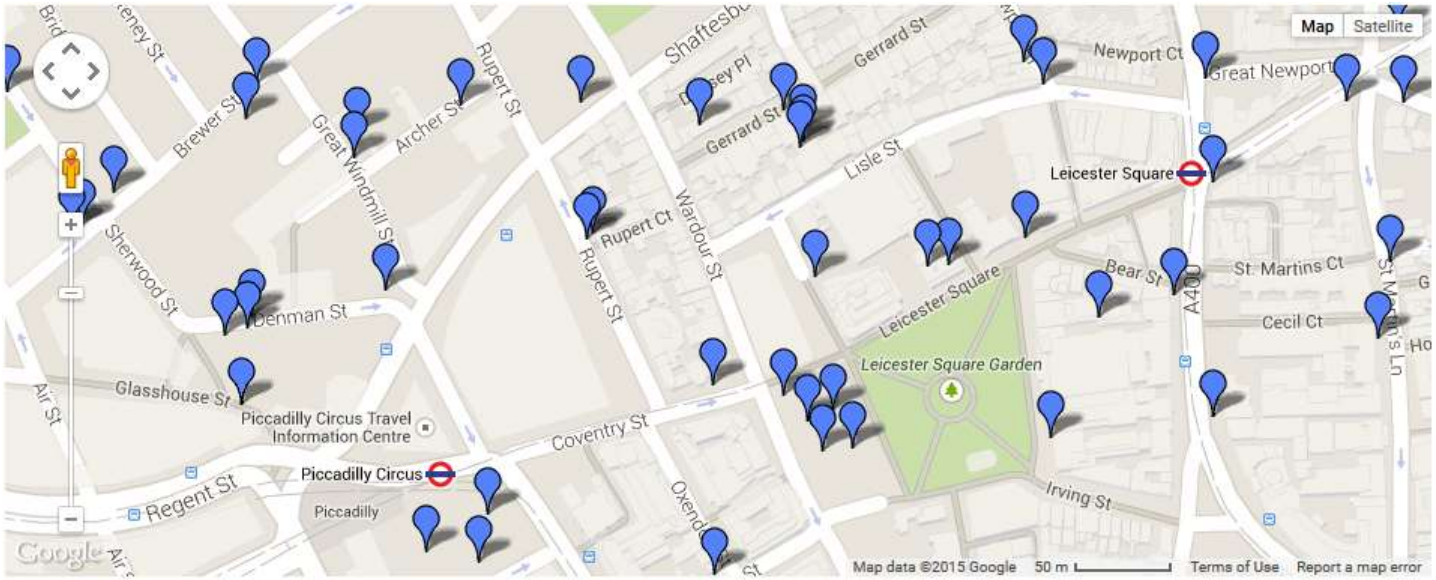
Examples…Beyond "Points"

	O2 + Vodafone	O2	Vodafone
Number of BSs	319	183	136
Number of rooftop BSs	95	62	33
Number of outdoor BSs	224	121	103
Average cell radius (m)	63.1771	83.4122	96.7577



Examples…Beyond "Points" (zoom in)





What Stochastic Geometry is Useful For ?

- Stochastic geometry is a rich branch of applied probability with several applications: material science, image analysis, stereology, astronomy, biology, forestry, geology, communications, etc.

- Stochastic geometry provides answers to questions such as:
  - How can one describe a (random) collection of points in one, two, or higher dimensions ?
  - How can one derive statistical properties of such a collection of points ?
  - How can one calculate statistical averages over all the possible realizations of such a random collection ?
  - How can one condition on having a point at a fixed location ?
  - Given an empirical set of points, which statistical model is likely to produce this point set ?

… In Communications…?

- Point processes are used to model the (spatial) locations of nodes (users, wireless terminals, base stations, access points, etc.) in (wireless) networks.

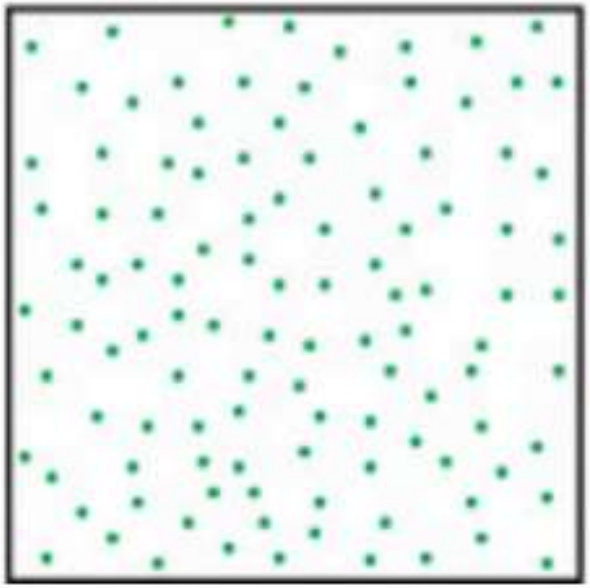
- Point process models permit statements about entire classes of (wireless) networks, instead of just about one specific configuration of the network.

- In some cases, distributions of relevant performance metrics over the point process can be calculated, in others, spatial averaging is performed, which yields expected values of certain performance metrics (e.g., the likelihood of transmission success).

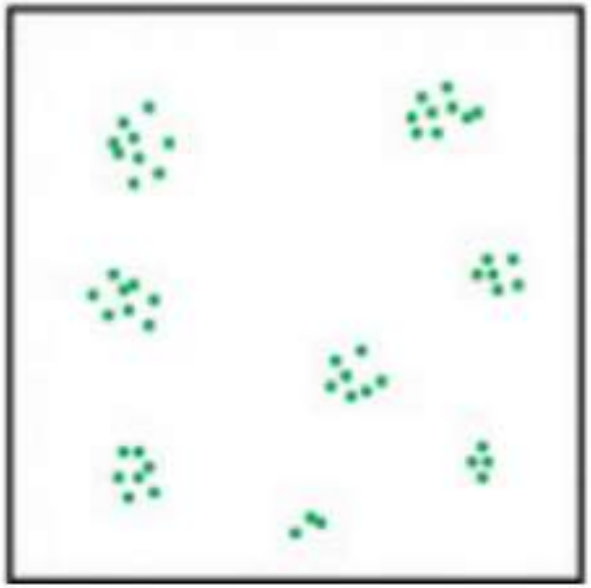
On (Complete) Spatial Randomness…

Uniform

Random



Clustered



What is a Point Process ? – Mathematical Definition

A point process is a countable random collection of points that reside in some measure space, usually the Euclidean space $\mathbb{R}^d$.

For simplicity, we often consider $d = 2$.

Notation

1. A point process is denoted by Φ
2. An instance (realization) of the point process is denoted by ϕ
3. The number of points of a point process in the set $A \subset \mathbb{R}^2$ is denoted by $\Phi(A)$

What is a Point Process ? – Mathematical Definition

Let $\mathbb{N}$ be the set of all sequences $\phi \subset \mathbb{R}^2$ satisfying

1. (Finite) Any bounded set $A \subset \mathbb{R}^2$ contains a finite number of points
2. (Simple) $x_i \neq x_j$ if $i \neq j$

Definition

A point process in $\mathbb{R}^2$ is a random variable taking values in the space $\mathbb{N}$

What is a Point Process ? – Mathematical Definition

A point process can be described by using two formalisms:

1. Random set formalism
2. Random measure formalism

Random set formalism

The point process is ragarded as a countable random set $\Phi = \{x_1, x_2, ...\} \subset \mathbb{R}^2$ consisting of random variables $x_i \in \mathbb{R}^2$ as its elements.

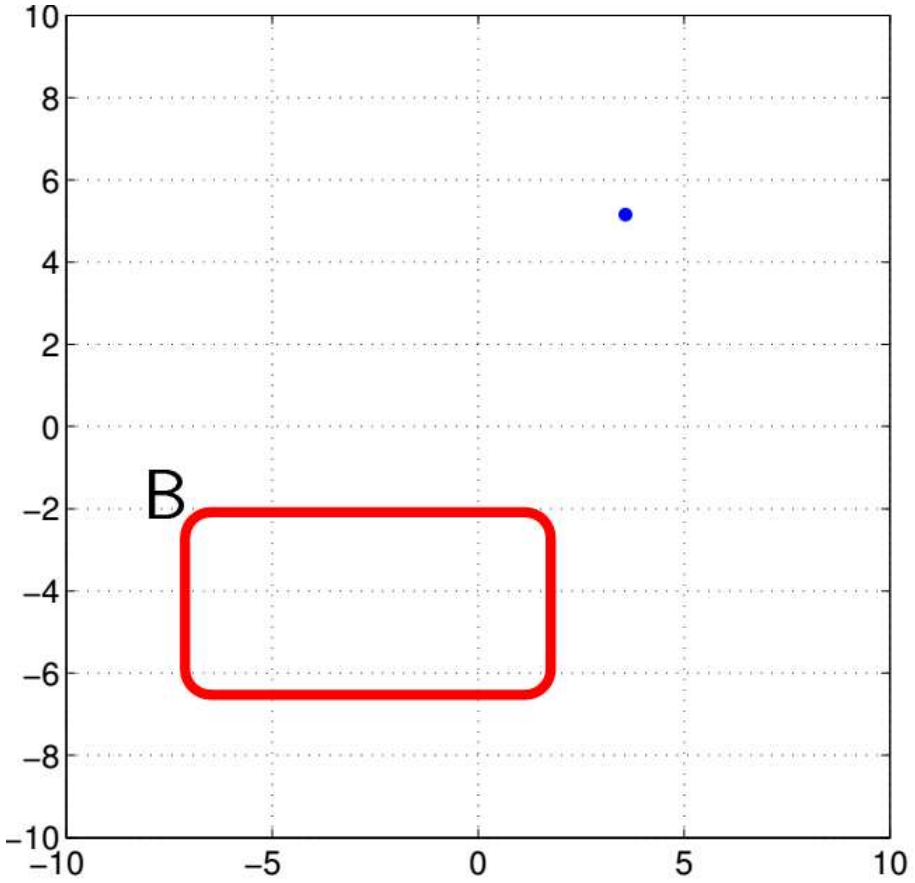
Random measure formalism

The point process is characterized by counting the number of points falling in sets $A \subset \mathbb{R}^2$, i.e., $\Phi(A)$. Hence $\Phi(A)$ is a random variable that assumes non-negative interger values. $\Phi(\cdot)$ is called (random) counting measure.

Starting Point: Point Process with a Single Point

Single - Point Point Processes

1. Contains only one random point
2. The random point x is uniformly distributed in a bounded set A



Thus, let $B \subset A,$ one has

$$\mathbb{P}\left(x \in B\right) = \frac{|B|}{|A|}$$

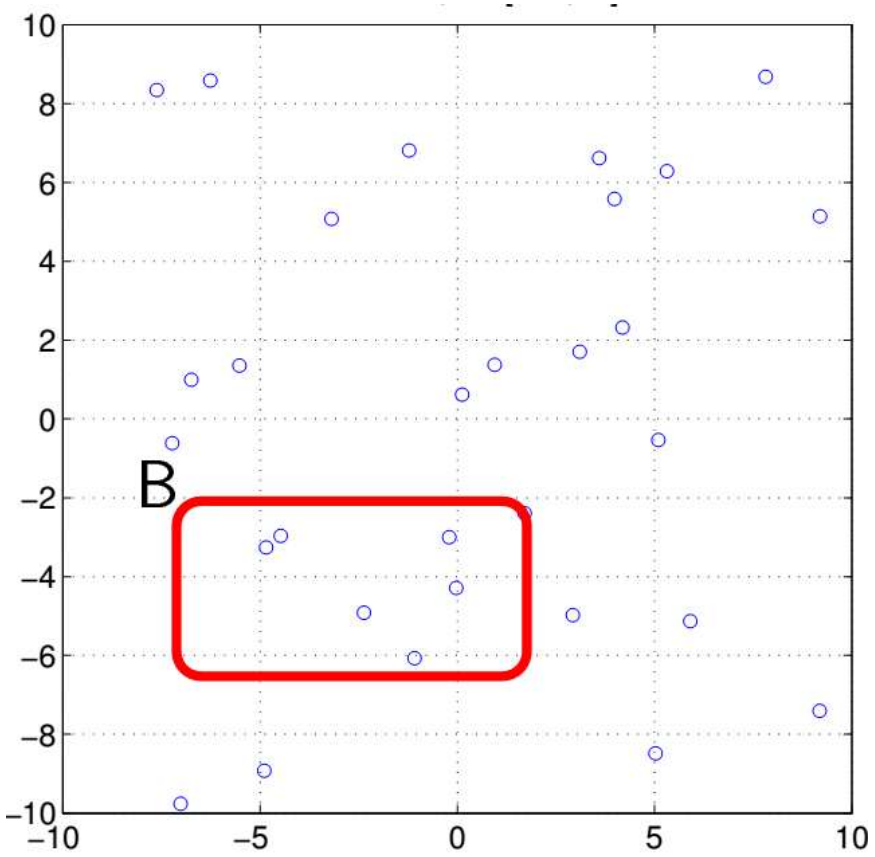
where $|A|$ denotes the area of A.

Binomial Point Process (BPP)

A **BPP** on a bounded set A $\left(|A| < +\infty\right)$ is the superposition of N independent and uniformly distributed points on the set A.

Let $B \subset A$, then: $\mathbb{P}\left(\Phi\left(B\right) = k\right)$

$$= \binom{\mathrm{N}}{k} \left(\frac{|B|}{|A|}\right)^{k} \left(1 - \frac{|B|}{|A|}\right)^{\mathrm{N}-k}$$



Equivalent Point Processes: Void Probability

Given two point processes, is there any simple approach to prove whether they are equivalent ?

Void Probability

Let a point process Φ. Its void probabilities over all bounded sets A are defined as $\mathbb{P}\left(\Phi\left(A\right)=0\right)$ for $A \subset \mathbb{R}^2$.

Equivalent point processes

1. A simple point process is determined by its void probabilities.
2. Two simple point processes are equivalent if they have the same void probability distributions for all bounded sets.

Stationarity, Isotropy, Motion-Invariance

Stationarity

Let a point process $\Phi=\{x_n\}$. Φ is said to be stationary if the translated point process $\Phi_x=\{x_n+x\}$ has the same distribution as Φ for every $x\in\mathbb{R}^2$.

Isotropy

Let a point process $\Phi=\{x_n\}$. Φ is said to be isotropic if the rotated point process $\mathbf{r}\Phi=\{\mathbf{r}x_n\}$ has the same distribution as Φ for every rotation $\mathbf{r}$ about the origin.

Motion - Invariant

A point process is motion-invariant if it is stationary and isotropic.

Stationarity and Intensity Measure

Density (Intensity) Measure

Let a stationary point process Φ. Its density is defined as follows:

$$\lambda = \frac{\mathbb{E}\{\Phi(A)\}}{|A|} \quad \text{for every } A \subset \mathbb{R}^2$$

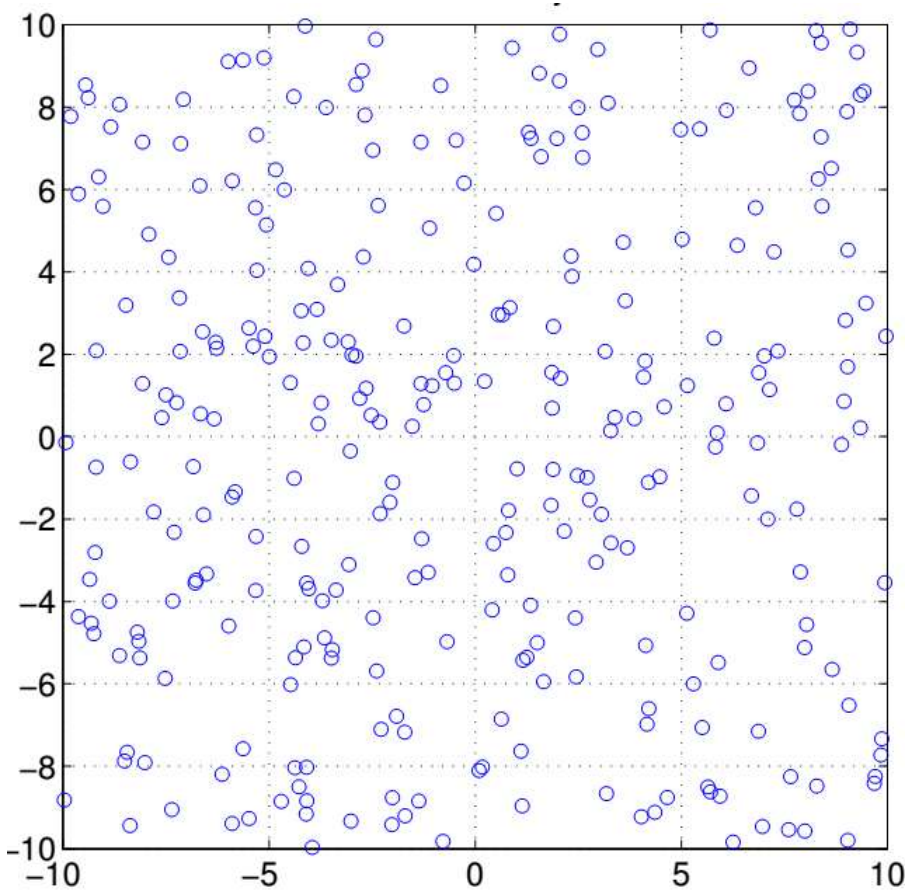
Remarks:

1. The density does not depend on the particular choice of the set A
2. Stationarity implies that the density is constant
3. The converse is, in general, not true: a constant density does not imply stationarity

Stationary (Homogeneous) Poisson Point Process (PPP)

- The most widely used model for the spatial locations of nodes
  - Most amicable for mathematical analysis
  - Considered the "Gaussian of point processes"

- No dependence between node locations
- Random number of nodes
- Defined on the entire plane (limiting case of a BPP)



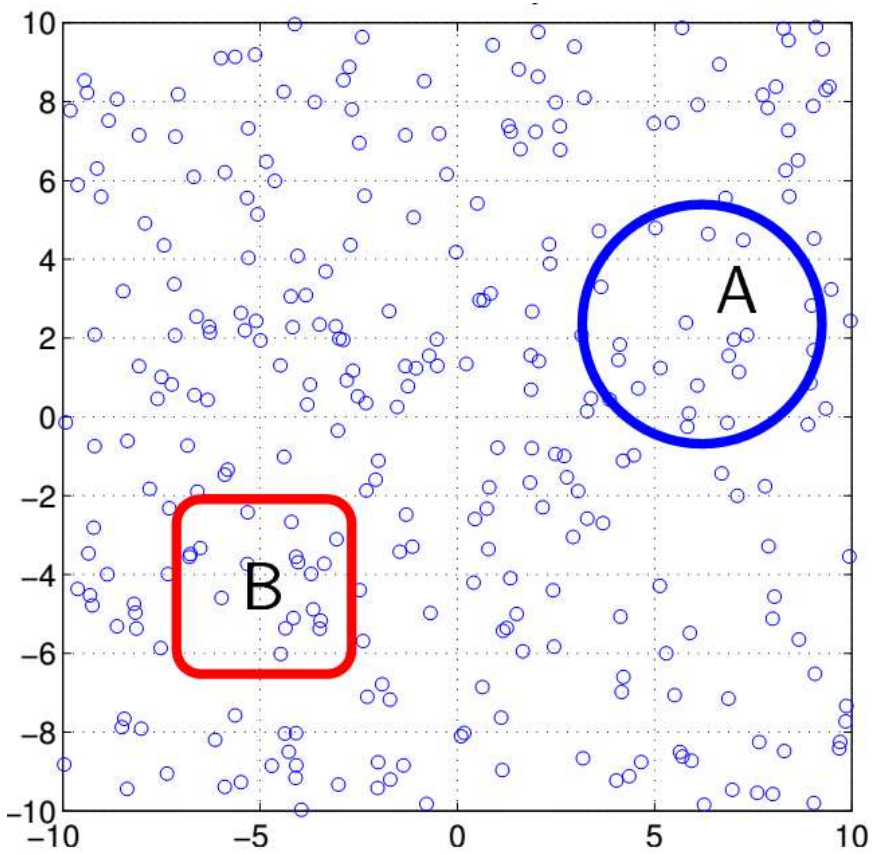
Homogeneous PPP: Formal Definition

A stationary point process Φ of density λ is PPP if:

1. The number of points in any bounded set $A \subset \mathbb{R}^2$ has a Poisson distribution with mean $\lambda|A|$, i.e.

$$\mathbb{P}\left(\Phi(A)=k\right)=\frac{\left(\lambda|A|\right)^k}{k!}\exp\left(-\lambda|A|\right)$$

2. The number of points in disjoint sets are independent, i.e., for every $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$ with $A \cap B = \varnothing$, $\Phi(A)$ and $\Phi(B)$ are independent



A homogeneous PPP is completely charaterized by a single number λ

Is the Density of a Homogeneous PPP Equal to λ ?

Proof :

Let Φ be a homogeneous PPP and $A \subset \mathbb{R}^2$. Then:

$$\frac{\mathbb{E}\{\Phi(A)\}}{|A|} = \frac{1}{|A|}\sum_{k=0}^{+\infty} k\mathbb{P}(\Phi(A)=k) = \frac{1}{|A|}\sum_{k=0}^{+\infty} k\frac{(\lambda|A|)^k}{k!}\exp(-\lambda|A|)$$

$$= \frac{\exp(-\lambda|A|)}{|A|}\sum_{k=0}^{+\infty} k\frac{(\lambda|A|)^k}{k!} = \frac{\exp(-\lambda|A|)}{|A|}\sum_{k=1}^{+\infty} k\frac{(\lambda|A|)^k}{k!}$$

$$= \frac{\exp(-\lambda|A|)}{|A|}(\lambda|A|)\sum_{k=1}^{+\infty}\frac{(\lambda|A|)^{k-1}}{(k-1)!} = \frac{\exp(-\lambda|A|)}{|A|}(\lambda|A|)\sum_{n=0}^{+\infty}\frac{(\lambda|A|)^n}{n!}$$

$$= \frac{\exp(-\lambda|A|)}{|A|}(\lambda|A|)\exp(\lambda|A|) = \lambda$$

Note: The result does not depend on the set A, as expected.

BPP vs. Homogeneous PPP

Let Φ be a homogeneous PPP and $A \subset \mathbb{R}^2$. Conditioned on $\Phi(A)$, i.e., the number of points in A, the points themselves are independently and uniformly distributed in A. In other words, conditioned on $\Phi(A)$, the points constitute a BPP in A.

Proof: Consider the void probability of $K \subset A \subset \mathbb{R}^2$. Let $\overline{K} = A \setminus K$.

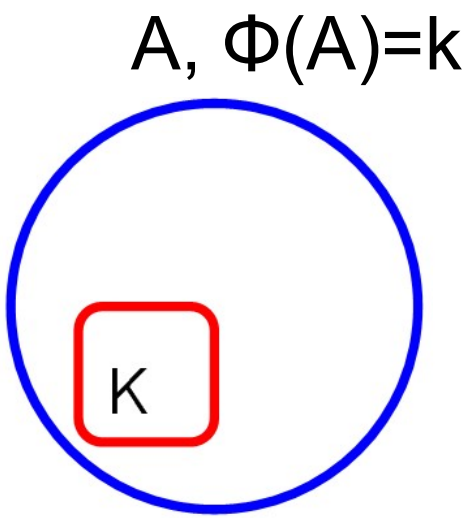
$$\mathbb{P}\left(\Phi(K)=0 \middle| \Phi(A)=k\right) = \frac{\mathbb{P}\left(\Phi(K)=0 \cap \Phi(A)=k\right)}{\mathbb{P}\left(\Phi(A)=k\right)}$$

$$= \frac{\mathbb{P}\left(\Phi(K)=0 \cap \Phi(\overline{K})=k\right)}{\mathbb{P}\left(\Phi(A)=k\right)} = \frac{\mathbb{P}\left(\Phi(K)=0\right)\mathbb{P}\left(\Phi(\overline{K})=k\right)}{\mathbb{P}\left(\Phi(A)=k\right)}$$

$$= \left[\frac{\left(\lambda|A|\right)^k}{k!}\exp\left(-\lambda|A|\right)\right]^{-1}\left[\exp\left(-\lambda|K|\right)\right]\left[\frac{\left(\lambda|\overline{K}|\right)^k}{k!}\exp\left(-\lambda|\overline{K}|\right)\right]$$

$$= \left[\frac{\left(\lambda|A|\right)^k}{k!}\exp\left(-\lambda|A|\right)\right]^{-1}\left[\exp\left(-\lambda\left(|A|-|\overline{K}|\right)\right)\right]\left[\frac{\left(\lambda|\overline{K}|\right)^k}{k!}\exp\left(-\lambda|\overline{K}|\right)\right]$$

$$= \frac{|\overline{K}|^k}{|A|^k} = \frac{\left(|A|-|K|\right)^k}{|A|^k} = \left(1-\frac{|K|}{|A|}\right)^k = \text{void probability of a BPP in } A$$



BPP vs. Homogeneous PPP: How to Simulate a PPP

How to simulate a homogeneous PPP of density λ on $A = [-L, L]^2$?

1. The number of points in the set A is a Poisson random variable with mean $\lambda |A|$.

2. Conditioned on the number of points, the points are distributed as a BPP.

How to simulated it in Matlab ?

$N = \text{poissrnd}(\lambda |A|)$

$\text{Points} = \text{unifrnd}(-L, L, N, 2)$

Inhomogeneous PPP – Just a Glimpse…

In homogeneous PPPs, the mean number of points per unit area does not vary over space, i.e., they have a constant density measure λ.

In several applications of interest, it may make sense to consider point processes with a location-dependent intensity function: $\lambda(x)$. Its interpretation is as follows: $\lambda(x)dx$ is the infinitesimal probability that there is a point of Φ in a region of infinitesimal area dx located at $x \in \mathbb{R}^2$.

The intensity measure of inhomogeneous PPPs is defined as follows:

$$\Lambda(A) = \int_A \lambda(x)dx \quad \text{for any bounded set } A$$

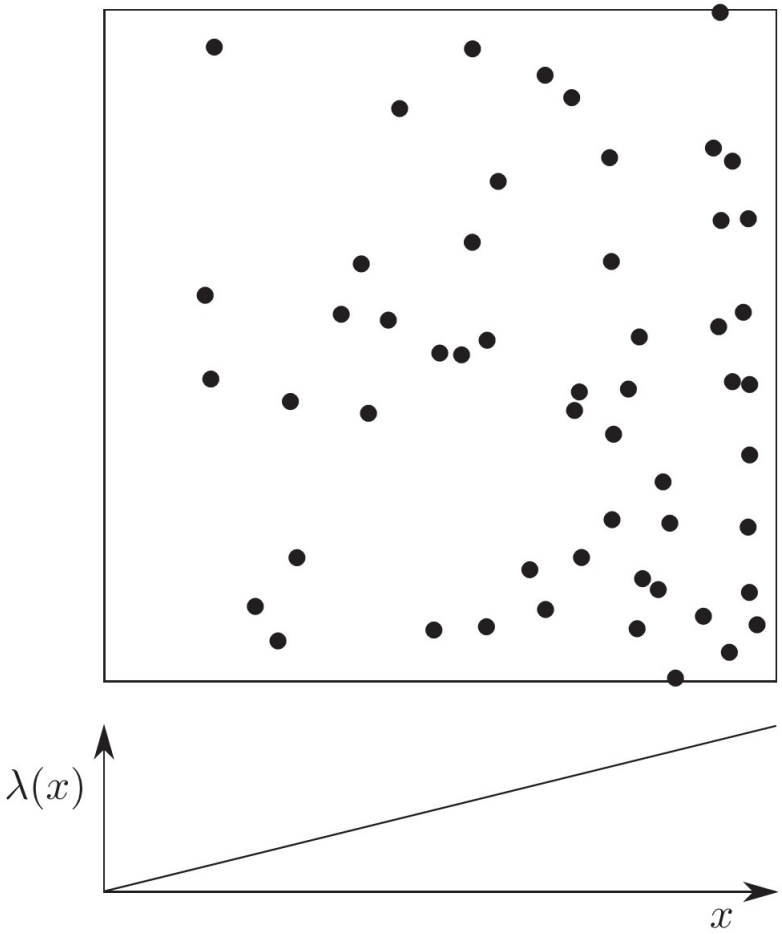
Inhomogeneous PPP – Just a Glimpse…

An inhomogeneous point process Φ of intensity function $\lambda(x)$ is PPP if:

1. The number of points in any bounded set $A \subset \mathbb{R}^2$ has a Poisson distribution with mean $\Lambda(A) = \int_A \lambda(x)\,dx$, i.e.

$$\mathbb{P}\left(\Phi(A) = k\right) = \frac{\left(\Lambda(A)\right)^k}{k!} \exp\left(-\Lambda(A)\right)$$

2. The number of points in disjoint sets are independent, i.e., for every $A \subset \mathbb{R}^2$ and $B \subset \mathbb{R}^2$ with $A \cap B = \varnothing$, $\Phi(A)$ and $\Phi(B)$ are independent



Inhomogeneous PPP – Just a Glimpse…

How to simulate an inhomogeneous PPP of intensity function $\lambda(x)$ on $A=[-L,L]^2$?

1. Assume that the intensity function is bounded by λ^*, i.e., $\lambda(x)\le\lambda^*$.
2. Generate a homogeneous PPP of density λ^* on A.
3. Sample the obtained random point pattern by deleting each point independently of the others with probability equal to $1-\lambda(x)/\lambda^*$.

Note: The sampling can be performed with the aid of an independent sequence $(u_1,u_2,\ldots)$ of random numbers uniformly distributed over $[0,1]$. More precisely the point x_k is deleted if $u_k>\lambda(x_k)/\lambda^*$.

Voronoi Cell and Voronoi Tessellation: Definitions

Voronoi Cell

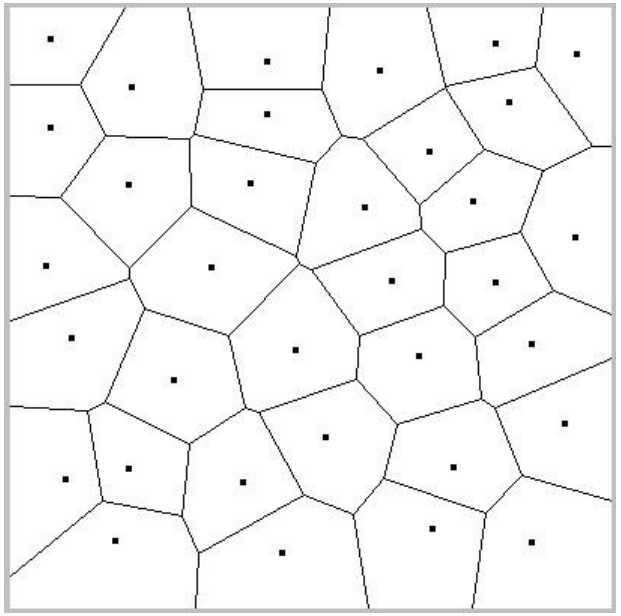
The Voronoi cell $V(x)$ of a point x of a general point process Φ consists of the locations whose distance to x is not greater than their distance from any other point of Φ :

$$V(x) = \left\{ y \in \mathbb{R}^2 : \|x - y\| \le \|z - y\| \quad \forall z \in \Phi \setminus \{x\} \right\}$$

$$= \left\{ y \in \mathbb{R}^2 : \|x - y\| \le \|y - \Phi\| \right\}$$

Voronoi Tessellation

The Voronoi tessellation or Voronoi diagram of Φ is the decomposition of the space into the Voronoi cells of Φ.

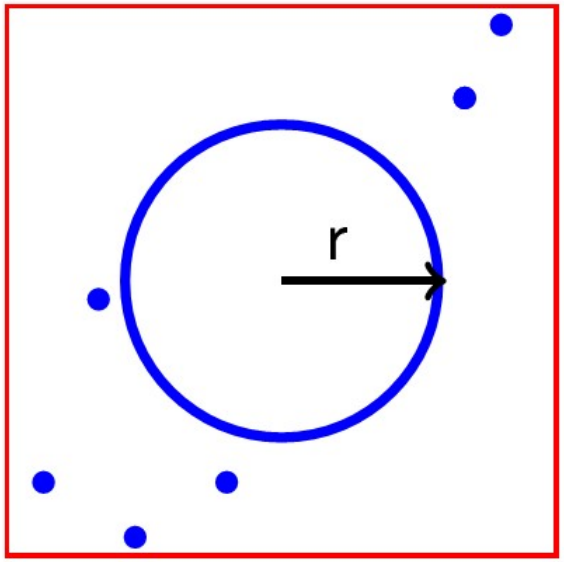


Void Probability of PPP – First Contact Distribution

Distribution of the distance of the nearest point to the origin

Let Φ be a homogeneous PPP of density λ. The Complementary Cumulative Distribution Function (CCDF) of the distance D of the nearest point of Φ to the origin is:

$$\mathrm{CCDF}_D(r) = \mathbb{P}(D \geq r) = \exp\left(-\lambda \pi r^2\right)$$



The Probability Density Function (PDF) of D is:

$$\mathrm{PDF}_D(r) = 2\lambda \pi r \exp\left(-\lambda \pi r^2\right)$$

Proof :

Let $B(o,r)$ be the ball of center the origin "o" and radius "r". Then:

$$\mathrm{CCDF}_D(r) = \mathbb{P}(D \geq r) = \mathbb{P}\left(B(o,r) \text{ is empty}\right) = \mathbb{P}\left(\Phi\left(B(o,r)\right) = 0\right)$$

$$= \exp\left(-\lambda \left|B(o,r)\right|\right) = \exp\left(-\lambda \pi r^2\right)$$

Sums over PPPs: The Campbell Theorem

Campbell Theorem

Let Φ be a PPP of density λ and $f(x): \mathbb{R}^2 \to \mathbb{R}^+$. Then:

$$\mathbb{E}\left\{\sum_{x\in\Phi} f(x)\right\} = \lambda \int_{\mathbb{R}^2} f(x)\,dx$$

Proof :

The proof is made of two parts:

1. First, we compute the expectation by conditioning on an area of radius R and on the number of points that fall in this finite area.
2. Then, we remove the conditioning with respect to the number of points and let the area go to infinity.

Rationale: Given a finite area, by conditioning on the number of points falling into it, the points are independent and uniformly distributed in that area, i.e., they constitute a BPP by definition of PPP.

Sums over PPPs: The Campbell Theorem

Detailed Proof :

Let $B(o,R)$ be the ball of radius R centered at the origin and $n = \Phi\left(B(o,R)\right)$ be the number of points in $B(o,R)$. Then:

$$\mathbb{E}\left\{\sum_{x\in\Phi} f(x)\right\} = \lim_{R\to\infty}\left\{\mathbb{E}\left\{\sum_{x\in\Phi\cap B(o,R)} f(x)\right\}\right\} = \lim_{R\to\infty}\left\{\mathbb{E}_n\left\{\mathbb{E}_{\setminus n}\left\{\sum_{x\in\Phi\cap B(o,R)} f(x)\,\middle|\, n=\Phi\left(B(o,R)\right)\right\}\right\}\right\}$$

- $$\mathbb{E}_{\setminus n}\left\{\sum_{x\in\Phi\cap B(o,R)} f(x)\,\middle|\, n=\Phi\left(B(o,R)\right)\right\} = n\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx$$

- $$\mathbb{E}_n\left\{n\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx\right\} = \mathbb{E}_n\{n\}\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx$$

$$= \lambda\left|B(o,R)\right|\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx$$

$$= \lambda\int_{B(o,R)} f(x)\,dx$$

$$\mathbb{E}\left\{\sum_{x\in\Phi} f(x)\right\} = \lim_{R\to\infty}\left\{\mathbb{E}\left\{\sum_{x\in\Phi\cap B(o,R)} f(x)\right\}\right\} = \lim_{R\to\infty}\left\{\lambda\int_{B(o,R)} f(x)\,dx\right\} = \lambda\int_{\mathbb{R}^2} f(x)\,dx$$

Products over PPPs: Probability Generating Functional

Probability Generating Functional (PGFL)

Let Φ be a PPP of density λ and $f(x): \mathbb{R}^2 \to [0,1]$ be a real value function. Then:

$$\mathbb{E}\left\{\prod_{x\in\Phi} f(x)\right\} = \exp\left(-\lambda \int_{\mathbb{R}^2} \left(1 - f(x)\right) dx\right)$$

Proof :

The proof is made of two parts:

1. First, we compute the expectation by conditioning on an area of radius R and on the number of points that fall in this finite area.
2. Then, we remove the conditioning with respect to the number of points and let the area go to infinity.

Rationale: Given a finite area, by conditioning on the number of points falling into it, the points are independent and uniformly distributed in that area, i.e., they constitute a BPP by definition of PPP.

Products over PPPs: Probability Generating Functional

Detailed Proof :

Let $B(o,R)$ be the ball of radius R centered at the origin and $n=\Phi\left(B(o,R)\right)$ be the number of points in $B(o,R)$. Then:

$$\mathbb{E}\left\{\prod_{x\in\Phi} f(x)\right\}=\lim_{R\to\infty}\left\{\mathbb{E}\left\{\prod_{x\in\Phi\cap B(o,R)} f(x)\right\}\right\}=\lim_{R\to\infty}\left\{\mathbb{E}_n\left\{\mathbb{E}_{\setminus n}\left\{\prod_{x\in\Phi\cap B(o,R)} f(x)\middle| n=\Phi\left(B(o,R)\right)\right\}\right\}\right\}$$

- $$\mathbb{E}_{\setminus n}\left\{\prod_{x\in\Phi\cap B(o,R)} f(x)\middle| n=\Phi\left(B(o,R)\right)\right\}=\left(\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx\right)^n$$

- $$\mathbb{E}_n\left\{\left(\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx\right)^n\right\}=\sum_{n=0}^{\infty}\left(\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx\right)^n\mathbb{P}\left(\Phi\left(B(o,R)\right)=n\right)$$

$$=\sum_{n=0}^{\infty}\left(\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx\right)^n\frac{\left(\lambda\left|B(o,R)\right|\right)^n}{n!}\exp\left(-\lambda\left|B(o,R)\right|\right)$$

$$=\exp\left(-\lambda\left|B(o,R)\right|\right)\exp\left(\lambda\left|B(o,R)\right|\int_{B(o,R)} f(x)\frac{1}{\left|B(o,R)\right|}dx\right)$$

$$=\exp\left(-\lambda\left|B(o,R)\right|\right)\exp\left(\lambda\int_{B(o,R)} f(x)dx\right)=\exp\left(-\lambda\int_{B(o,R)}\left(1-f(x)\right)dx\right)$$

$$\mathbb{E}\left\{\prod_{x\in\Phi} f(x)\right\}=\lim_{R\to\infty}\left\{\mathbb{E}\left\{\prod_{x\in\Phi\cap B(o,R)} f(x)\right\}\right\}=\lim_{R\to\infty}\left\{\exp\left(-\lambda\int_{B(o,R)}\left(1-f(x)\right)dx\right)\right\}=\exp\left(-\lambda\int_{\mathbb{R}^2}\left(1-f(x)\right)dx\right)$$

Sums and Products over Inhomogeneous PPPs

Campbell Theorem

Let Φ be a PPP of intensity function $\lambda(x)$, i.e., $\Lambda(dx)=\lambda(x)dx$ and $f(x):\mathbb{R}^2 \to \mathbb{R}^+$. Then:

$$\mathbb{E}\left\{\sum_{x\in\Phi} f(x)\right\} = \int_{\mathbb{R}^2} f(x)\Lambda(dx) = \int_{\mathbb{R}^2} f(x)\lambda(x)dx$$

Proof : The same as for the homogeneous case.

Probability Generating Functional (PGFL)

Let Φ be a PPP of intensity funtion $\lambda(x)$, i.e., $\Lambda(dx)=\lambda(x)dx$ and $f(x):\mathbb{R}^2 \to [0,1]$ be a real value function. Then:

$$\mathbb{E}\left\{\prod_{x\in\Phi} f(x)\right\} = \exp\left(-\int_{\mathbb{R}^2}(1-f(x))\Lambda(dx)\right) = \exp\left(-\int_{\mathbb{R}^2}(1-f(x))\lambda(x)dx\right)$$

Proof : The same as for the homogeneous case.

Displacement Theorem of (Inhomogeneous) PPP

Definition: A probability measure is a real-valued function defined on a set of events in a probability space that satisfies measure properties, i.e., it returns values in $[0,1]$ and satisfies the countable additivity property.

Random Transformations of Point Processes

Let a point process Φ on $\mathbb{R}^d$. Let A_p be a bounded set in $\mathbb{R}^{d_p}$. Let $p\left(x, A_p\right)$ be a probability kernel from $\mathbb{R}^d$ to $\mathbb{R}^{d_p}$ for every $x \in \mathbb{R}^d$ of Φ, i.e., a probability measure on $\mathbb{R}^{d_p}$. Φ_p on $\mathbb{R}^{d_p}$ is called the transformed point process of Φ by the probability kernel $p\left(x, A_p\right) = \mathbb{P}\left(x_p \in A_p \middle| \Phi\right)$, where $x_p \in \mathbb{R}^{d_p}$ is a point of Φ_p, i.e., the transformed version of x according to the probability kernel.

Displacement Theorem of (Inhomogeneous) PPP

In other words, Φ_p is obtained by randomly and independently displacing each point of Φ on $\mathbb{R}^d$ to some new locations on $\mathbb{R}^{d_p}$ according to the kernel $p\left(x, A_p\right)$, which denotes the probability that the displaced version of x (i.e., x_p) lies in A_p.

Displacement Theorem $\rightarrow$ Independent displacements preserve the Poissonness

If Φ is a PPP of intensity measure $\Lambda(dx)=\lambda(x)dx$, then Φ_p is a PPP of intensity measure $\Lambda_p(dx)$ equal to:

$$\Lambda_p\left(A_p\right)=\int_{\mathbb{R}^d} p\left(x, A_p\right)\Lambda(dx)=\int_{\mathbb{R}^d}\mathbb{P}\left(x_p\in A_p\middle|\Phi\right)\Lambda(dx)$$

$$=\int_{\mathbb{R}^d}\mathbb{P}\left(x_p=l_p(x)\in A_p\right)\lambda(x)dx=\int_{\mathbb{R}^d}\mathbf{1}_{A_p}\left(l_p(x)\right)\lambda(x)dx$$

Note: $\mathbf{1}_{A_p}\left(l_p(x)\right)=1$ if $l_p(x)\in A_p$ and $\mathbf{1}_{A_p}\left(l_p(x)\right)=0$ otherwise

Displacement Theorem of (Inhomogeneous) PPP

$$x_p = l_p(x),\ l_p(\cdot): \mathbb{R}^2 \to \mathbb{R}$$

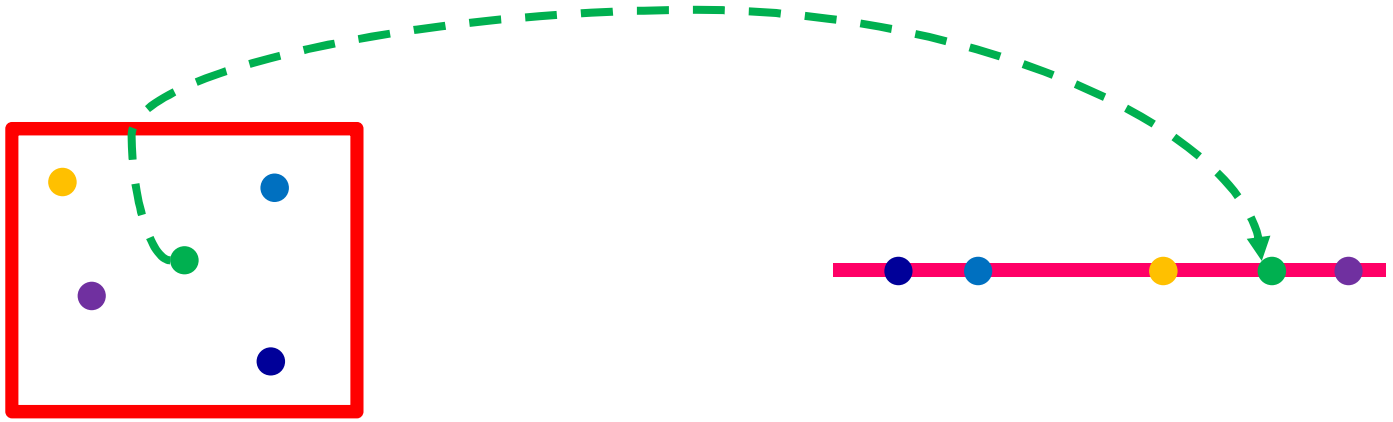
$$\Lambda_p([0,y)) = \int_{\mathbb{R}^2} \mathbf{1}_{[0,y)}(l_p(x))\lambda(x)dx = \int_0^{+\infty}\int_0^{2\pi} \mathbf{1}_{[0,y)}(l_p(r,\theta))\lambda(r,\theta)rdrd\theta$$

$$(\Rightarrow l_p(r,\theta) = r^\alpha \text{ and } \lambda(r,\theta) = \lambda) \quad = 2\pi\lambda\int_0^{+\infty} \mathbf{1}_{[0,y)}(r^\alpha)rdr = 2\pi\lambda\int_0^{y^{1/\alpha}} rdr = \pi\lambda y^{2/\alpha}$$

A homogeneous PPP in 2-D is transformed into an inhomogeneous PPP in 1-D

Displacement Theorem of (Inhomogeneous) PPP

$$x_p = l_p(x),\ l_p(\cdot): \mathbb{R}^2 \to \mathbb{R}$$



$$\Lambda_p\left([0,y)\right) = \int_{\mathbb{R}^2} \mathbf{1}_{[0,y)}\left(l_p(x)\right)\lambda(x)\,dx = \int_0^{+\infty}\int_0^{2\pi} \mathbf{1}_{[0,y)}\left(l_p(r,\theta)\right)\lambda(r,\theta)\,r\,dr\,d\theta$$

$$\left(\Rightarrow l_p(r,\theta) = r^{\alpha}/T \text{ and } \lambda(r,\theta) = \lambda \text{ and } \mathbb{P}(T \le t) = \mathrm{CDF}_T(t)\right)$$

$$= 2\pi\lambda\mathbb{E}_T\left\{\int_0^{+\infty} \mathbf{1}_{[0,y)}\left(\frac{r^{\alpha}}{T}\middle| T\right) r\,dr\right\} = 2\pi\lambda\mathbb{E}_T\left\{\int_0^{(yT)^{1/\alpha}} r\,dr\right\}$$

$$= \pi\lambda\mathbb{E}_T\left\{(yT)^{2/\alpha}\right\} = \pi\lambda y^{2/\alpha}\mathbb{E}_T\left\{T^{2/\alpha}\right\}$$

Displacement Theorem of (Inhomogeneous) PPP

Proof :

Consider the summation $S = \sum_{x_p \in \Phi_p} f\left(x_p\right)$ for any measurable function $f\left(\cdot\right)$.

Consider the (rather general) displacement $x_p = l_p\left(x, T_x\right),$ where $x \in \Phi$ and T_x are independent distributed random variables (that, however, may depend on x) whose distribution is $\mathbb{P}\left(T_x \le t\right) = \mathrm{CDF}_{T_x}\left(t\right)$.

If Φ_p was a PPP, from the PGFL theorem, we would have:

$$\mathbb{E}\left\{\exp\left(-S\right)\right\} = \mathbb{E}\left\{\exp\left(-\sum_{x_p \in \Phi_p} f\left(x_p\right)\right)\right\} = \mathbb{E}\left\{\prod_{x_p \in \Phi_p} \exp\left(-f\left(x_p\right)\right)\right\}$$

$$= \exp\left(-\int_{\mathbb{R}^{d_p}} \left(1 - \exp\left(-f\left(x_p\right)\right)\right) \Lambda_p\left(dx_p\right)\right)$$

Displacement Theorem of (Inhomogeneous) PPP

Let us compute $\mathbb{E}\left\{\exp\left(-S\right)\right\}$ without assuming that Φ_p is a PPP:

$$\mathbb{E}\left\{\exp\left(-S\right)\right\} = \mathbb{E}\left\{\exp\left(-\sum_{x_p\in\Phi_p} f\left(x_p\right)\right)\right\} = \mathbb{E}\left\{\prod_{x_p\in\Phi_p}\exp\left(-f\left(x_p\right)\right)\right\}$$

$$= \mathbb{E}\left\{\prod_{x\in\Phi}\exp\left(-f\left(l_p\left(x,T_x\right)\right)\right)\right\} = \mathbb{E}_{\Phi}\left\{\prod_{x\in\Phi}\mathbb{E}_{T_x}\left\{\exp\left(-f\left(l_p\left(x,T_x\right)\right)\right)\right\}\right\}$$

Let define $\tilde{f}\left(x\right) = \mathbb{E}_{T_x}\left\{\exp\left(-f\left(l_p\left(x,T_x\right)\right)\right)\right\}$, we have:

$$\mathbb{E}\left\{\exp\left(-S\right)\right\} = \mathbb{E}_{\Phi}\left\{\prod_{x\in\Phi}\tilde{f}\left(x\right)\right\} = \exp\left(-\int_{\mathbb{R}^d}\left(1-\tilde{f}\left(x\right)\right)\Lambda\left(dx\right)\right)$$

$$= \exp\left(-\int_{\mathbb{R}^d}\left(1-\mathbb{E}_{T_x}\left\{\exp\left(-f\left(l_p\left(x,T_x\right)\right)\right)\right\}\right)\Lambda\left(dx\right)\right)$$

$$= \exp\left(-\int_{\mathbb{R}^d}\int_{\mathbb{T}}\left(1-\exp\left(-f\left(l_p\left(x,t\right)\right)\right)\right)\mathrm{CDF}_{T_x}\left(dt\right)\Lambda\left(dx\right)\right)$$

$$= \exp\left(-\int_{\mathbb{R}^{d_p}}\left(1-\exp\left(-f\left(x_p\right)\right)\right)\Lambda_p\left(dx_p\right)\right)$$

the last identity holds if $\Lambda_p\left(dx_p\right) = \mathrm{CDF}_{T_x}\left(dt\right)\Lambda\left(dx\right) = p\left(x,dx_p\right)\Lambda\left(dx\right)$.

Marked Point Processes

A point process is made into a marked point process by attaching a characteristic (a mark) to each point of the process. Thus a marked point process on $\mathbb{R}^2$ is a random sequence $y_n = \{x_n, m_m\}$ for $n = 1, 2, ...,$ where the points x_n constitute the point process Φ, i.e., $x_n \in \Phi \subset \mathbb{R}^2$ (unmarked or ground process) and m_m are the marks corrresponsing to the respective points x_n. The marks belong to a given space and have some given distribution.

Examples:

- x is the center of an atom and m is the type of atom
- x is the location of a tree and m is the type of tree
- x is the location of a transmitter and m is the transmit power
- x is the location of a transmitter and m is the channel gain

Marking Theorem for Inhomogeneous PPP

Independent Marks

A marked point process is said to be independently marked if, given the locations of the points of the ground point process $\Phi = \{x_i\} \subset \mathbb{R}^d$, the marks are mutually independent random vectors on $\mathbb{R}^l$ and if the conditional distribution of the mark m of a point $x \in \Phi$ depends only on the point x it is attached to, i.e., $\mathbb{P}\left(m \in \cdot \middle| \Phi\right) = \mathbb{P}\left(m \in \cdot \middle| x\right) = F_x\left(dm\right)$, where $F_x\left(dm\right)$ on $\mathbb{R}^l$ is the probability kernel (distribution) of the marks.

Marking Theorem of PPPs

Let a ground PPP Φ with intensity measure $\Lambda\left(dx\right)$ on $\mathbb{R}^d$ and marks with distributions $F_x\left(dm\right)$ on $\mathbb{R}^l$. The independently marked point process Φ_M is a PPP on $\mathbb{R}^d \times \mathbb{R}^l$ with intensity measure equal to:

$$\Lambda_M\left(d\left(x,m\right)\right) = F_x\left(dm\right)\Lambda\left(dx\right)$$

Proof : It is the same as for the displacement theorem.

Bottom Line…

- **Independent displacements of a PPP result in a PPP**
- **Independent markings of a PPP result in a PPP**
- **These transformations occur in several applications…**
- **To deal with them, apply the constructive proof used to prove the displacement theorem**
- **You will be able to compute "sums over PPPs", "products over PPPs", etc. of a large class of "practical transformations" of PPPs…**

Example: Independent Thinning

Independent Thinned Point Processes

Let a point process Φ. The point process Φ_{thin} obtained from Φ by randomly and indepenently removing some fractions of its points with probability $1 - p(x)$ is called thinned point process with retention probability $p(x)$.

Thinning Theorem of PPPs

The thinning by retention probability $p(x)$ of an inhomogeneous PPP of intensity measure $\Lambda(dx)$ is an inhomogenous PPP of intensity measure:

$$\Lambda_{thin}(dx) = p(x)\Lambda(dx)$$

Proof : It is an application of the displacement theorem. Let's do it...

Example: Independent Thinning

Consider the summation $S = \sum_{x \in \Phi} \mu_x f(x)$ for any measurable function $f(\cdot)$, where $\mathbb{P}(\mu_x = 1) = p(x)$ & $\mathbb{P}(\mu_x = 0) = 1 - p(x)$.

Let us compute $\mathbb{E}\{\exp(-S)\}$:

$$\mathbb{E}\{\exp(-S)\} = \mathbb{E}\left\{\exp\left(-\sum_{x \in \Phi} \mu_x f(x)\right)\right\} = \mathbb{E}\left\{\prod_{x \in \Phi} \exp\left(-\mu_x f(x)\right)\right\}$$

$$= \mathbb{E}_{\Phi}\left\{\prod_{x \in \Phi} \mathbb{E}_{\mu_x}\left\{\exp\left(-\mu_x f(x)\right)\right\}\right\} = \mathbb{E}_{\Phi}\left\{\prod_{x \in \Phi} p(x)\exp\left(-f(x)\right) + \left(1 - p(x)\right)\right\}$$

Let define $\tilde{f}(x) = p(x)\exp\left(-f(x)\right) + \left(1 - p(x)\right)$, we have:

$$\mathbb{E}\{\exp(-S)\} = \mathbb{E}_{\Phi}\left\{\prod_{x \in \Phi} \tilde{f}(x)\right\} = \exp\left(-\int_{\mathbb{R}^2} \left(1 - \tilde{f}(x)\right)\Lambda(dx)\right)$$

$$= \exp\left(-\int_{\mathbb{R}^2} \left(1 - p(x)\exp\left(-f(x)\right) - \left(1 - p(x)\right)\right)\Lambda(dx)\right)$$

$$= \exp\left(-\int_{\mathbb{R}^2} \left(1 - \exp\left(-f(x)\right)\right) p(x)\Lambda(dx)\right)$$

$\Rightarrow$ PGFL of a PPP with intensity measure $\Lambda_{\text{thin}}(dx) = p(x)\Lambda(dx)$

Independent Thinning: An Illusory Paradox

A Bernoulli trial is an idealized coin flip. The probability of heads is p and the probability of tails is $q = 1 - p$. Sequences of Bernoulli trials are independent.

Denote the numbers of heads and tails observed in a sequence of $n > 1$ independent Bernoulli trials by n_h and n_t, respectively. The sequence of Bernoulli trials is performed (conceptually) many times, so the observed numbers n_h and n_t are realizations of random variables, denoted by N_h and N_t, respectively. If exactly n trials are always performed, the random variables N_h and N_t are not independent because of the deterministic constraint $n_h + n_t = n$.

However, if the sequence length n is a realization of a Poisson distributed random variable, denoted by N, then N_h and N_t are independent random variables! The randomized constraint $n_h + n_t = n$ holds, but it is not enough to induce any dependence whatever between N_h and N_t.

Independent Thinning: An Illusory Paradox

N is a Poisson random variable with density λ and its realization n is the length of the number of Bernoulli trials performed. Then $n = n_h + n_t$, where n_h and n_t are the observed numbers of heads and tails. The random variables N_h and N_t are **independent** Poisson distributed with mean intensities $p\lambda$ and $(1-p)\lambda$.

Let us prove the independence:

$$\mathbb{P}(N = n, N_h = n_h, N_t = n_t) = \mathbb{P}(N = n)\,\mathbb{P}(N_h = n_h, N_t = n_t \,|\, N = n)$$

$$= \left[\frac{\lambda^n}{n!}\exp(-\lambda)\right]\left[\binom{n}{n_h} p^{n_h}(1-p)^{n_t}\right] = \left[\frac{\lambda^{n_h}\lambda^{n_t}}{n!}\exp(-\lambda)\right]\left[\frac{n!}{n_h!\,n_t!} p^{n_h}(1-p)^{n_t}\right]$$

$$= \left(\frac{(p\lambda)^{n_h}}{n_h!}\exp(-\lambda)\right)\left(\frac{((1-p)\lambda)^{n_t}}{n_t!}\right)\left[\frac{\exp(-p\lambda)}{\exp(-p\lambda)}\right]$$

$$= \left(\frac{(p\lambda)^{n_h}}{n_h!}\exp(-p\lambda)\right)\left(\frac{((1-p)\lambda)^{n_t}}{n_t!}\exp(-(1-p)\lambda)\right)$$

Palm Theory and Conditioning

- **Palm theory formalizes the notion of the conditional distribution of a general point process given that it has a point at some location.**
- **Palm probability/measure is the probability of an event given that the point process contains a point at some location.**
- **Palm theory formalizes the notion of the "typical point" of a point process. Informally, the typical point results from a selection procedure in which every point has the same chance of being selected.**
- **On the other hand, a point chosen according to some sampling procedure, such as the point closest to the origin, is not typical, because it has been selected in a specific manner.**
- **Palm distribution is the conditional point process distribution given that a point exists at a specific location.**

Palm Distribution: Notation

Consider the event (or property) E of a point process Φ.

The following notations are equivalent and used interchangeably:

$$\begin{aligned}\mathbb{P}\left(\Phi \text{ has property } E \middle\| x\right) &= \mathbb{P}\left(\Phi \text{ has property } E \middle| x \in \Phi\right) \\ &= \mathbb{P}\left(\Phi \in E \middle| x \in \Phi\right) \\ &= \mathbb{P}^x\left(E\right) \\ &= \mathbb{P}_x\left(E\right)\end{aligned}$$

Reduced Palm Distribution: Definition and Notation

Rationale

1. When calculating Palm probabilities it is more natural not to consider the point of the point process that we condition on.

2. Consider a network whose nodes form a point process. Assume that we want to identify one of them as the intended transmitter, while the other act as the interferers. The computation of the sum interference from all the interferers requires the conditioning on the location of the intended transmitter and its exclusion from the set of interferers for computing the distribution of the sum interference.

Reduced Palm Distribution

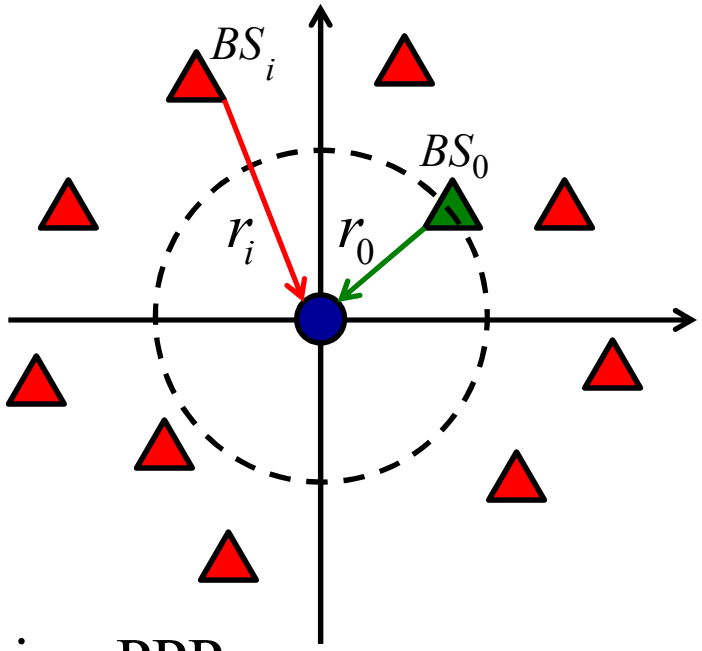
Consider the event (or property) E of a point process Φ.

The reduced Palm distribution is the probability that Φ has property E conditioning on a point of Φ being located at x and not counting it, i.e., the point on which we condition is not included in the distribution.

The following notations are equivalent and used interchangeably:

$$\mathbb{P}\left(\Phi \setminus \{x\} \in E \middle| x \in \Phi\right) = \mathbb{P}\left(\Phi \setminus \{x\} \in E \middle\| x\right) = \mathbb{P}^{!x}\left(E\right) = \mathbb{P}_x^{!}\left(E\right)$$

Reduced Palm Distribution: Application Example



Φ is a PPP

$$\mathrm{P}_{\mathrm{cov}} = \Pr\left\{\mathrm{SINR} > T\right\}$$

$$\mathrm{SINR} = \frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)}$$

$$I_{agg}\left(r_0\right) = \sum_{i \in \Phi \setminus BS_0} P\left|h_i\right|^2 r_i^{-\alpha}$$

$$\mathrm{P}_{\mathrm{cov}} = \Pr\left\{\frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)} > T\right\} = \ldots$$

Reduced Palm Distribution of PPP: Slivnyak Theorem

Slivnyak - Mecke Theorem

The reduced Palm distribution of a PPP is equal to its original distribution:

$$\mathbb{P}^{!x}\left(E\right)=\mathbb{P}\left(E\right)$$

Note: This implies that, for a PPP, a new point can be added or a point can be removed from the point process without disturbing the distribution of the other points of the process. This originates from their complete spatial randomness.

Reduced Palm Distribution: Sums over Point Processes

Campbell - Mecke Theorem

Let Φ be a point process of intensity measure $\Lambda(dx)$ and $\mathbb{E}^{!x}\{\cdot\}$ be the expectation under the reduced Palm distribution. Let $f(\cdot)$ be a real-valued function.

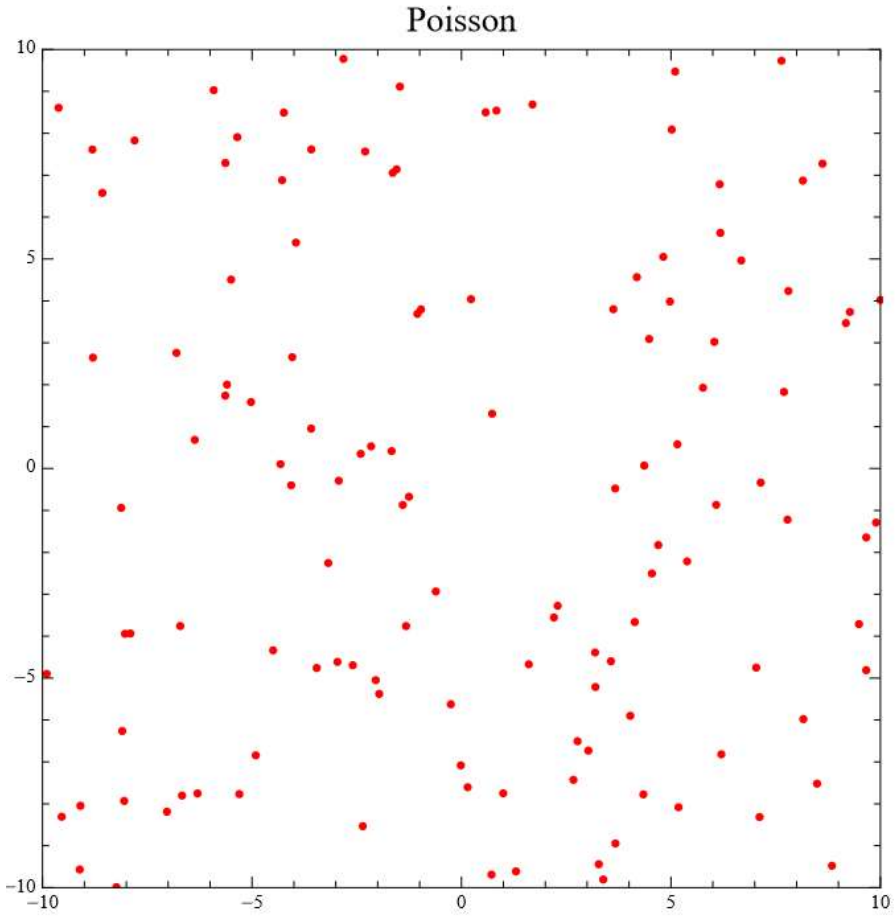
The following holds:

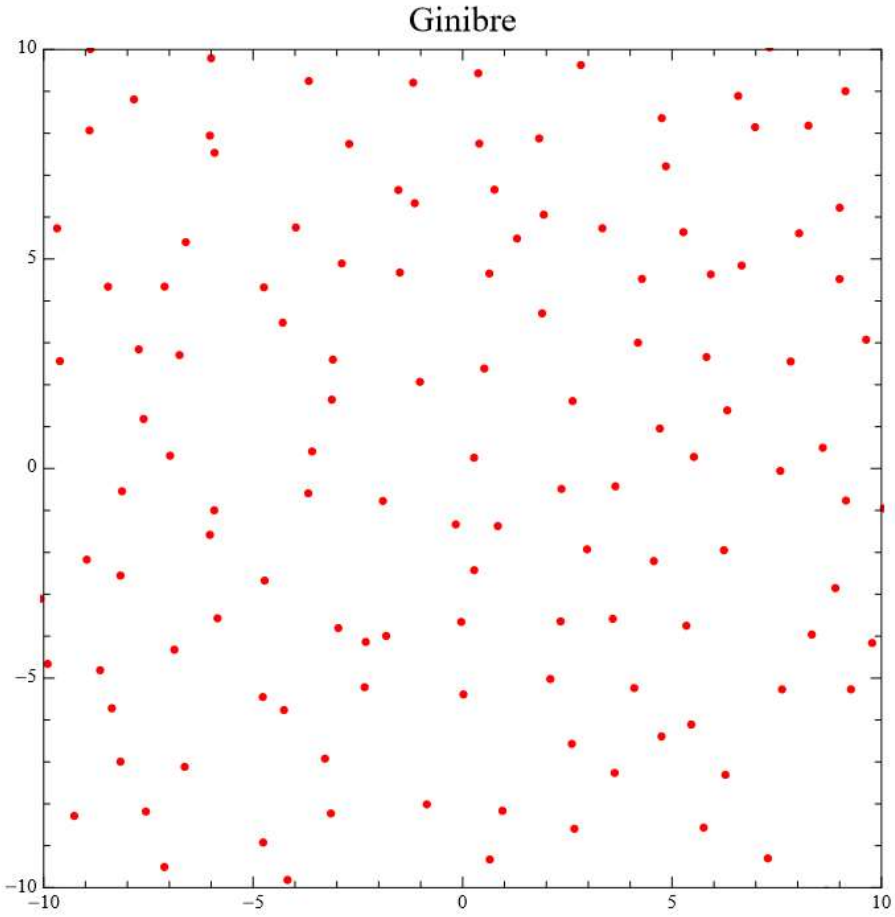
$$\mathbb{E}\left\{\sum_{x\in\Phi} f\left(x,\Phi\setminus\{x\}\right)\right\} = \int_{\mathbb{R}^2} \mathbb{E}^{!x}\left\{f\left(x,\Phi\right)\right\}\Lambda(dx)$$

Campbell - Mecke Theorem of PPPs

$$\mathbb{E}\left\{\sum_{x\in\Phi} f\left(x,\Phi\setminus\{x\}\right)\right\} = \int_{\mathbb{R}^2} \mathbb{E}\left\{f\left(x,\Phi\right)\right\}\Lambda(dx)$$

Counter-Example that the PPP is Special: Beta-Ginibre





Counter-Example that the PPP is Special: Beta-Ginibre

Proposition 1. *Let* $\Phi_c = \{X_i\}_{i\in\mathbb{N}}$ *be a scaled* β*-GPP. For* $k \in \mathbb{N}$*, let* Q_k *be a random variable with probability density function*

$$f_{Q_k}(q) = \frac{q^{k-1}e^{-\frac{c}{\beta}q}}{(\beta/c)^k\Gamma(k)},$$

i.e., $Q_k \sim \text{gamma}(k, \beta/c)$*, with* Q_k *independent of* Q_j *if* $k \neq j$*. Then the set* $\{|X_i|^2\}_{i\in\mathbb{N}}$ *has the same distribution as the set* Ξ *obtained by retaining from* $\{Q_k\}_{k\in\mathbb{N}}$ *each* Q_k *with probability* β *independently of everything else* .

Counter-Example that the PPP is Special: Beta-Ginibre

Proposition 2. (The Palm measure of the scaled β-Ginibre point process). *For a scaled β-GPP Φ_c, the Palm measure of Φ_c is the law of the process obtained by adding the origin and deleting the point X if it belongs (which occurs with probability β) to the process Φ_c, where $|X|^2 = Q_1$.*

From Propositions 1 and 2, we observe that the Palm distribution of the squared moduli Q_k is closely related to the non-Palm version, the only difference being that Q_1 is removed if it is included in Ξ.

Playing with Point Processes

spatstat analysing spatial point patterns

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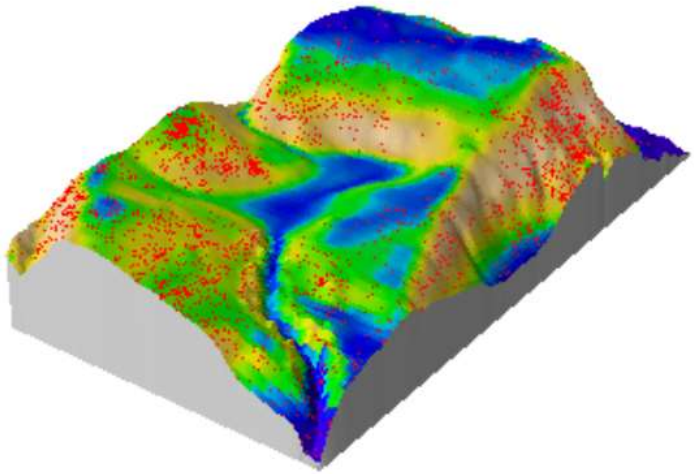

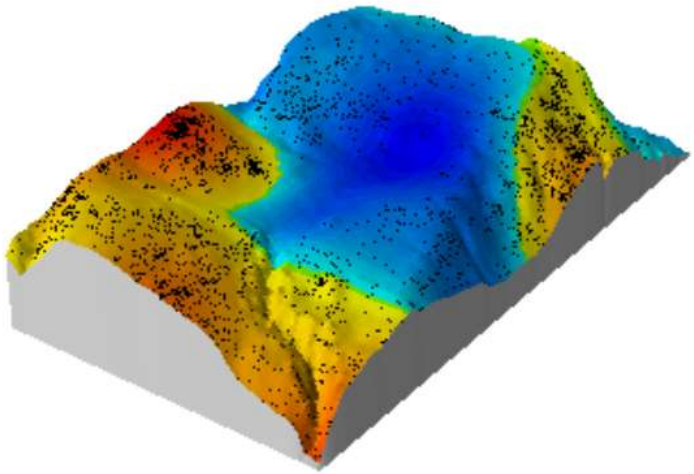

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Playing with Point Processes

Package 'spatstat'

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Date 2016-07-08

Title Spatial Point Pattern Analysis, Model-Fitting, Simulation, Tests

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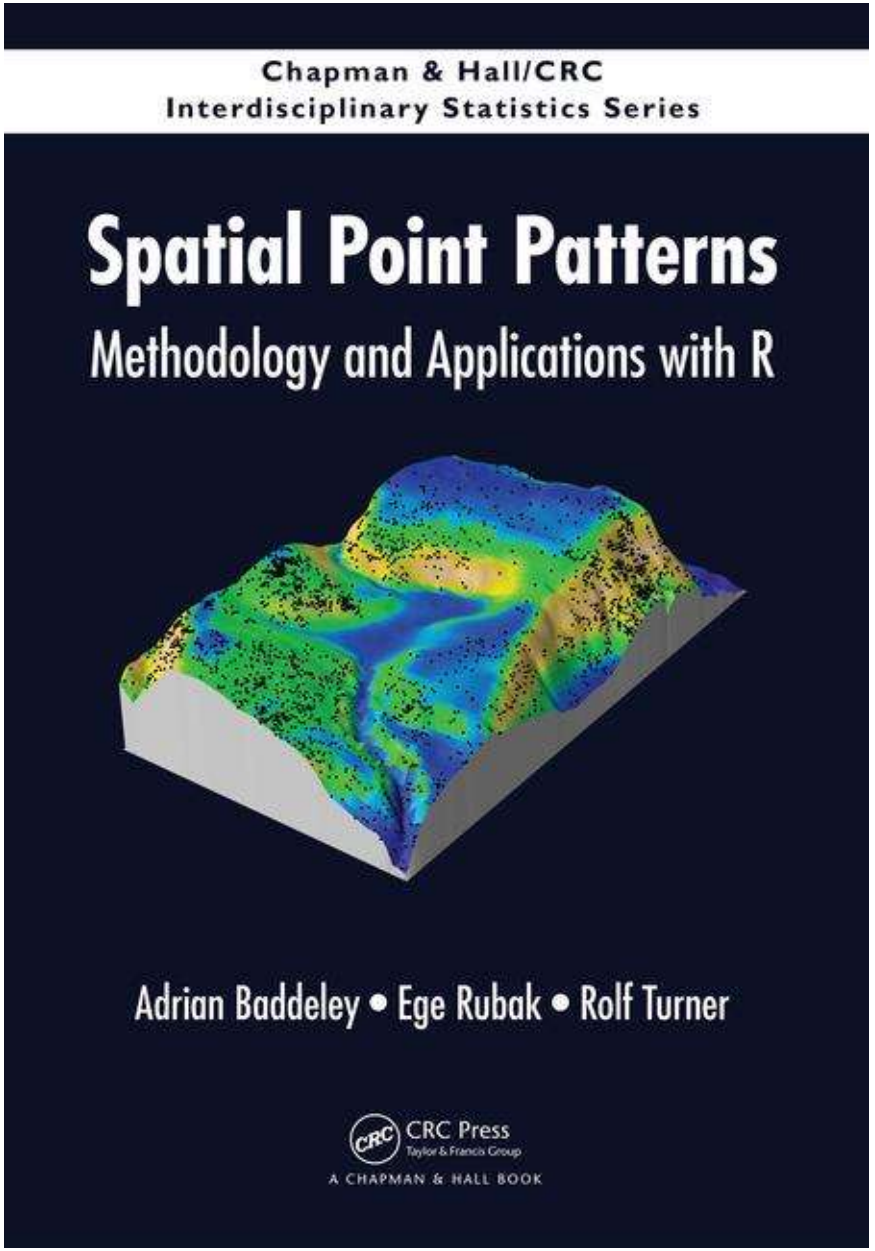
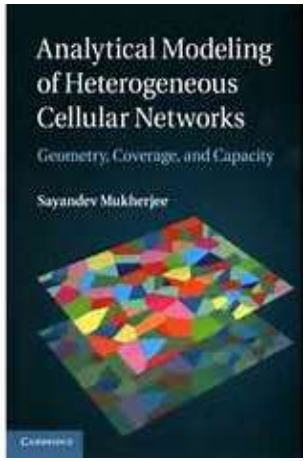
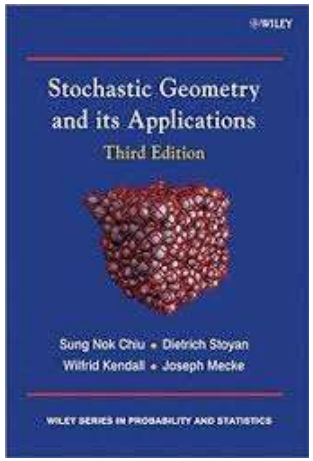
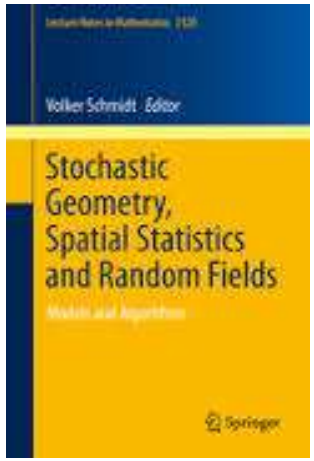
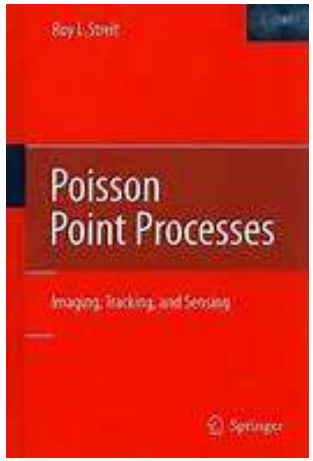


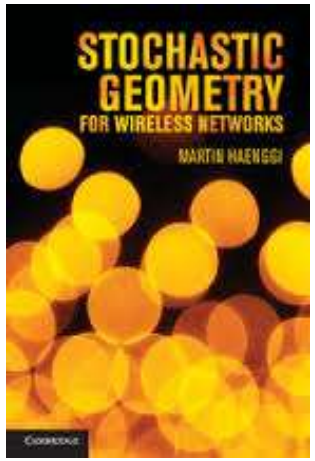
Useful Material

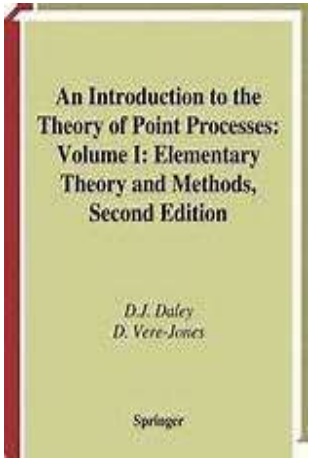


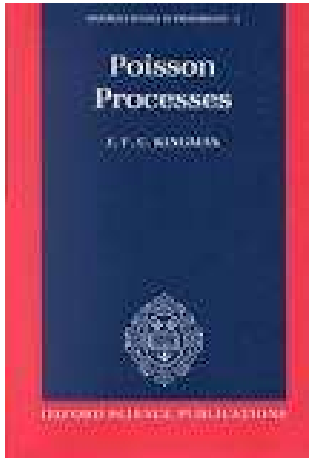


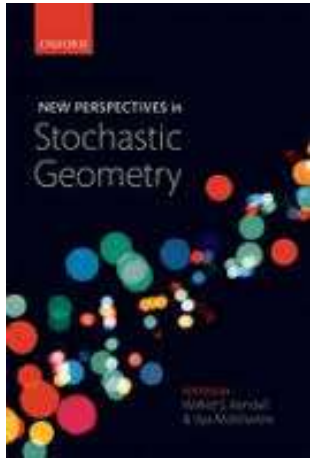


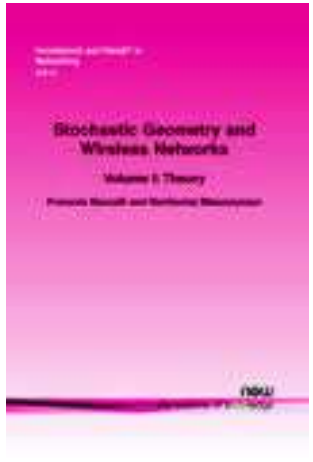


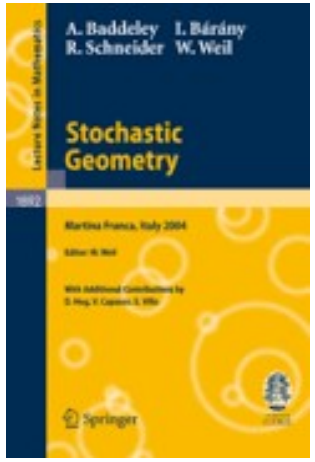












"Cappuccino" Point Process: The Time Has Come…

Thank You for Your Attention

- **ETN-5Gwireless (H2020-MCSA, grant 641985)**

  An European Training Network on 5G Wireless Networks

  http://cordis.europa.eu/project/rcn/193871_en.html (Jan. 2015, 4 years)

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