System-Level Analysis of Cellular Networks

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H2020-MCSA

Course on Random Graphs and Wireless Commun. Networks
Oriel College, Oxford University, UK, Sep. 5-6, 2016

5G-PPP – 5G Network Vision

More information at

www.5g-ppp.eu

5G Vision

The 5G Infrastructure Public Private Partnership:

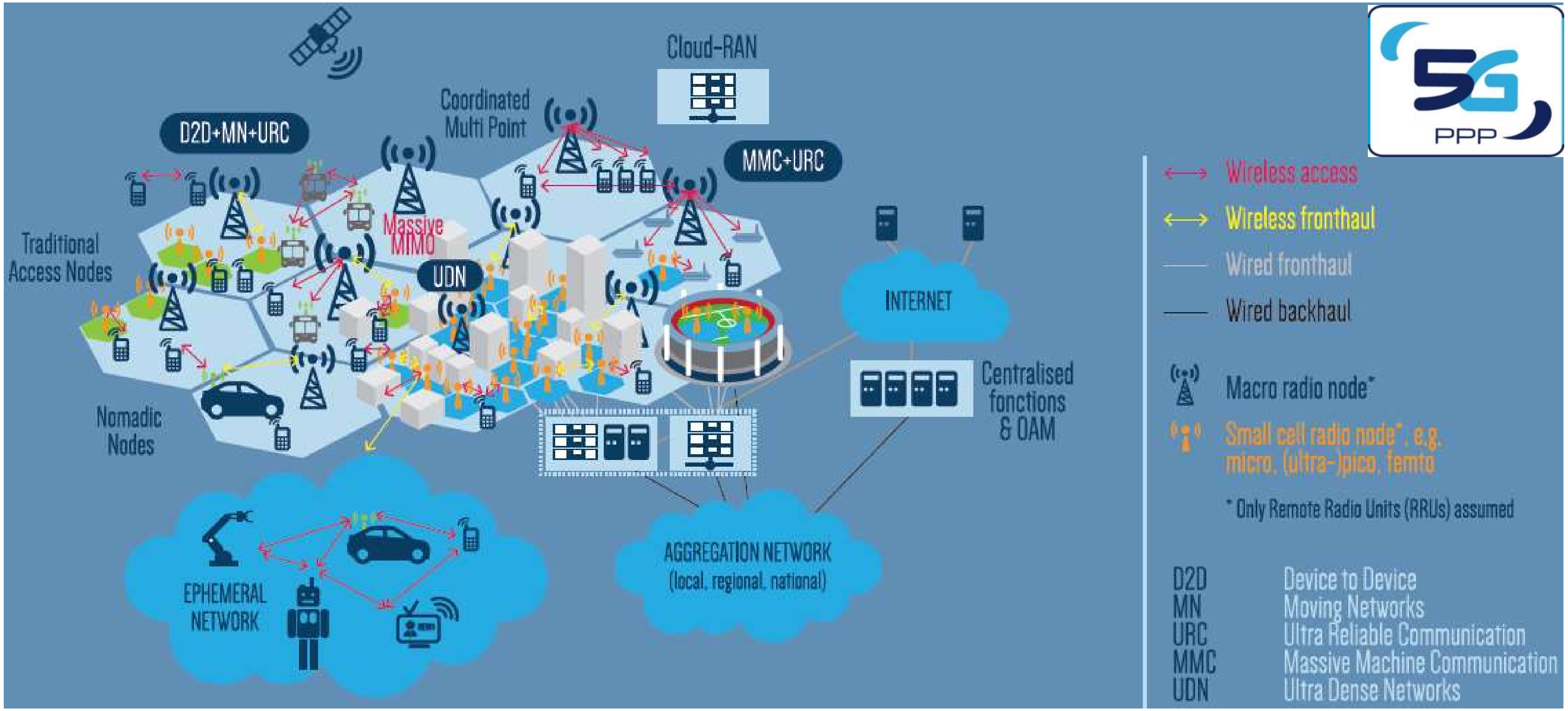
the next generation of

communication networks and services.

5G infrastructure

5G-PPP 5G Vision Document, "The next-generation of communication networks and services", March 2015. Available: http://5g-ppp.eu/wp-content/uploads/2015/02/5G-Vision-Brochure-v1.pdf.

5G-PPP – 5G Network Vision



5G-PPP 5G Vision Document, "The next-generation of communication networks and services", March 2015. Available: http://5g-ppp.eu/wp-content/uploads/2015/02/5G-Vision-Brochure-v1.pdf.

The 5G (Cellular) Network of the Future

- ☐ Buzzword 1: Densification
  1. Access Points (*Network Topology*, HetNets)
  2. Radiating Elements (Large-Scale/Massive MIMO)

- ☐ Buzzword 2: Spectral vs. Energy Efficiency Trade-Off
  1. Shorter Transmission Distance (Relaying, Femto, D2D)
  2. Total Power Dissipation (Single-RF MIMO, Antenna Muting)
  3. RF Energy Harvesting, Wireless Power Transfer, Full-Duplex

- ☐ Buzzword 3: Spectrum Scarcity
  1. Cognitive Radio and Opportunistic Communications
  2. mmWave Cellular Communications

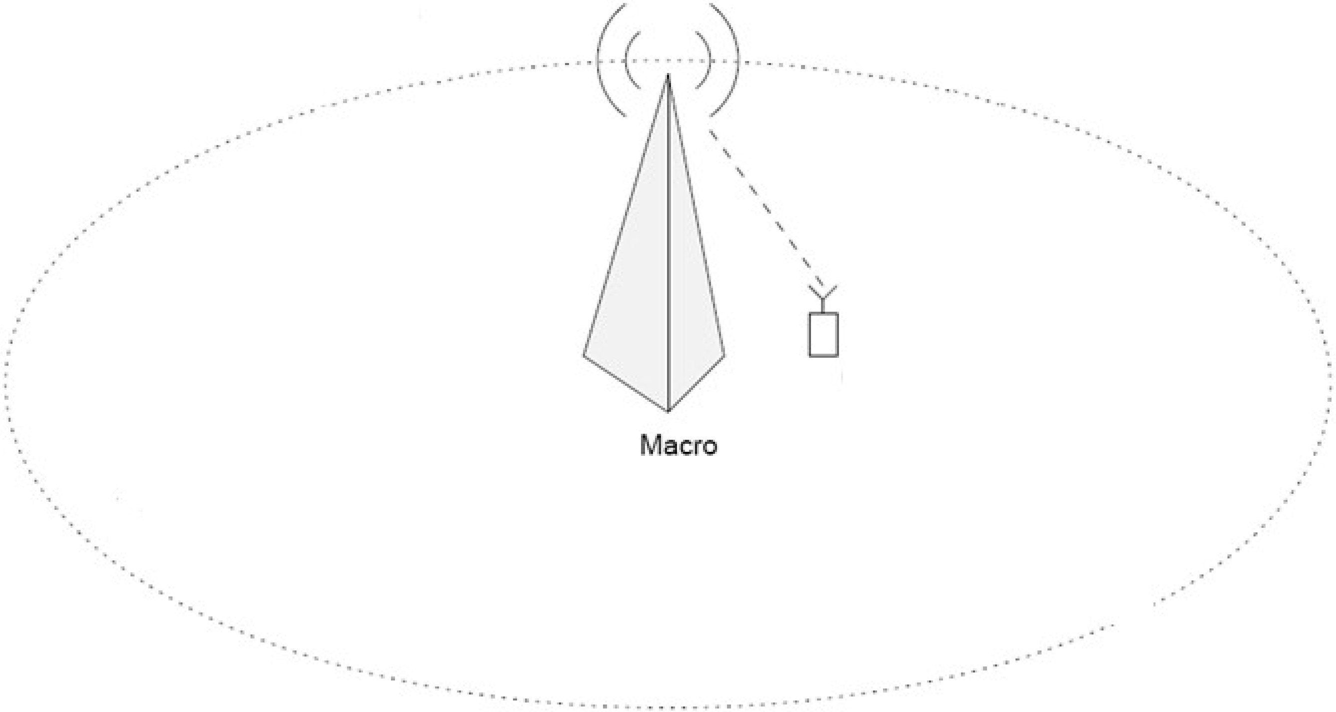
- ☐ Buzzword 4: Software-Defined, Centrally-Controlled, Shared, Virtualized
  1. SDN, NFV, Network Resource Virtualization (NRV)

This Lecture: System-Level Analysis of 5G Networks

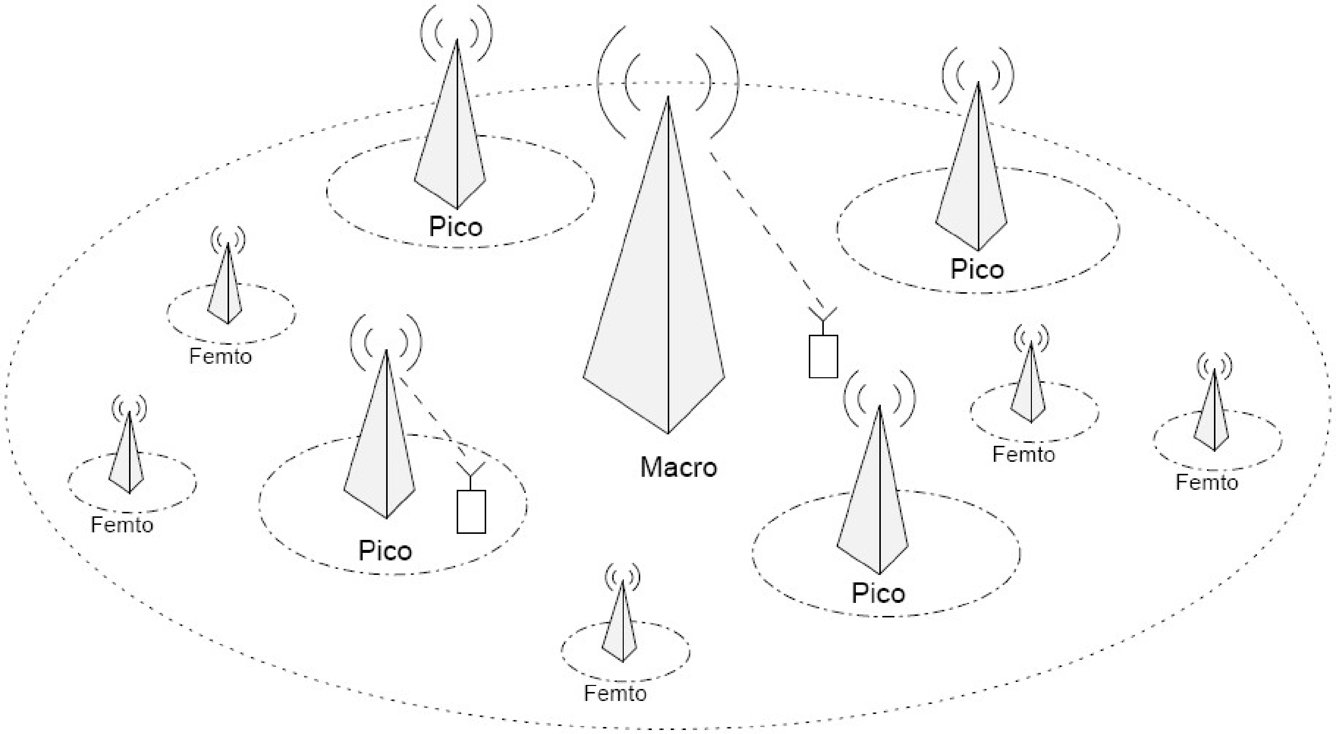
- Stochastic Geometry for Modeling Cellular Networks
  - Why do we need Stochastic Geometry ?
  - Can Stochastic Geometry model practical network deployments ?
  - How to use Stochastic Geometry for performance evaluation ?
  - Quick survey of recently proposed mathematical approaches…

- Cellular "applications": Not covered in this lecture
  - HetNets
  - Massive MIMO
  - mmWave cellular
  - Relaying
  - Wireless power transfer
  - etc… etc…

Why? - Densification of Base Stations (your parents net)

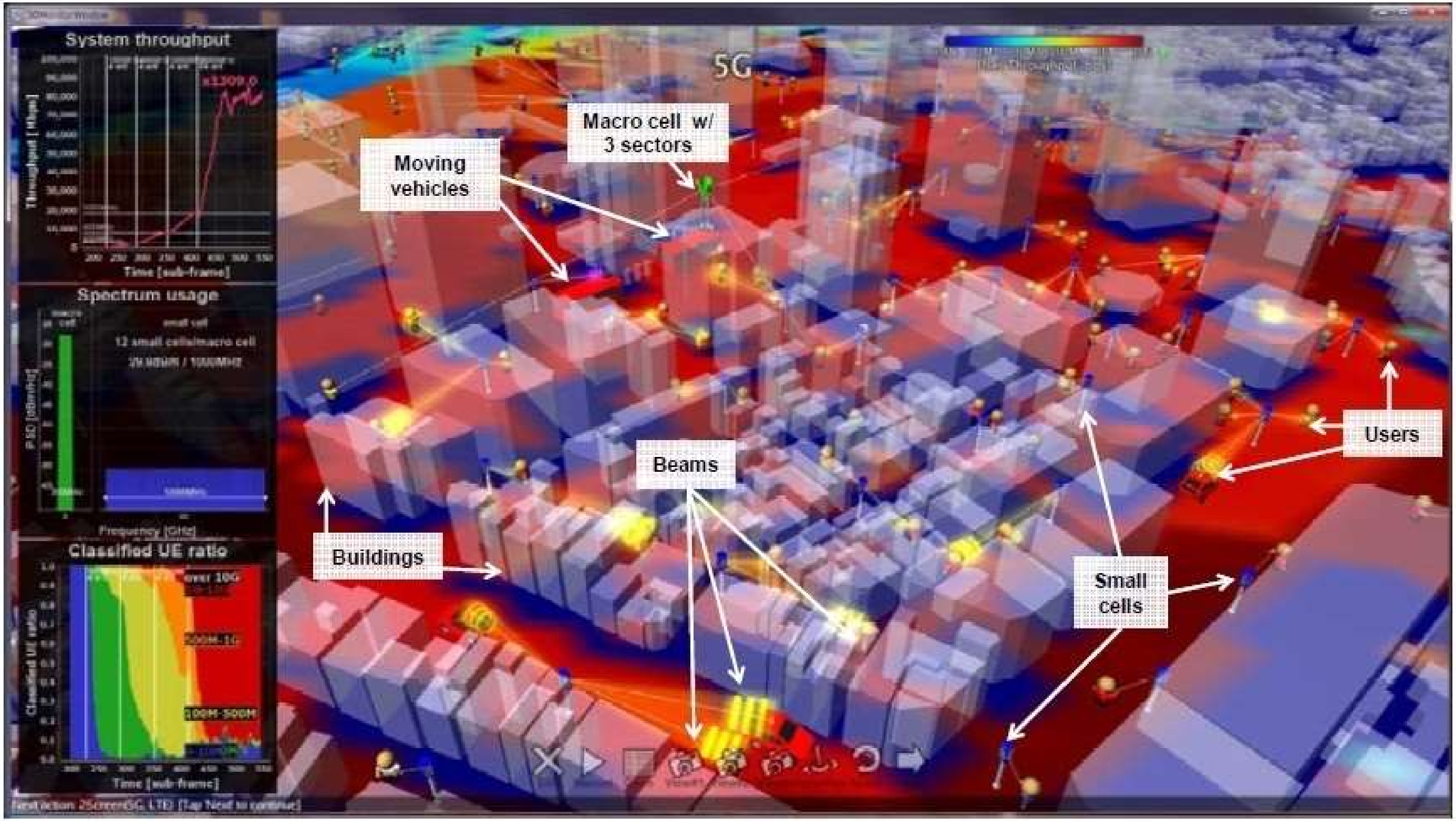


Why? - Densification of Base Stations (your kids net)



Modeling Cellular Networks – In Industry

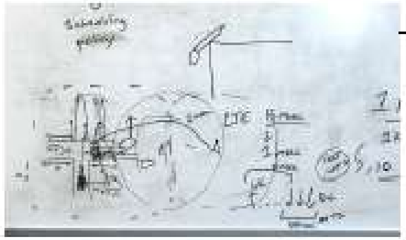
The NTT DOCOMO 5G Real-Time Simulator



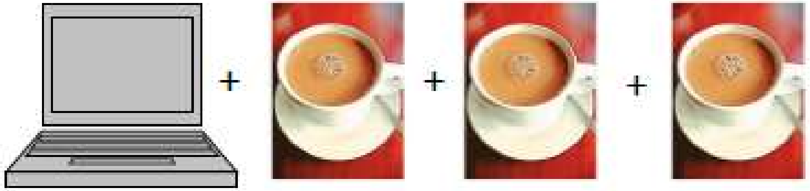
DOCOMO 5G White Paper, "5G Radio Access: Requirements, Concept and Technologies", July 2014.

Life of a 3GPP Simulation Expert (according to Samsung)

Inspiration strikes!



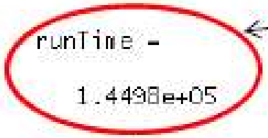
Weeks writing/debugging code:



There's a reason the server is called "grumpy":

```
[t.novlan@grumpy ~]$ bhosts
HOST_NAME          STATUS        JL/U    MAX  NJOBS
grumpy             unavail          -      1      0
mpisrv01dal        ok               -     24      5
mpisrv02dal        unavail          -      1      0
mpisrv03dal        ok               -     20     14
mpisrv04dal        ok               -     24     18
mpisrv05dal        ok               -     20      0
```

Come back after waiting this long:

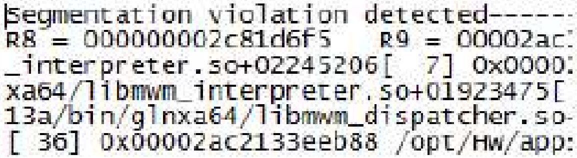


```
runTime =

    1.4498e+05
```

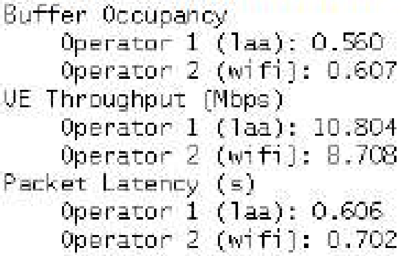
seconds!

Hope you don't find this:

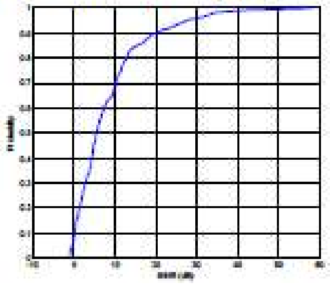


```
Segmentation violation detected------
R8 = 000000002c81d6f5   R9 = 00002ac
_interpreter.so+02245206[  7] 0x0000
xa64/libmwm_interpreter.so+01923475[
13a/bin/glnxa64/libmwm_dispatcher.so-
[ 36] 0x00002ac2133eeb88 /opt/Hw/app:
```

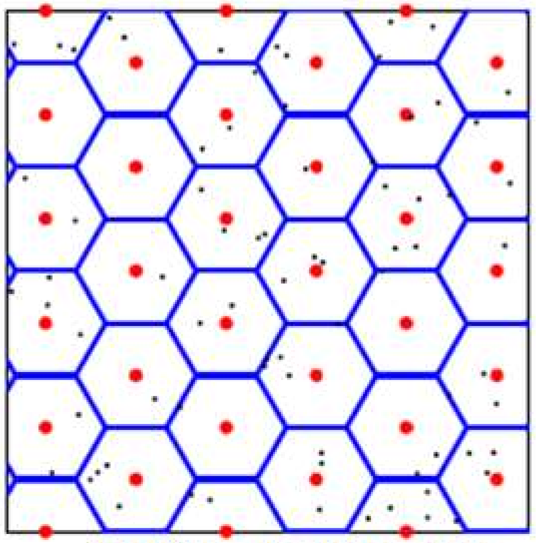
All for a few numbers and a MATLAB plot:



```
Buffer Occupancy
     Operator 1 (laa): 0.560
     Operator 2 (wifi): 0.607
UE Throughput (Mbps)
     Operator 1 (laa): 10.804
     Operator 2 (wifi): 8.708
Packet Latency (s)
     Operator 1 (laa): 0.606
     Operator 2 (wifi): 0.702
```



Modeling Cellular Networks – In Academia

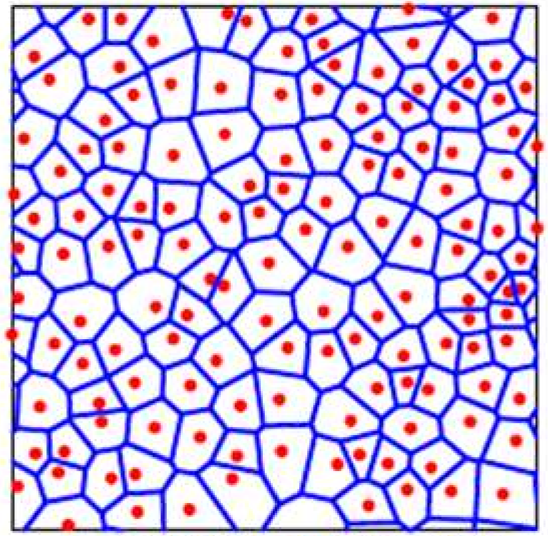


Traditional grid model

- ☐ **Conventional approaches to the analysis and design of cellular networks (abstraction models) are:**
  - **The Wyner model**
  - **The single-cell interfering model or dominant interferers model**
  - ***The regular hexagonal or square grid model***

  D. H. Ring and W. R. Young, "The hexagonal cells concept", Bell Labs Technical Journal, Dec. 1947. http://www.privateline.com/archive/Ringcellreport1947.pdf.

Modeling Cellular Networks – In Academia



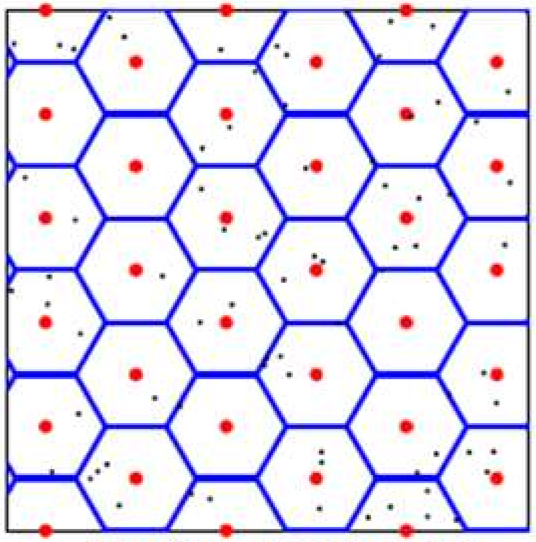
Actual 4G network today

← Reality

vs.

Abstraction

Modeling →



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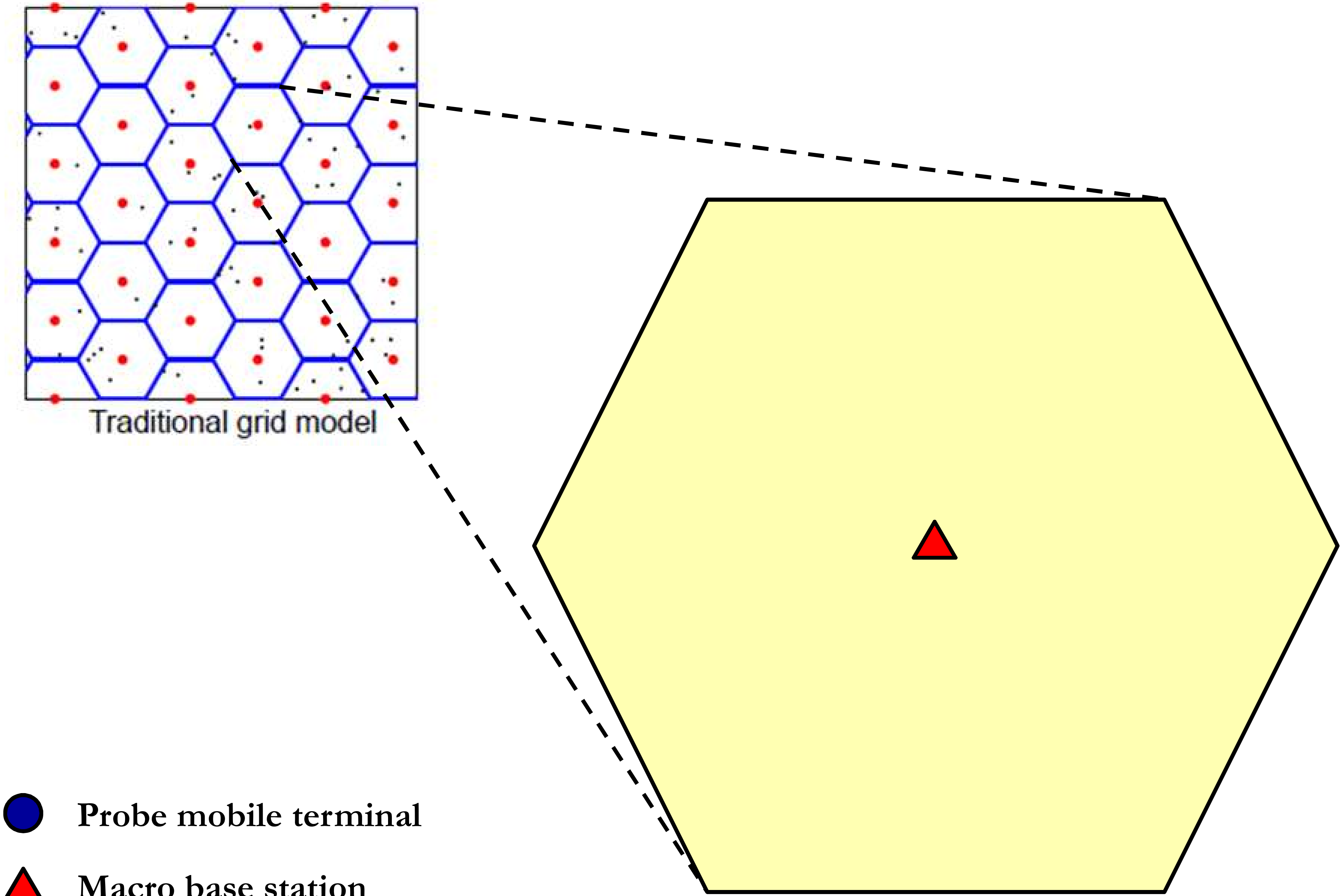
The Conventional Grid-Based Approach

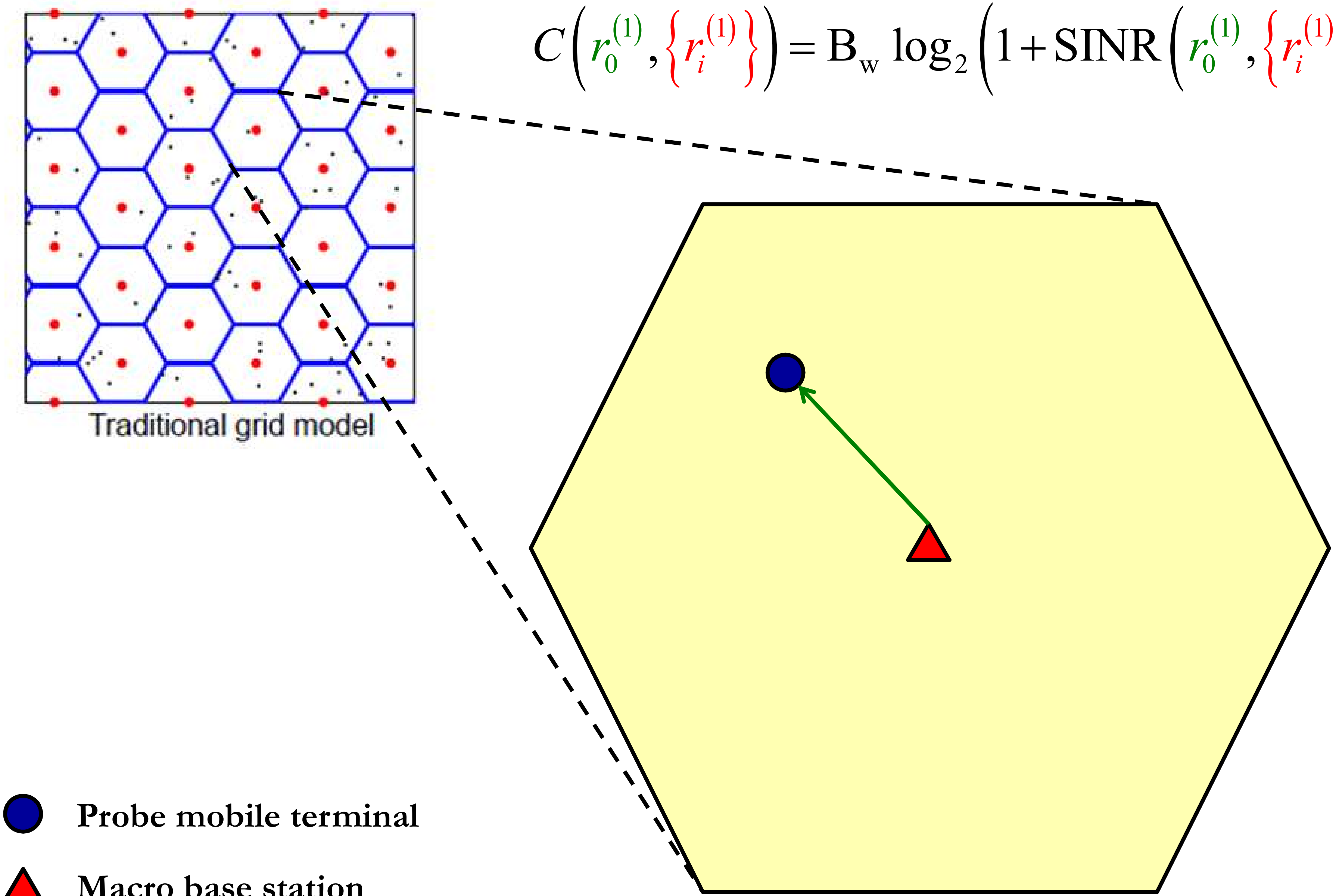


● **Probe mobile terminal**

▲ **Macro base station**

The Conventional Grid-Based Approach

$$C\left(r_0^{(1)},\left\{r_i^{(1)}\right\}\right)=\mathrm{B_w}\log_2\left(1+\mathrm{SINR}\left(r_0^{(1)},\left\{r_i^{(1)}\right\}\right)\right)$$



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The Conventional Grid-Based Approach

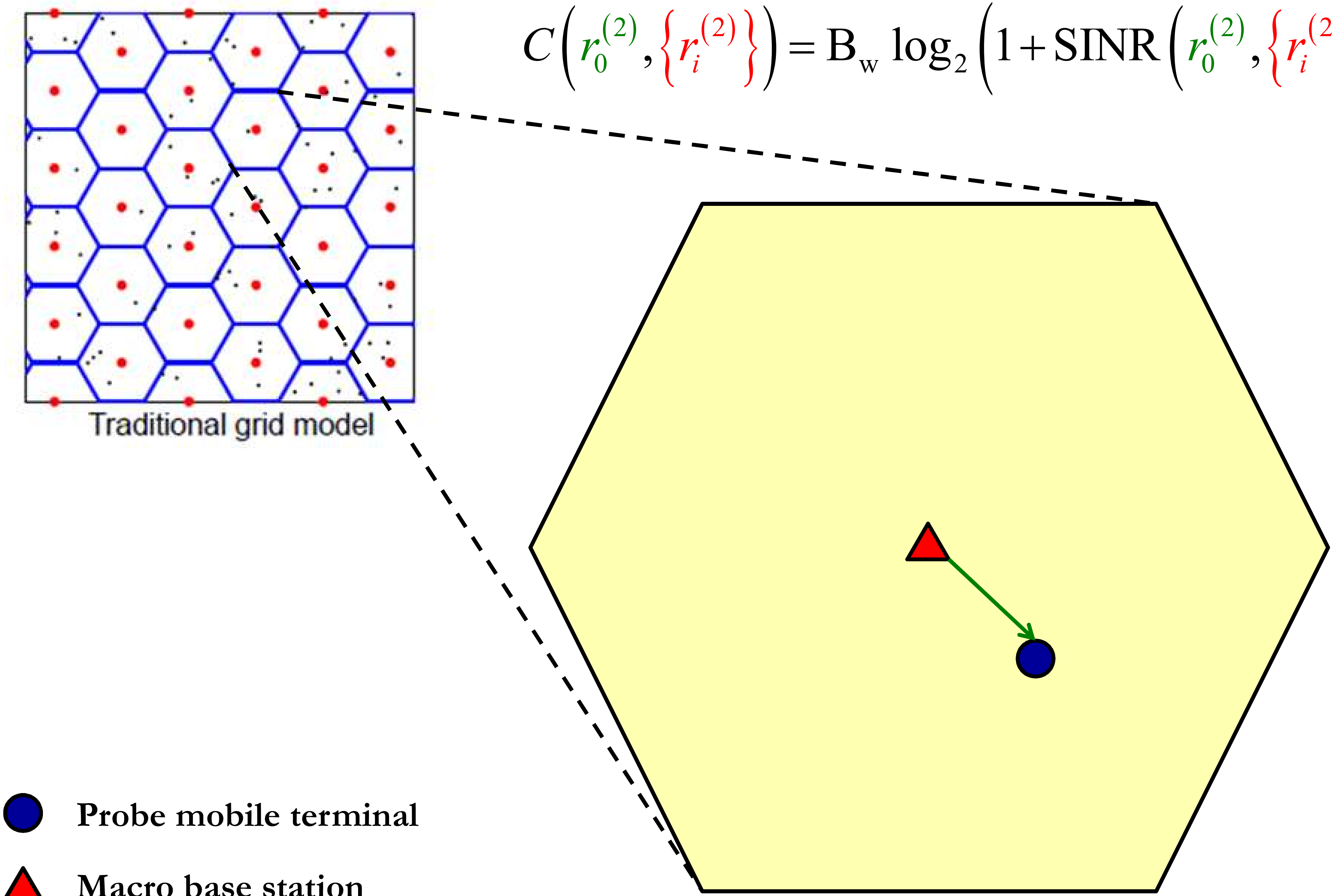
… Signal-to-Interference-Plus-Noise Ratio (SINR) …

$$\mathrm{SINR} = \frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)} \qquad I_{agg}\left(r_0\right) = \sum_{i \in \Phi \backslash BS_0} P\left|h_i\right|^2 r_i^{-\alpha}$$

$$\begin{aligned} \mathrm{CCDF}\left(T\right) &= \mathrm{P}_{\mathrm{cov}}\left(T\right) = \Pr\left\{\mathrm{SINR} > T\right\} \\ &= \Pr\left\{\frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)} > T\right\} = \ldots \end{aligned}$$

The Conventional Grid-Based Approach

$$C\left(r_0^{(2)},\left\{r_i^{(2)}\right\}\right)=\mathrm{B_w}\log_2\left(1+\mathrm{SINR}\left(r_0^{(2)},\left\{r_i^{(2)}\right\}\right)\right)$$



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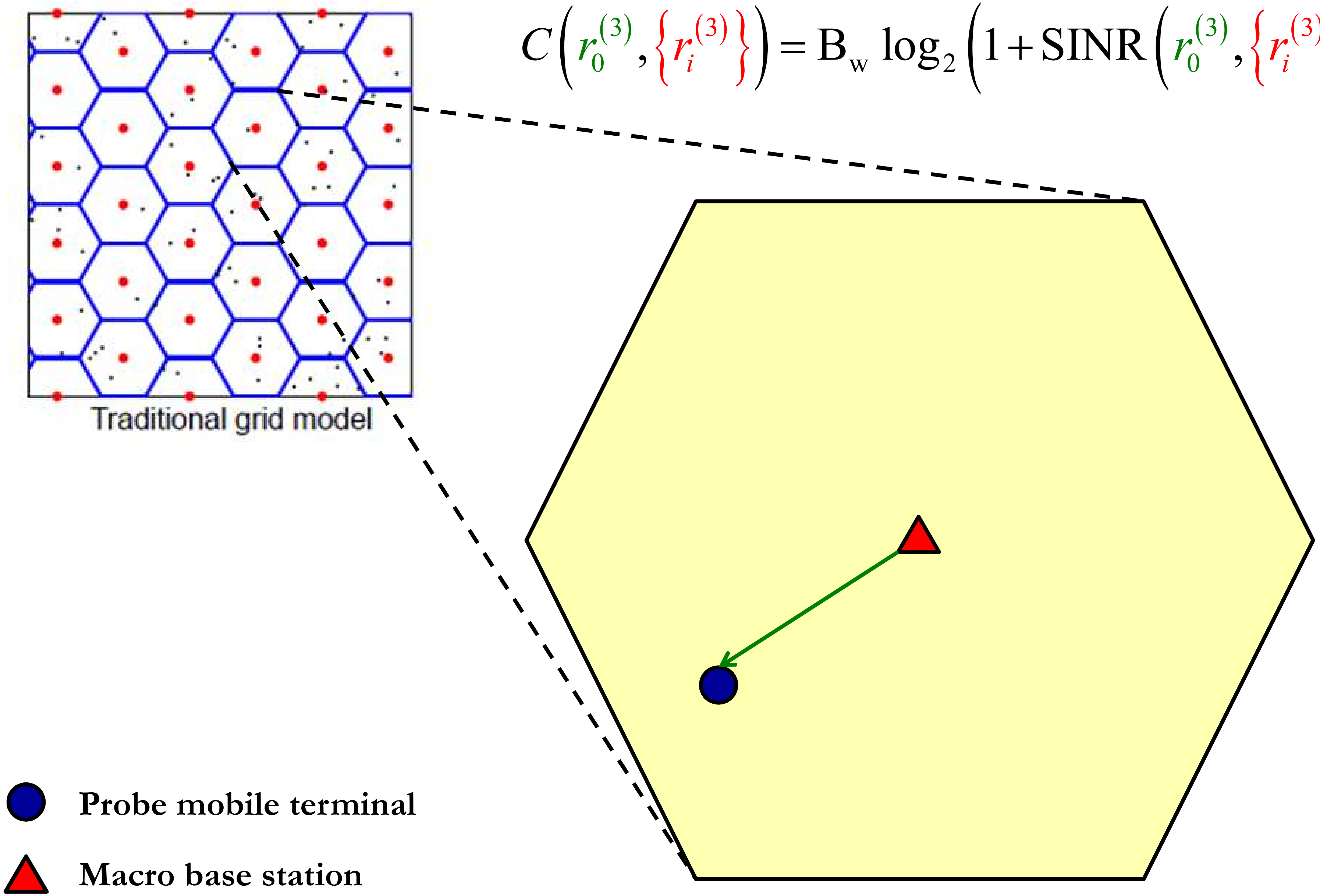

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The Conventional Grid-Based Approach

$$\bar{C} = \mathrm{E}_{r_0,\{r_i\}}\left\{C\left(r_0,\{r_i\}\right)\right\} \approx \frac{1}{N}\sum_{n=1}^{N} C\left(r_0^{(n)},\left\{r_i^{(n)}\right\}\right)$$

$$= \frac{1}{N}\sum_{n=1}^{N} \mathrm{B_w} \log_2\left(1+\mathrm{SINR}\left(r_0^{(n)},\left\{r_i^{(n)}\right\}\right)\right)$$

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Simple enough… So, where is the issue?

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Simple enough… So, where is the issue?

The answer:

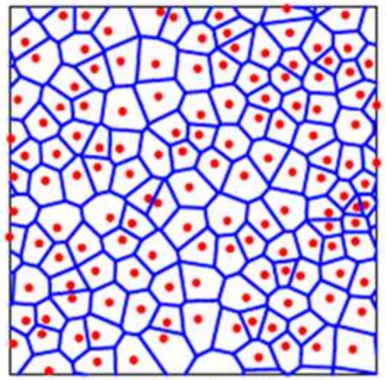
…this spatial expectation

cannot **be computed mathematically…**

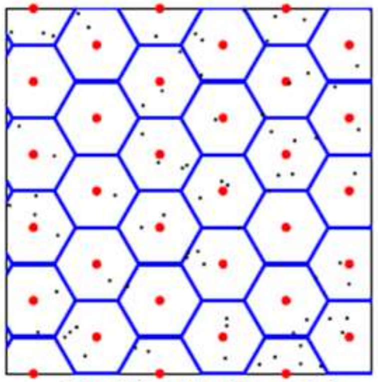
The Conventional Grid-Based Approach: (Some) Issues

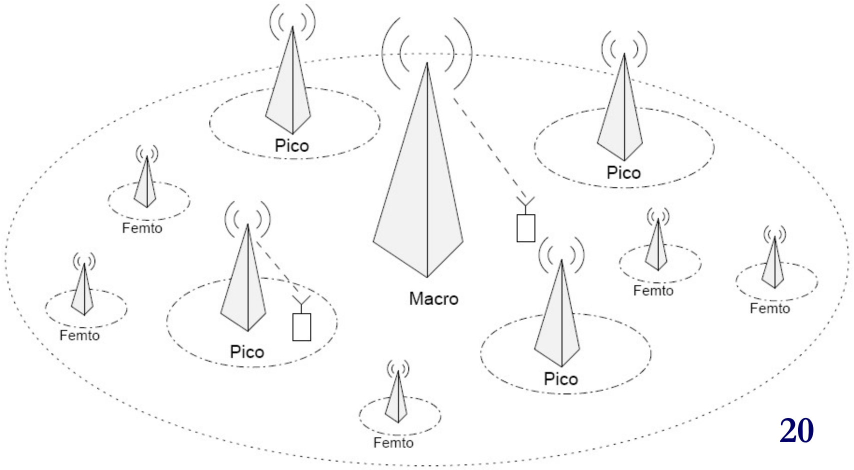
- **Advantages:**
  - Dozens of system parameters can be modeled and tuned in such simulations, and the results have been sufficiently accurate as to enable the evaluation of new proposed techniques and guide field deployments

- **Limitations:**
  - Actual coverage regions deviate from a regular grid
  - Mathematical modeling and optimization are not possible. Any elegant and insightful Shannon formulas for cellular networks?
  - The abstraction model is not scalable for application to ultra-dense HetNets (different densities, transmit powers, access technologies, etc…)

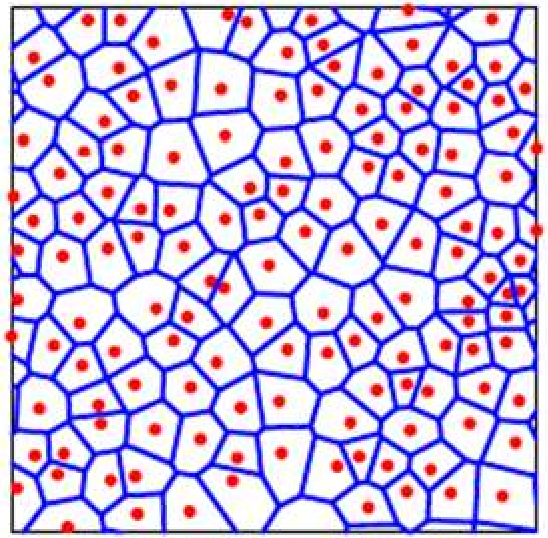

Actual 4G network today

VS.


Traditional grid model

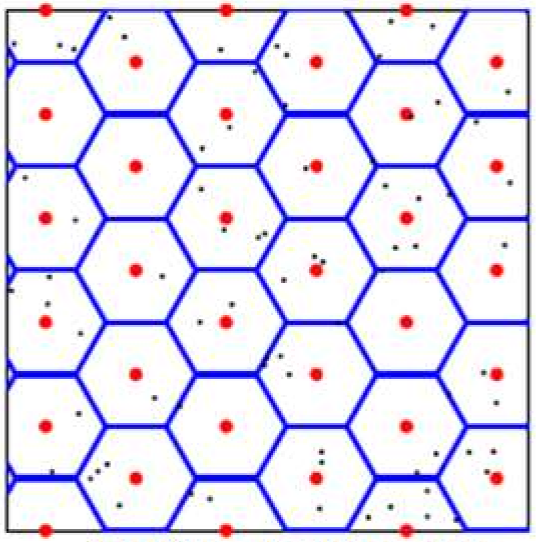


Let's Change the Abstraction Model, Then…



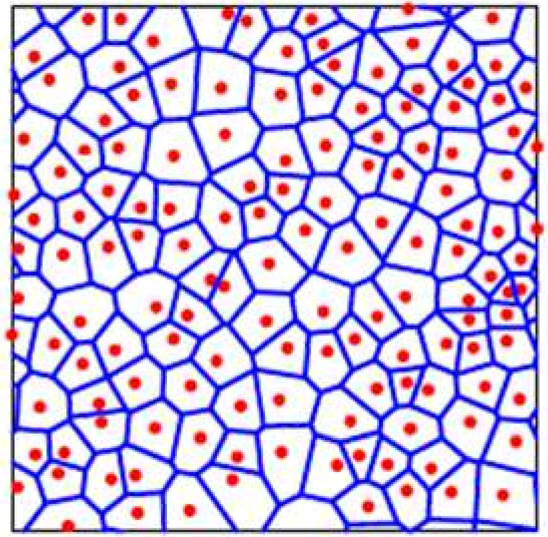
Actual 4G network today

Regular deployment



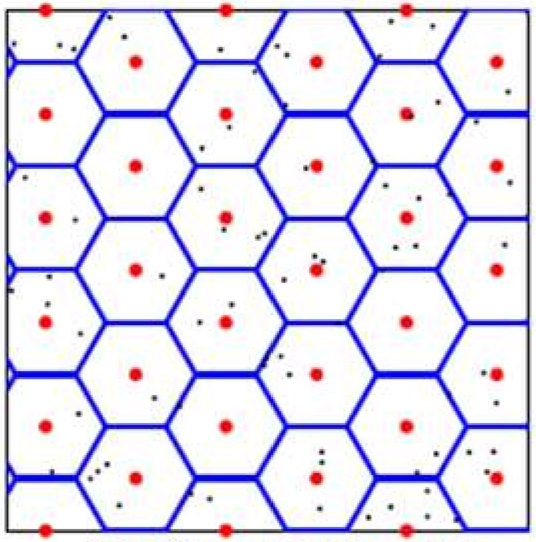
Traditional grid model

Let's Change the Abstraction Model, Then…



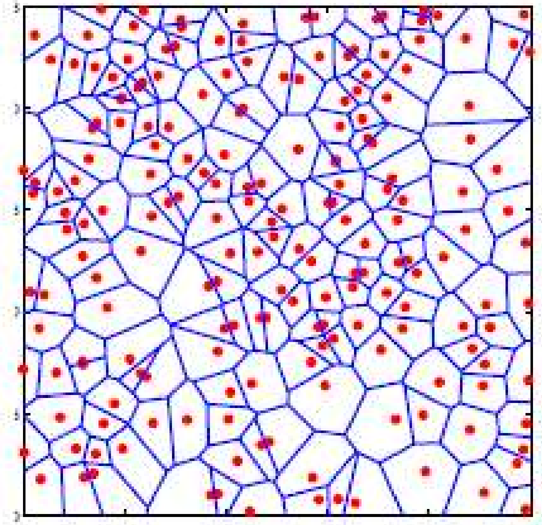
Actual 4G network today

Regular deployment



Traditional grid model

Random deployment (PPP)



Completely random BSs

Stochastic Geometry Based Abstraction Model

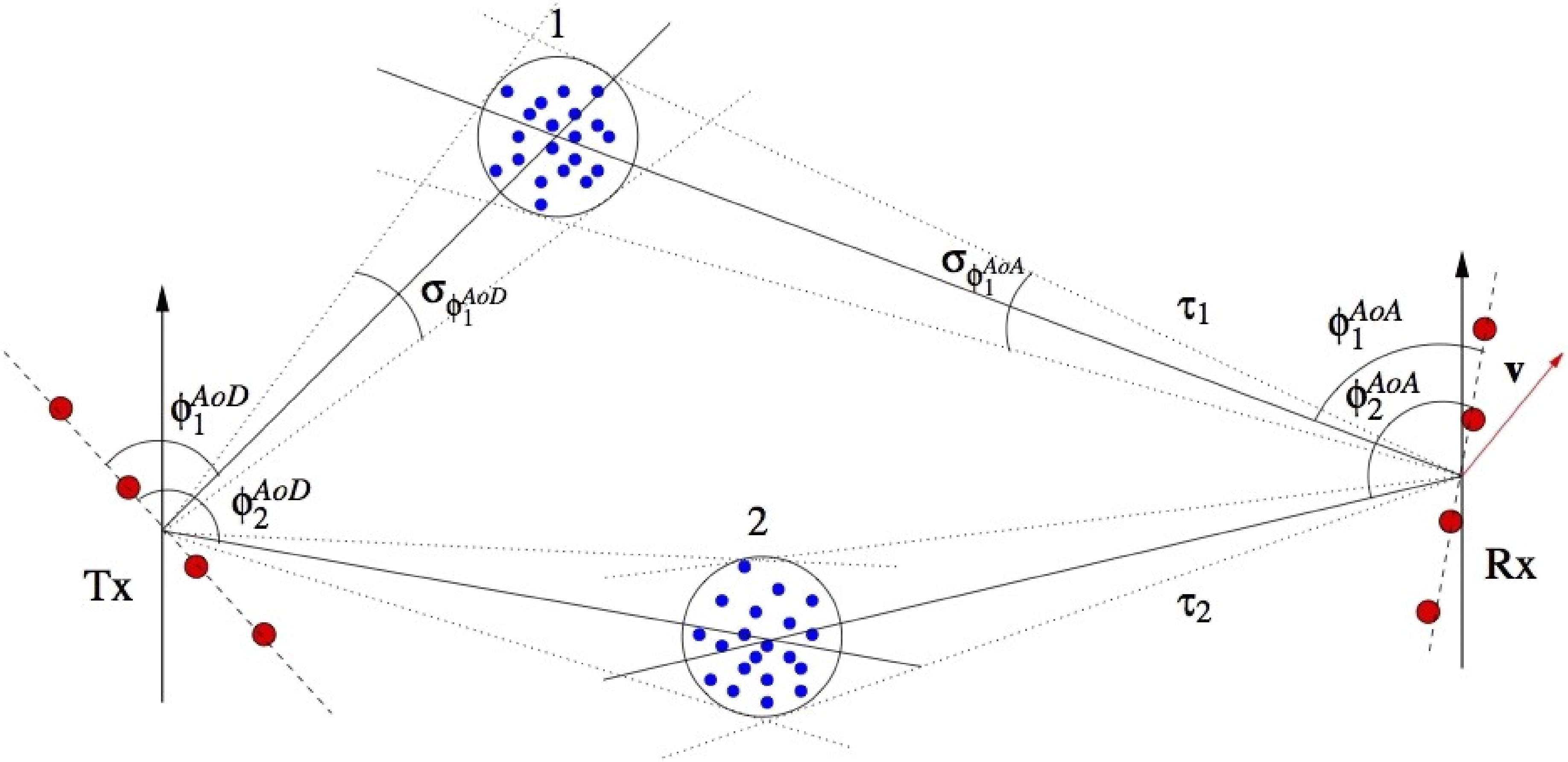
An Emerging (Tractable) Approach

- ❑ **A RANDOM SPATIAL MODEL for Heterogeneous Cellular Networks (HetNets):**
  - ➢ K-tier network with BS locations modeled as independent marked Poisson Point Processes (PPPs)
  - ➢ The PPP model is surprisingly good for 1-tier as well (macro BSs): lower/upper bound to reality and trends still hold
  - ➢ The PPP model makes even more sense for HetNets due to less regular BSs placements for lower tiers (femto, etc.)

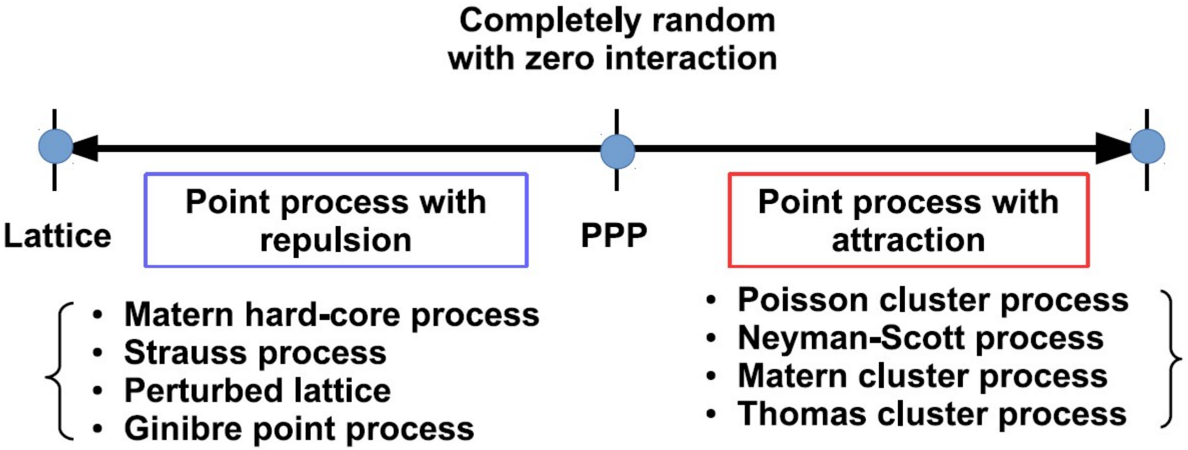
Stochastic Geometry
emerges as a powerful tool for the
analysis, design and optimization
of ultra-dense HetNets

The PPP: Does it Make Sense?

- **Additive White Gaussian Noise. Does it?**
- **Independent and Identically Distributed Rayleigh Fading. Does it?**
- **etc…**



Beyond the PPP: Possible, but Math is More Complicated



Completely random with zero interaction

Lattice

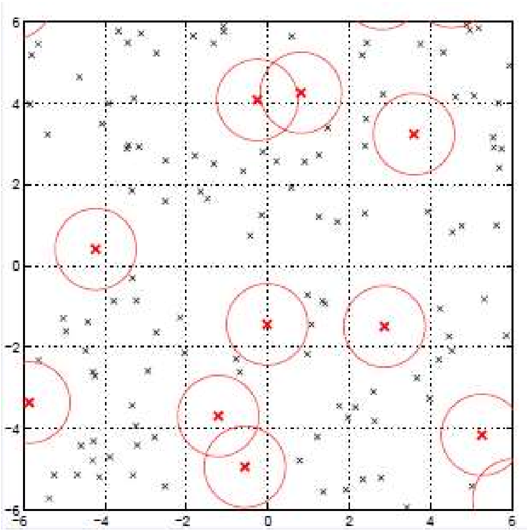
Point process with repulsion

PPP

Point process with attraction

- Matern hard-core process
- Strauss process
- Perturbed lattice
- Ginibre point process

- Poisson cluster process
- Neyman-Scott process
- Matern cluster process
- Thomas cluster process



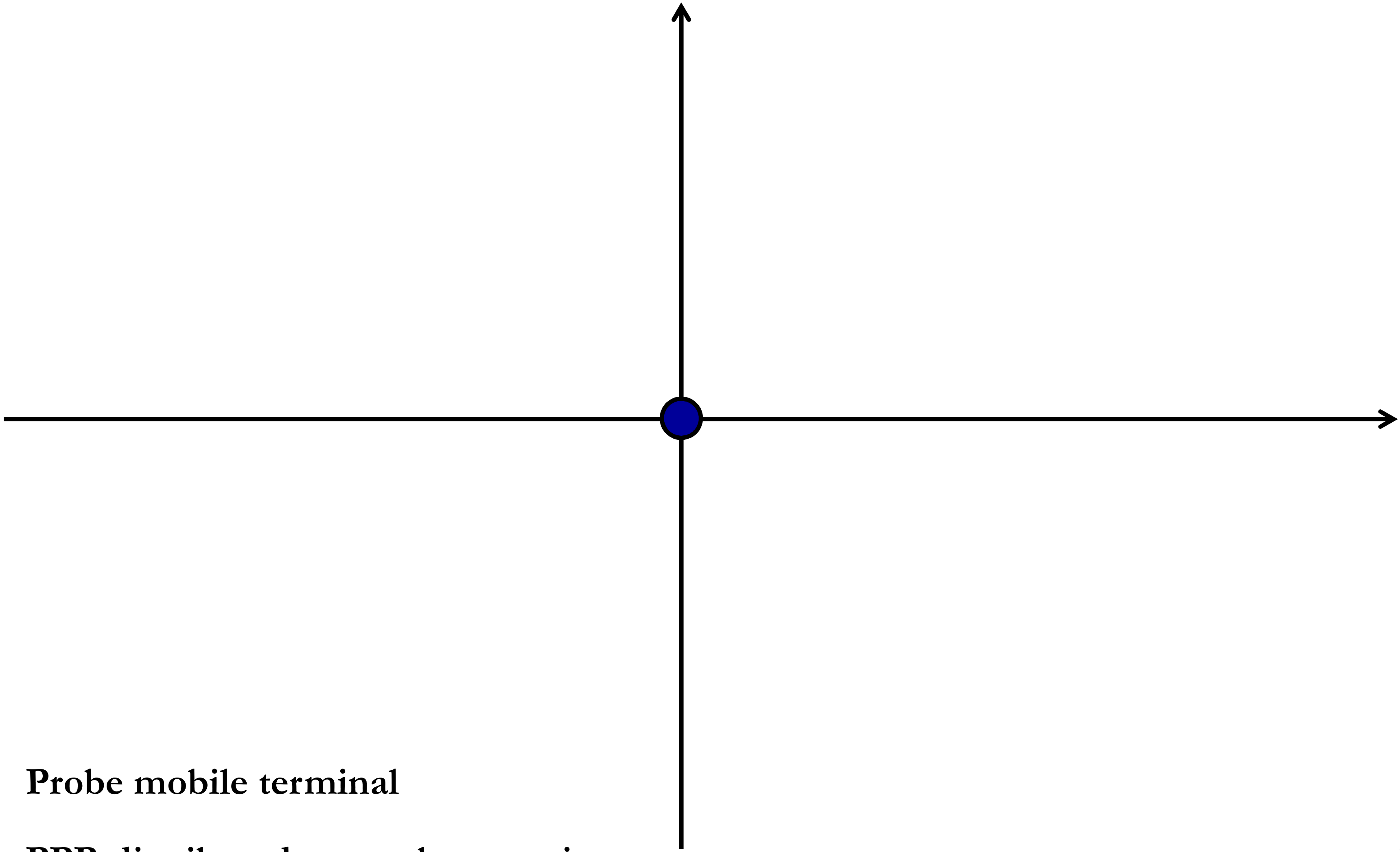
Matern Hard-Core PP

Take a homogeneous PPP and remove any pairs of points that are closer to each other than a predefined minimum distance R

Y. J. Chun, M. O. Hasna, A. Ghrayeb, and M. Di Renzo, "On modeling heterogeneous wireless networks using non-Poisson point processes", IEEE Commun. Mag., submitted. [Online]. Available: http://arxiv.org/pdf/1506.06296.pdf.

PPP-based Abstraction

How It Works (Downlink – 1-tier)

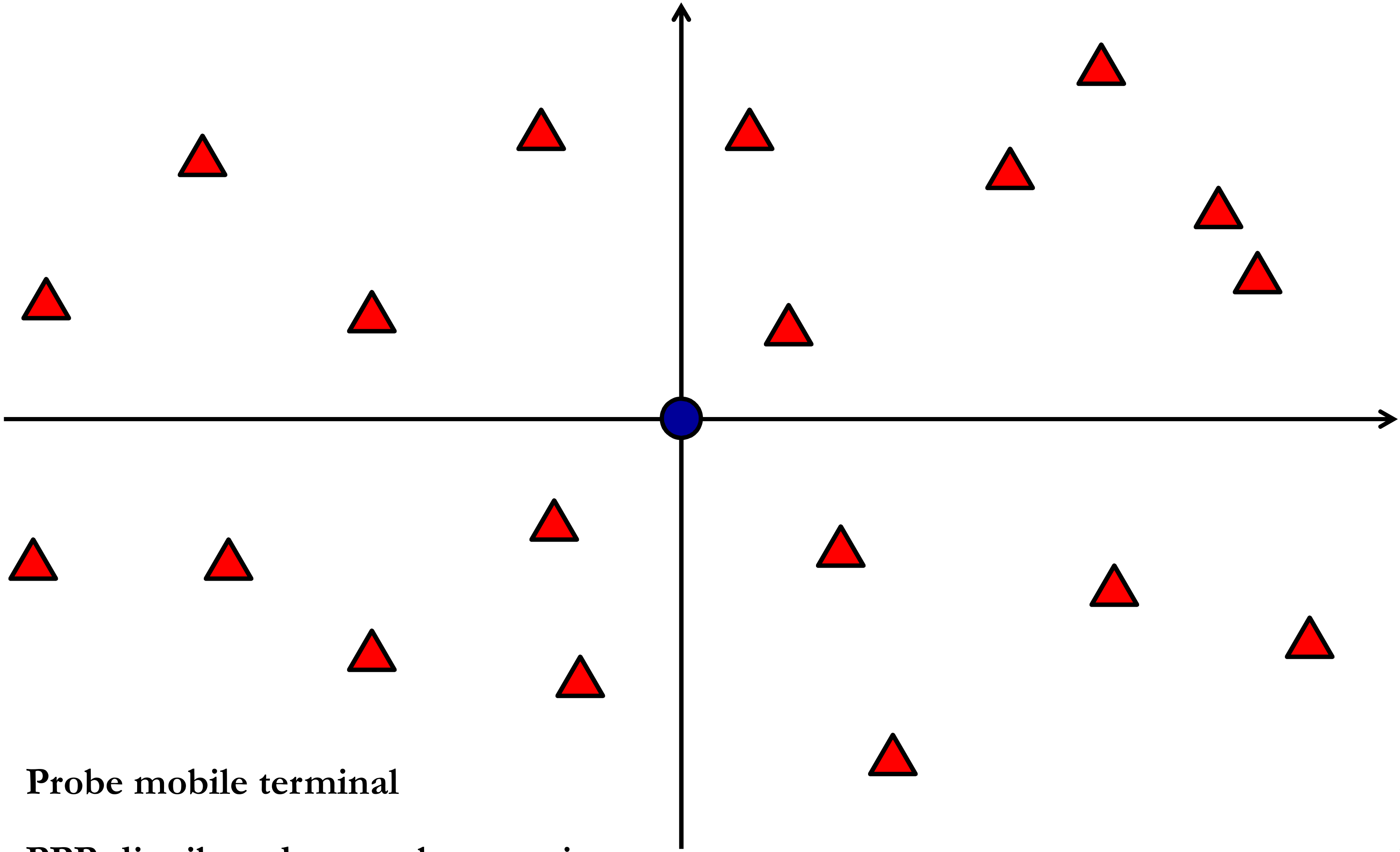


● **Probe mobile terminal**

▲ **PPP-distributed macro base station**

PPP-based Abstraction

How It Works (Downlink – 1-tier)



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PPP-based Abstraction

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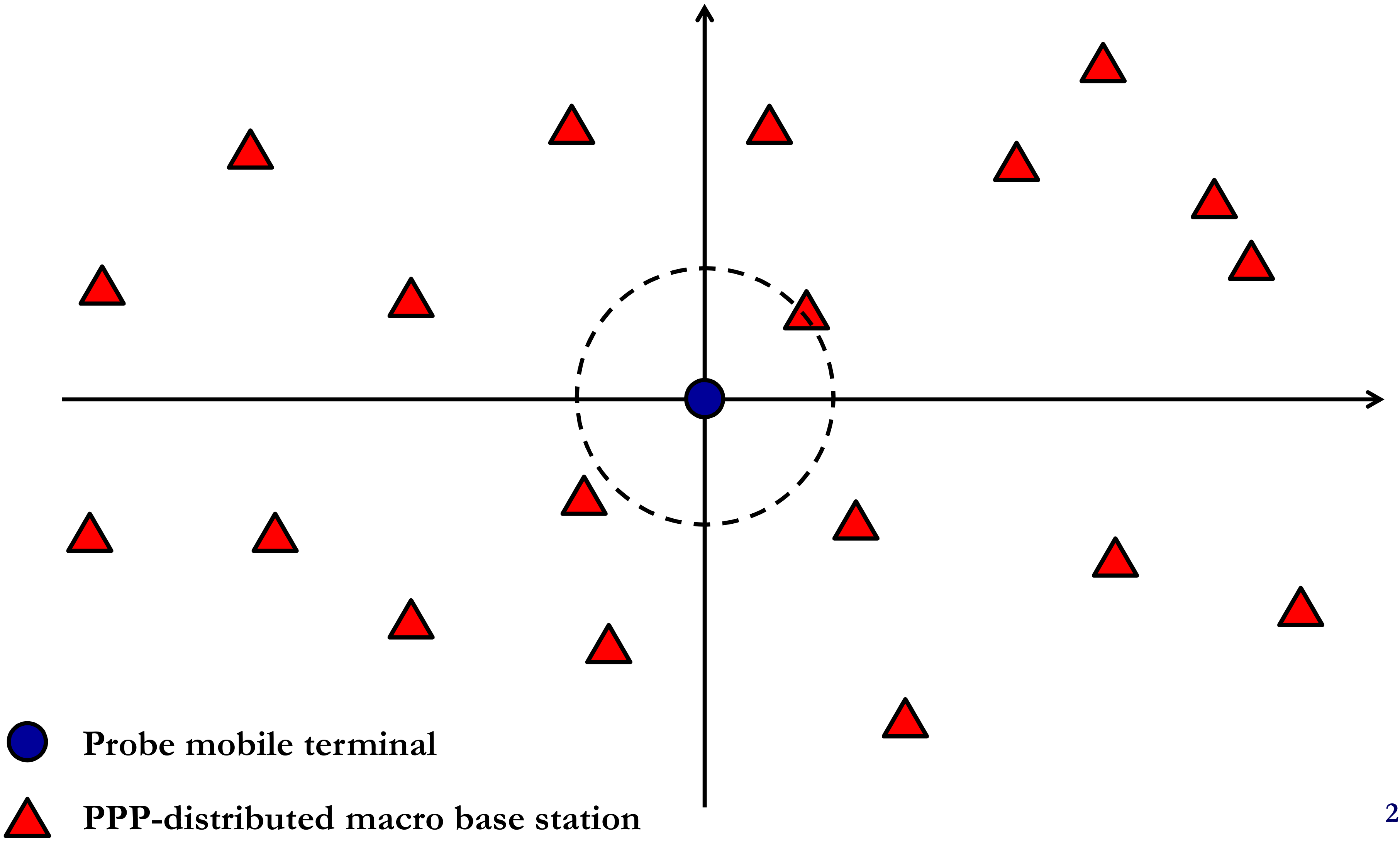

PPP-based Abstraction

$$C\left(r_0^{(1)},\left\{r_i^{(1)}\right\}\right)=\mathrm{B_w}\log_2\left(1+\mathrm{SINR}\left(r_0^{(1)},\left\{r_i^{(1)}\right\}\right)\right)$$

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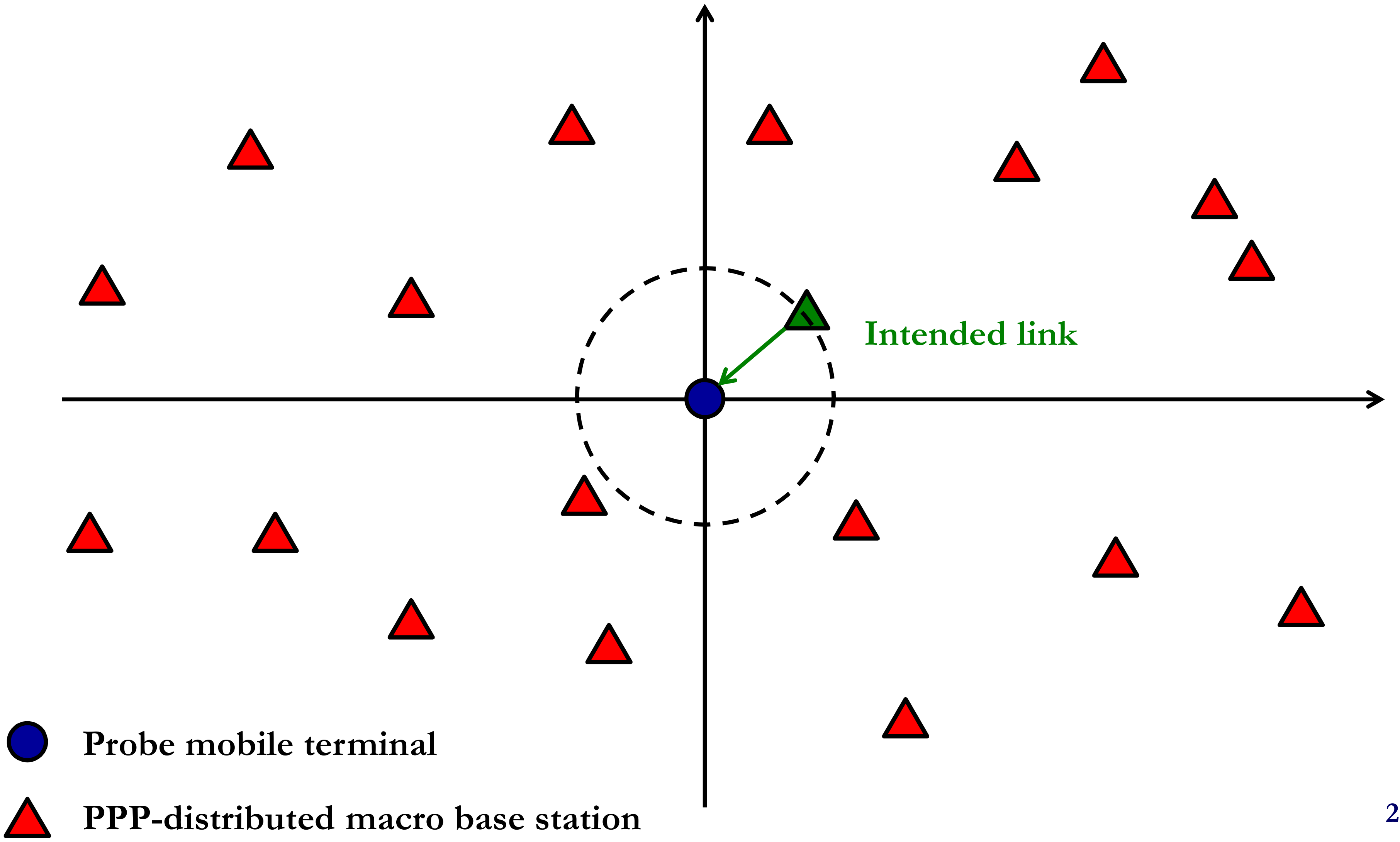

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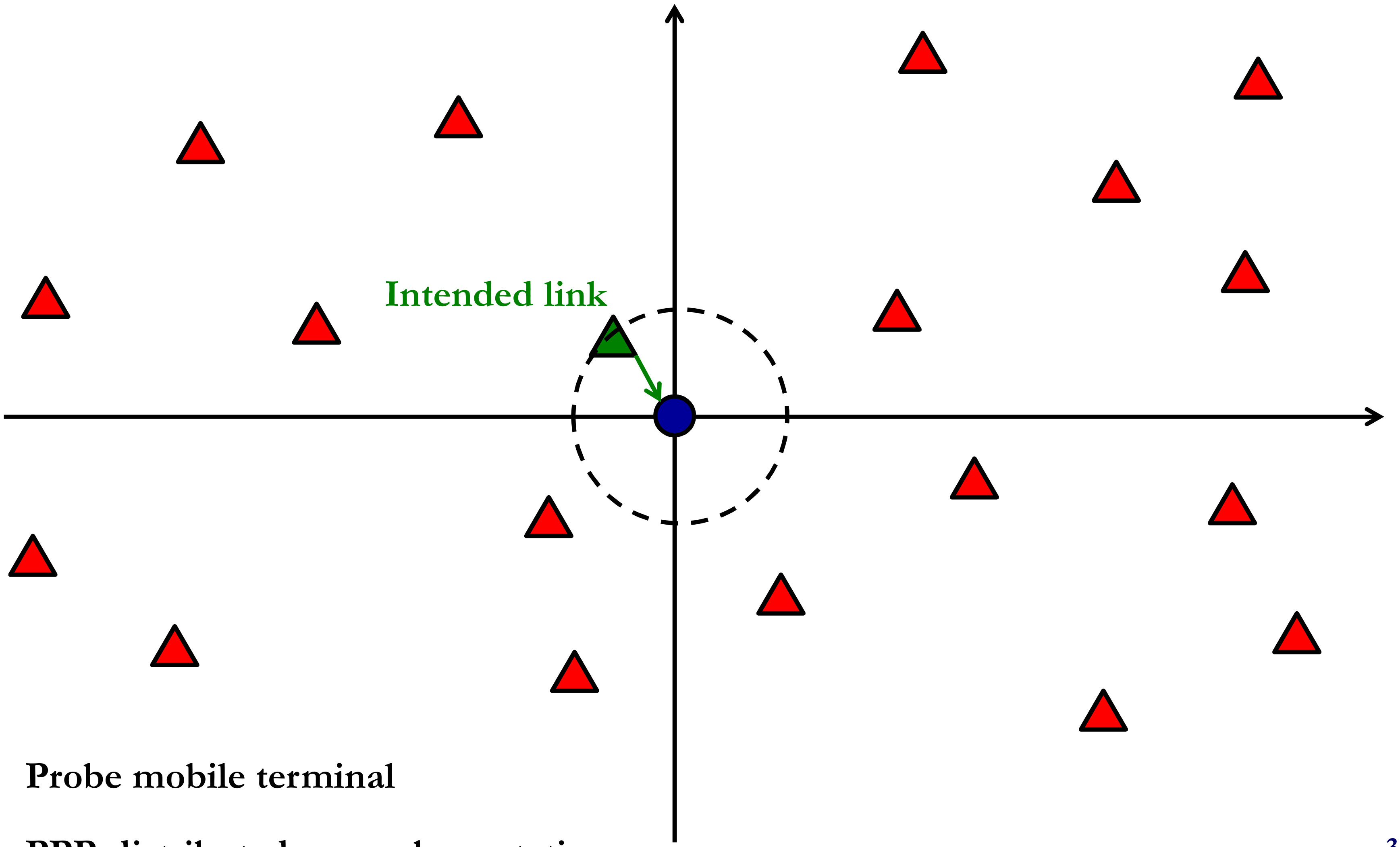
How It Works (Downlink – 1-tier)



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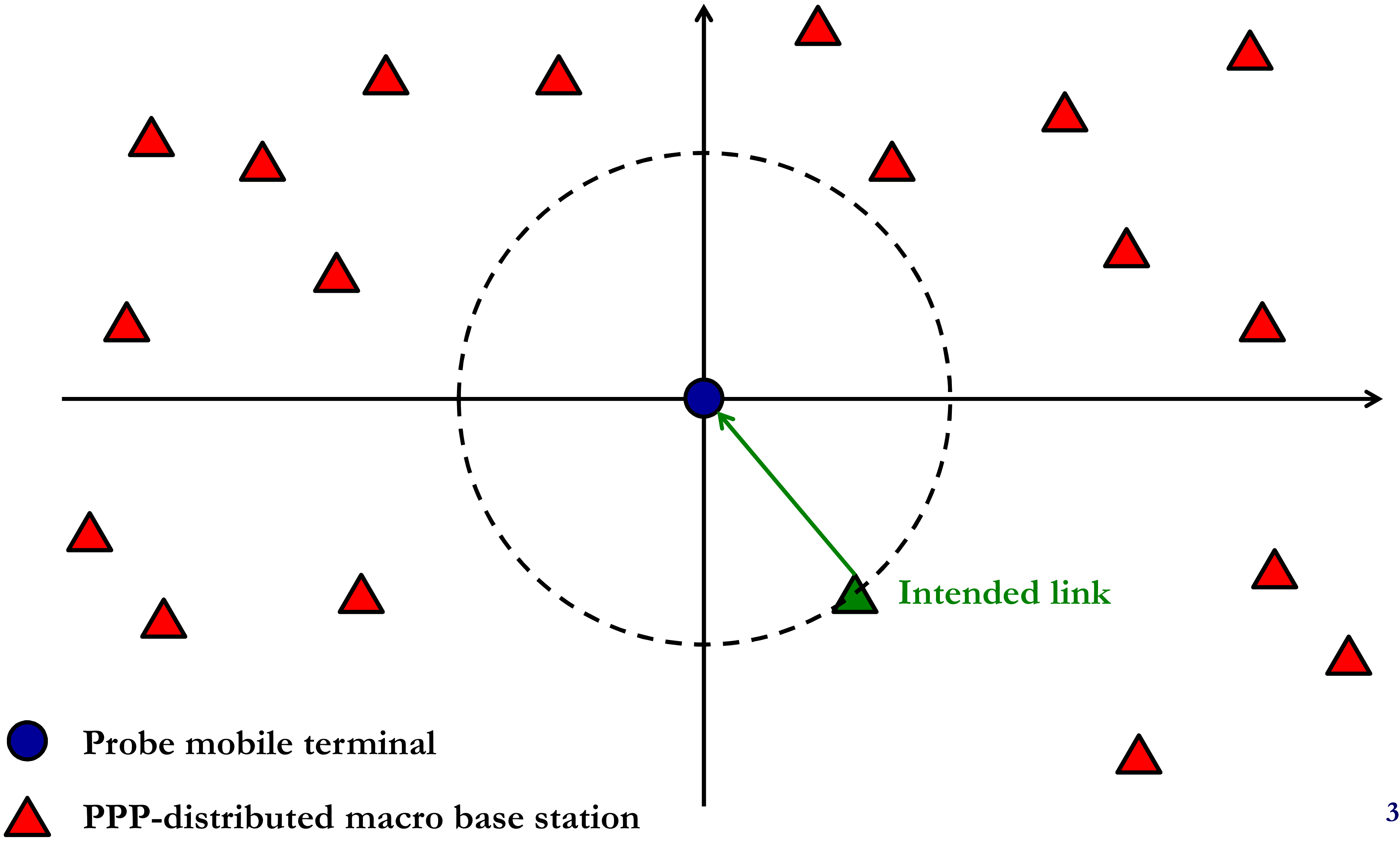

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Are you kidding me? … What makes it different?

PPP-based Abstraction

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Are you kidding me? … What makes it different?

The answer:

…this spatial expectation

can **be computed mathematically…**

… On Abstraction Modeling …

George Edward Pelham Box
(18 October 1919 – 28 March 2013)
Statistician
Fellow of the Royal Society (UK)
Director of the Statistical Research Group
(Princeton University)
Emeritus Professor
(University of Wisconsin-Madison)

"…all models are wrong, but some are useful…"

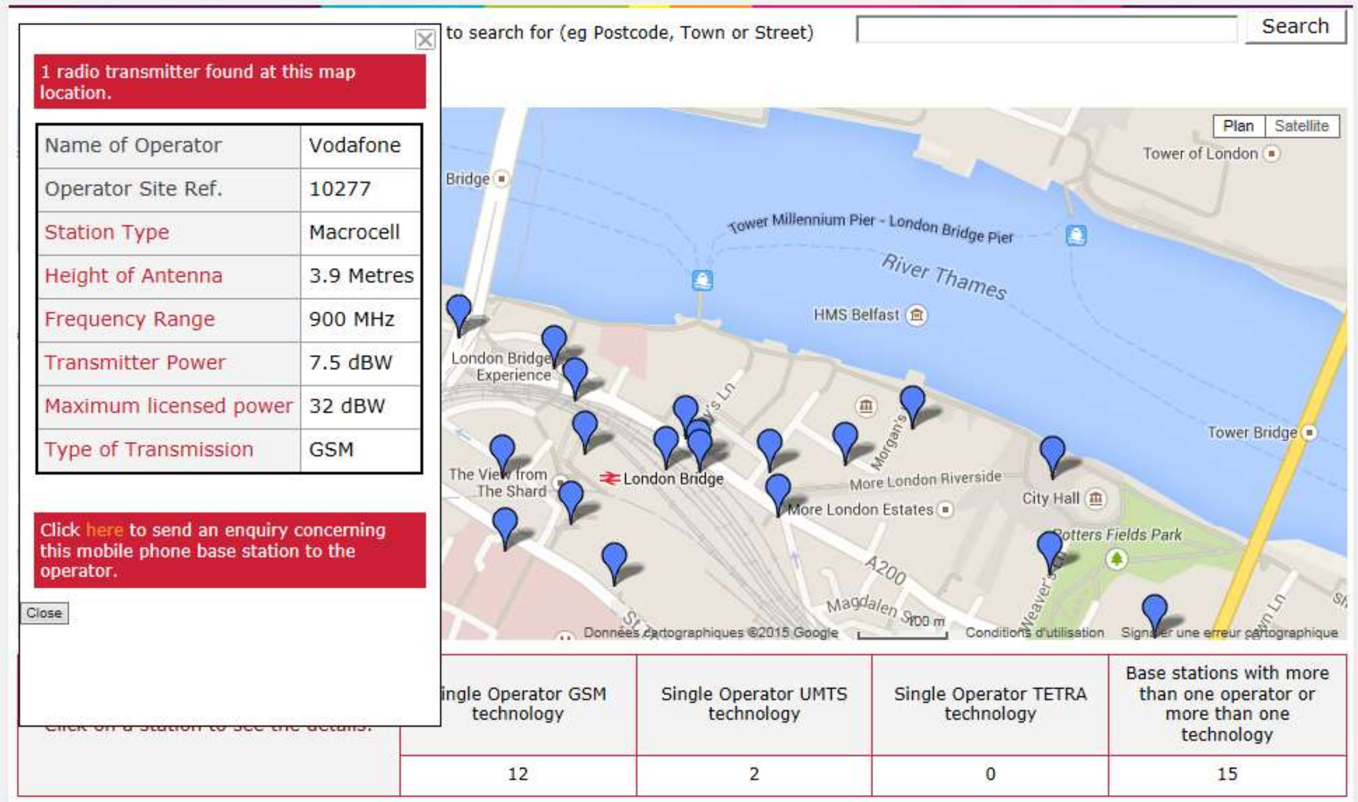
Is This Abstraction Model Accurate?

- **Methodology:**

Is This Abstraction Model Accurate?

- **Methodology:**
  - Actual base station locations from OFCOM (UK)

**OFCOM:
London
“London
Bridge area”**



to search for (eg Postcode, Town or Street)

1 radio transmitter found at this map location.

Name of Operator	Vodafone
Operator Site Ref.	10277
Station Type	Macrocell
Height of Antenna	3.9 Metres
Frequency Range	900 MHz
Transmitter Power	7.5 dBW
Maximum licensed power	32 dBW
Type of Transmission	GSM

Click here to send an enquiry concerning this mobile phone base station to the operator.

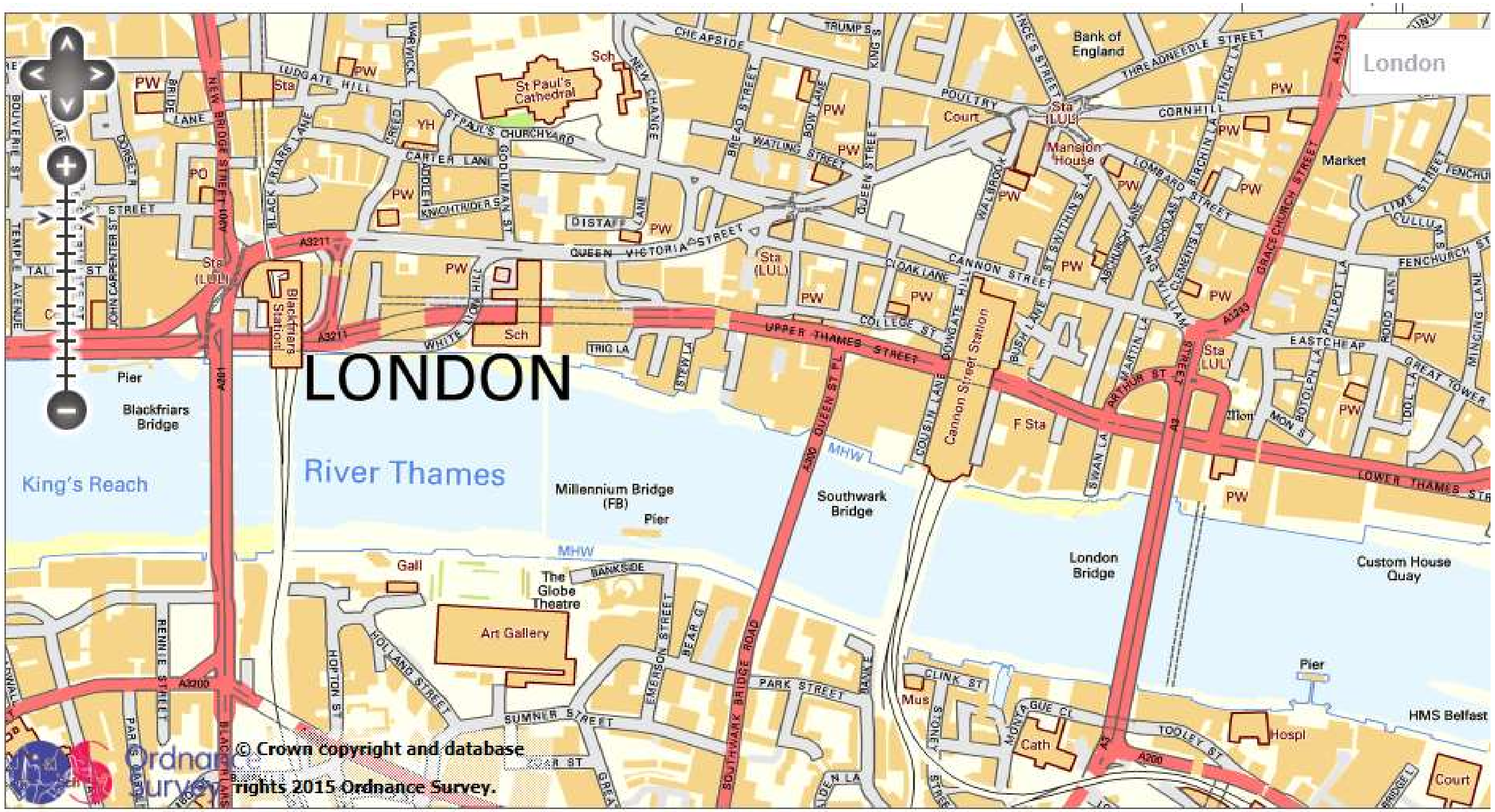
ingle Operator GSM technology	Single Operator UMTS technology	Single Operator TETRA technology	Base stations with more than one operator or more than one technology
12	2	0	15

OFCOM: http://stakeholders.ofcom.org.uk/sitefinder/sitefinder-dataset/

Is This Abstraction Model Accurate?

- **Methodology:**
  - Actual base station locations from **OFCOM** (UK)
  - Actual building footprints from **ORDNANCE SURVEY** (UK)

ORDNANCE SURVEY: London "London Bridge area"

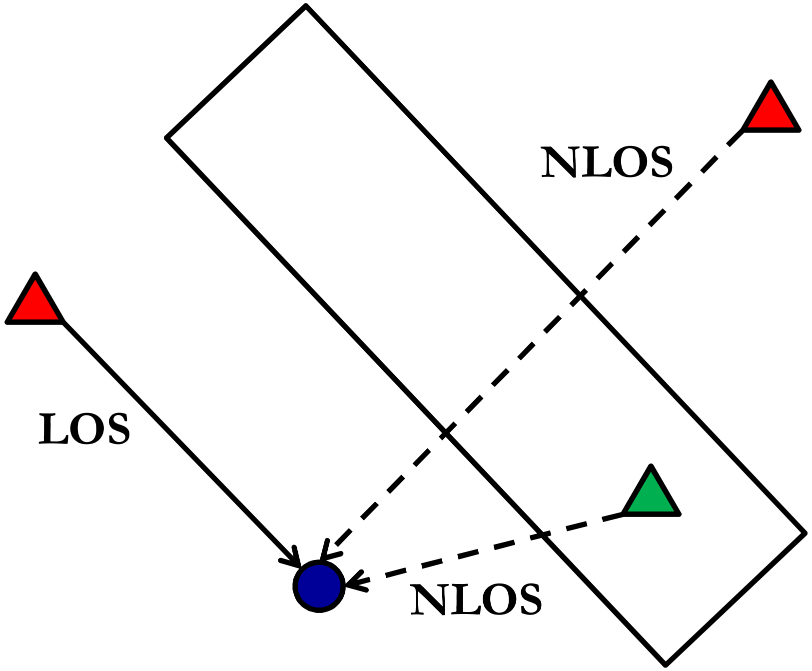


OFCOM: http://stakeholders.ofcom.org.uk/sitefinder/sitefinder-dataset/

ORDNANCE SURVEY: https://www.ordnancesurvey.co.uk/opendatadownload/products.html

Is This Abstraction Model Accurate?

- **Methodology:**
  - Actual base station locations from **OFCOM** (UK)
  - Actual building footprints from **ORDNANCE SURVEY** (UK)
  - Channel model added on top (1-state and 2-state with LOS/NLOS)



● Mobile terminal

▲ Base station (outdoor)

▲ Base station (rooftop)

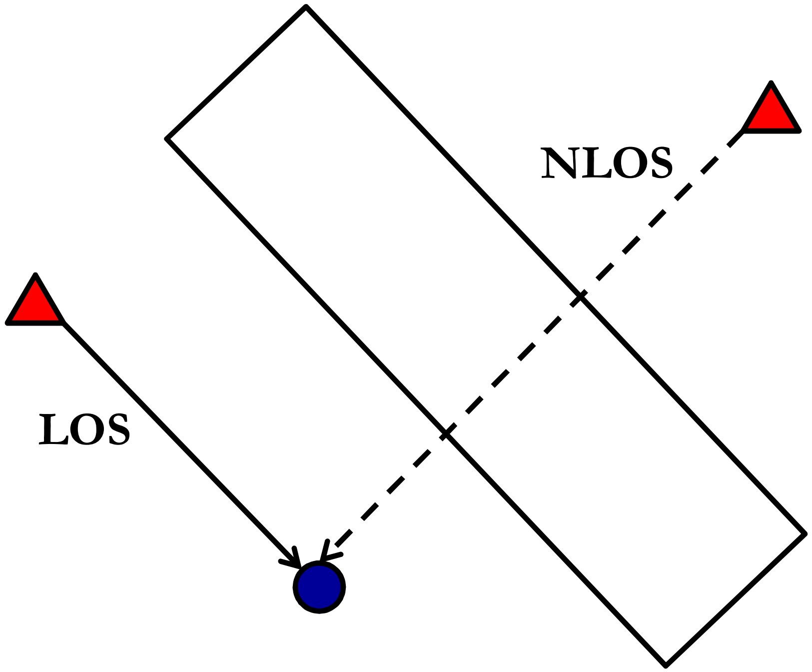
2-state: the location of MTs and BSs and the location/shape of buildings determine LOS/NLOS conditions

1-state: all links are either in LOS or NLOS regardless of the topology

OFCOM: http://stakeholders.ofcom.org.uk/sitefinder/sitefinder-dataset/
ORDNANCE SURVEY: https://www.ordnancesurvey.co.uk/opendatadownload/products.html

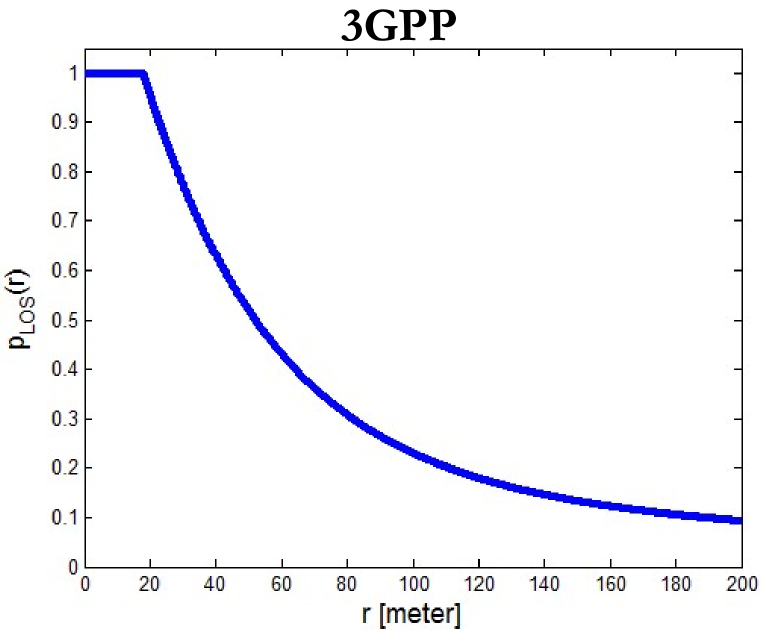
An Example of Blockage Model (3GPP)

... Impact of LOS/NLOS ...



● Mobile terminal

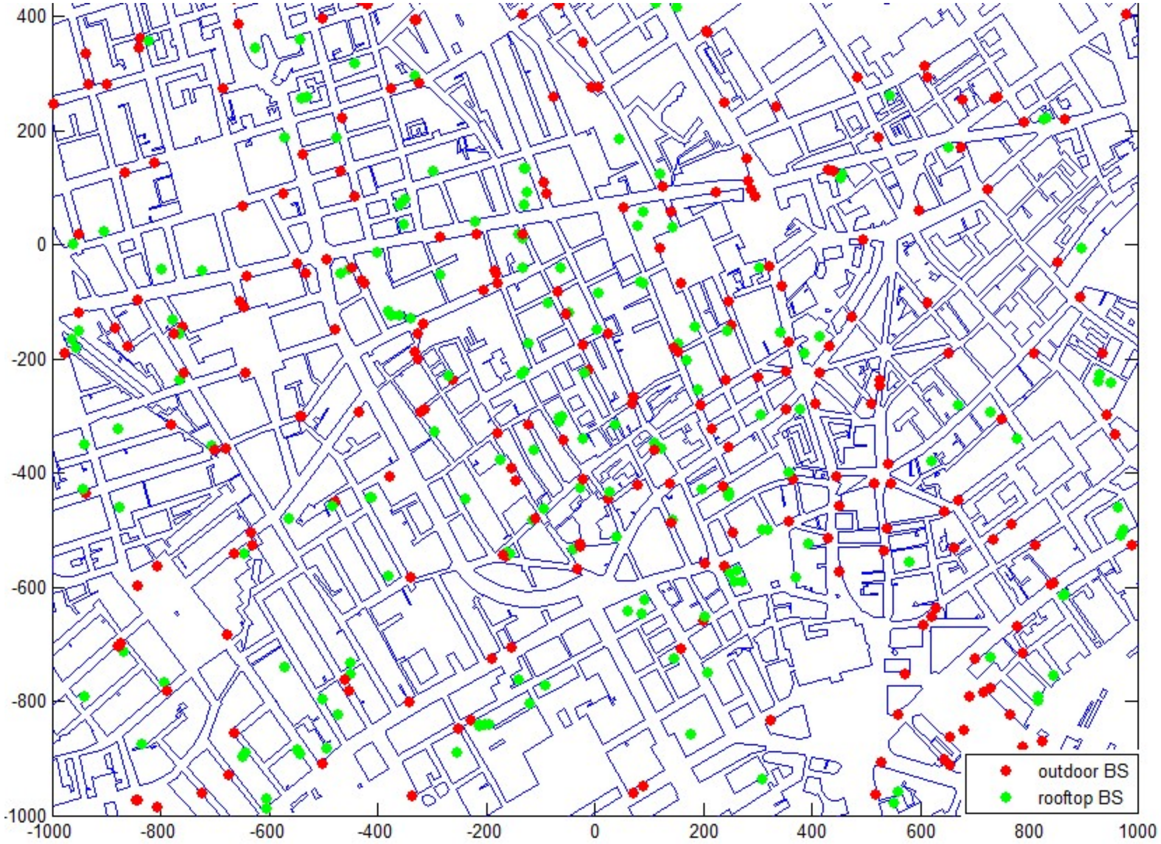
▲ Base station



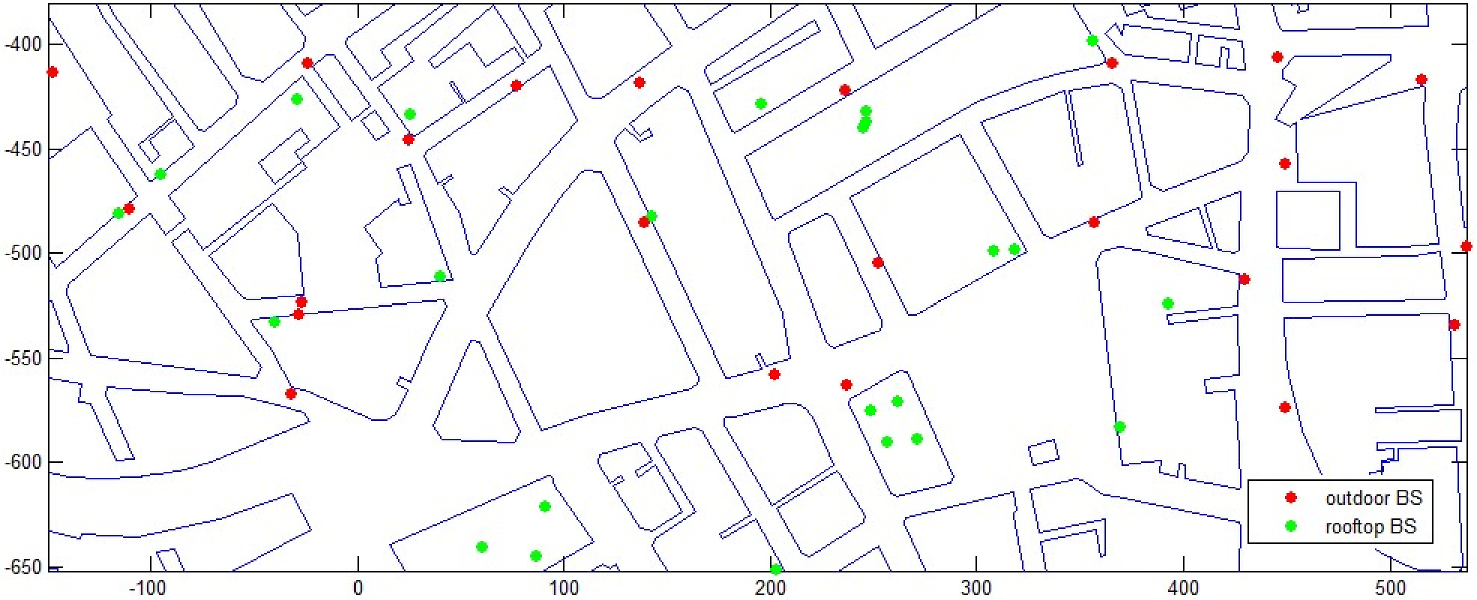
$$p_{\mathrm{LOS}}(r) = \min\left\{\frac{18}{r}, 1\right\}\left(1 - e^{-\frac{r}{36}}\right) + e^{-\frac{r}{36}}$$

The London Case Study (1/7)

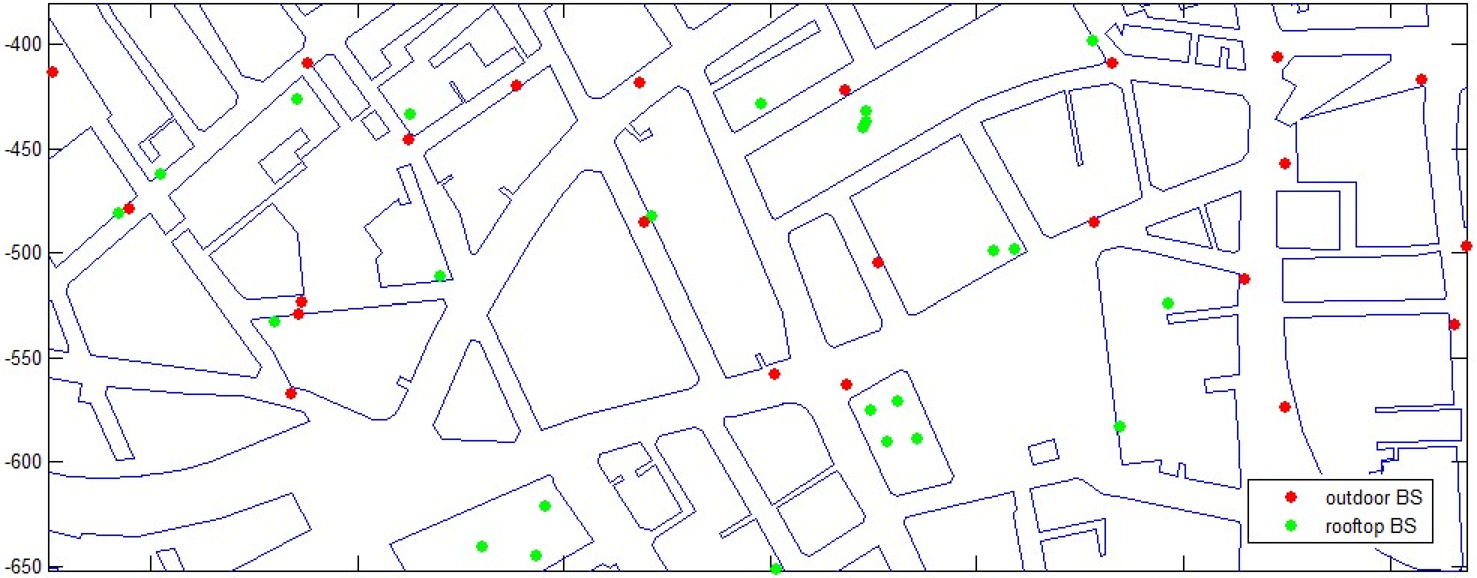
	O2 + Vodafone	O2	Vodafone
Number of BSs	319	183	136
Number of rooftop BSs	95	62	33
Number of outdoor BSs	224	121	103
Average cell radius (m)	63.1771	83.4122	96.7577



The London Case Study (2/7)



The London Case Study (3/7)

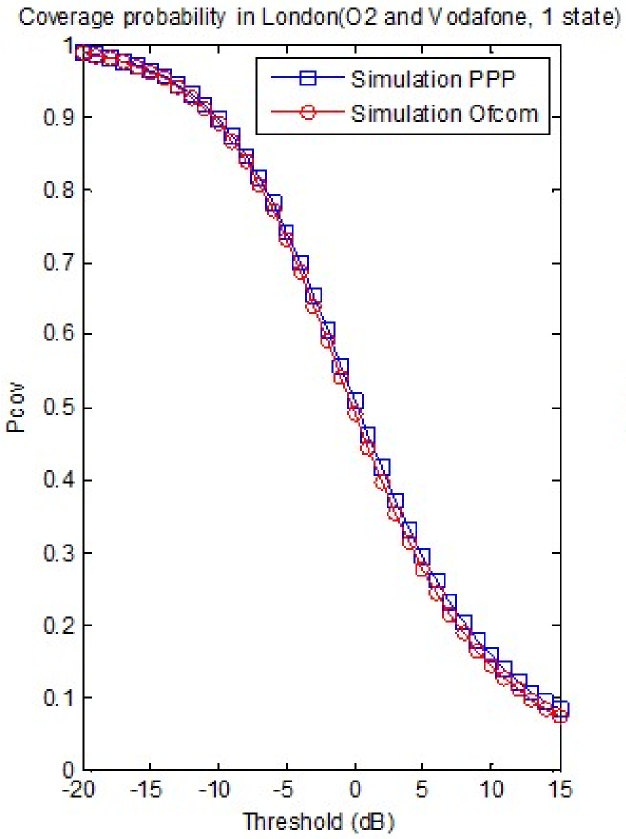


The London Case Study (4/7)

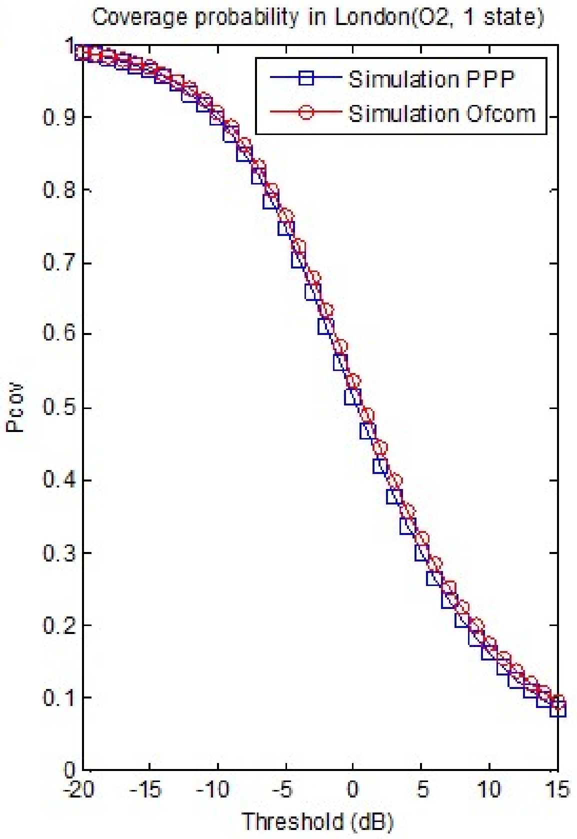
PPP Accuracy: 1-State Channel Model

- **OFCOM: Actual base station locations, (actual building footprints), actual channels**
- **PPP: Random base station locations, (actual building footprints), actual channels**

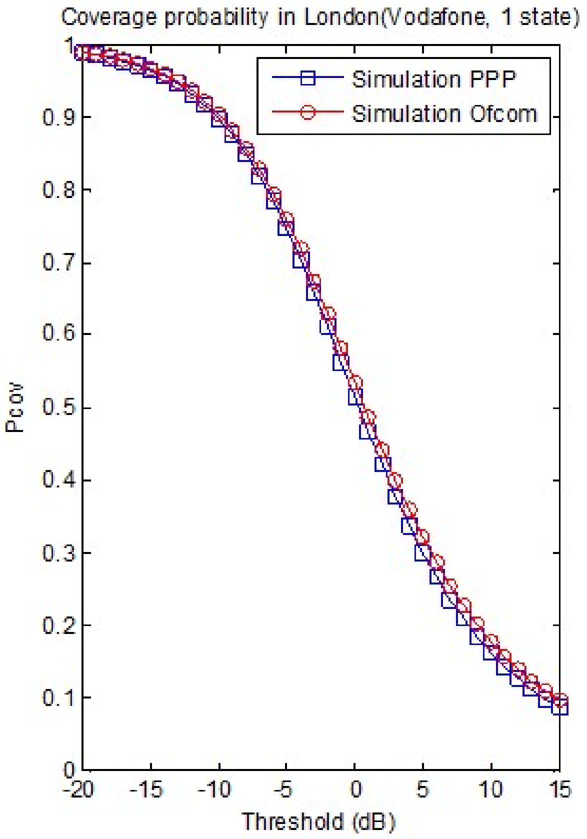
O2+VODAFONE



O2



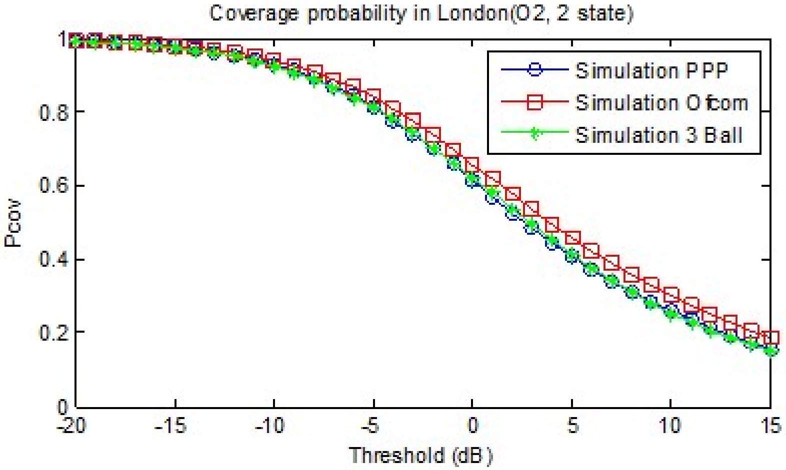
VODAFONE



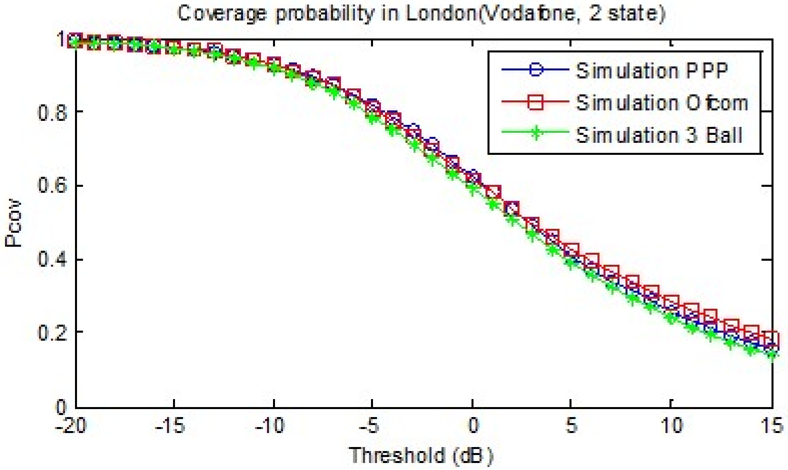
The London Case Study (5/7)

PPP Accuracy: 2-State Channel Model

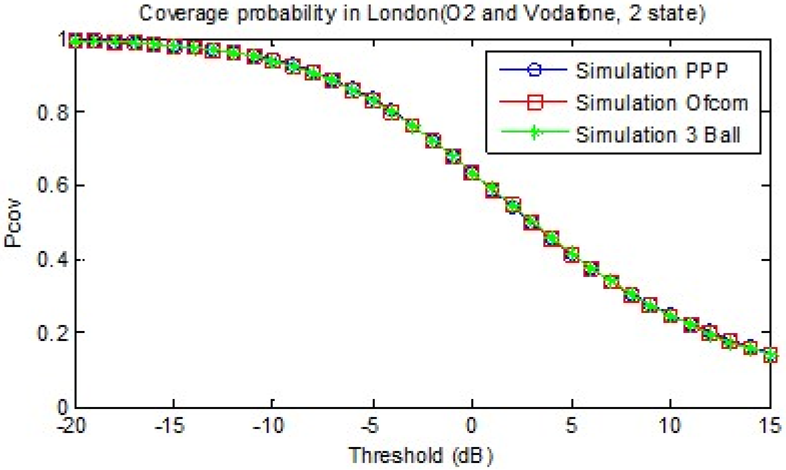
O2



VODAFONE

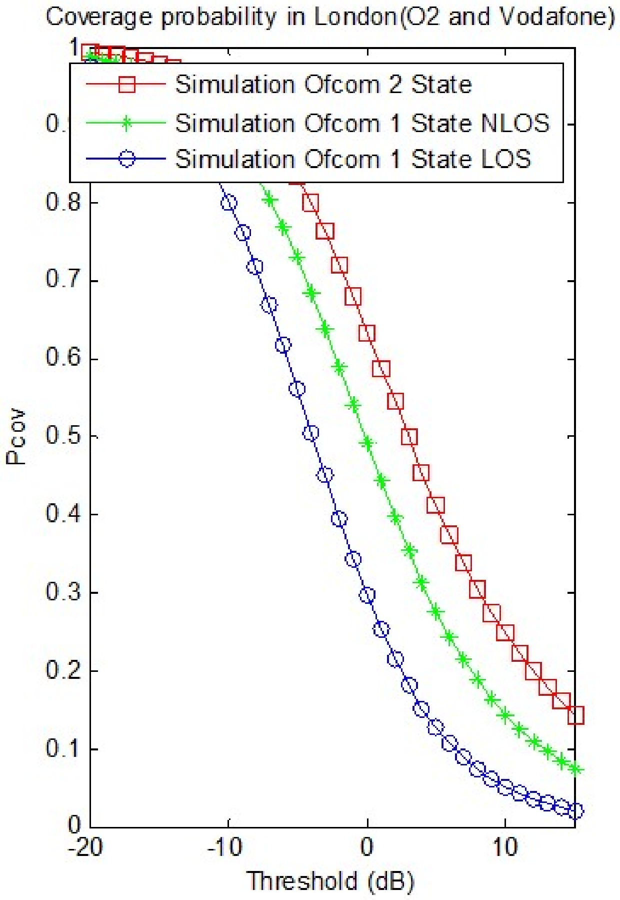


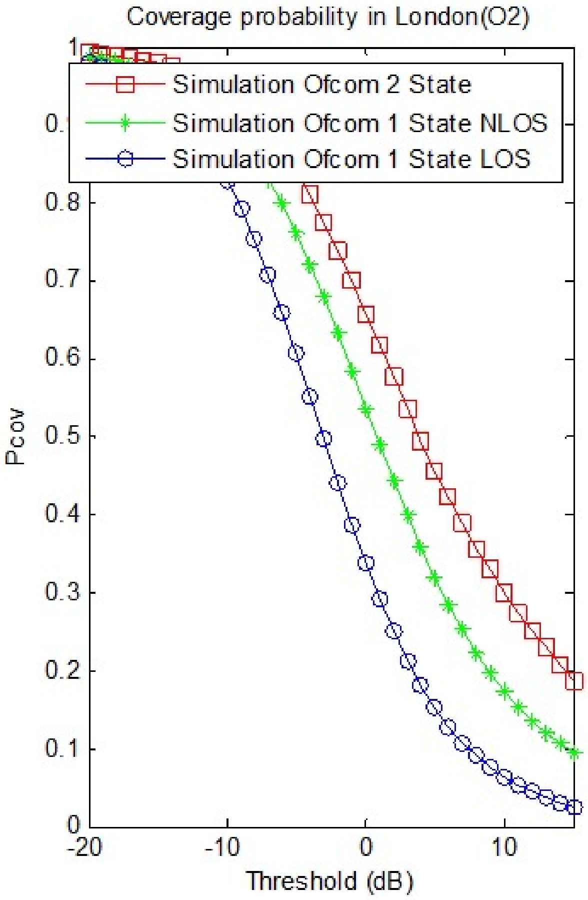
O2+VODAFONE

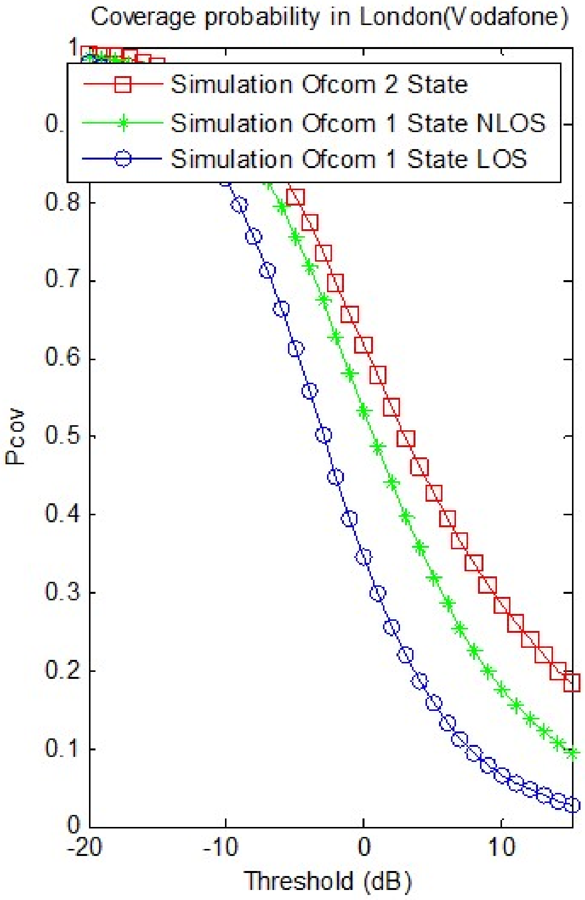


The London Case Study (6/7)

- **1-State vs. 2-State Channel Models:**
  - **Only LOS → Worse coverage, as interference is enhanced**
  - **Only NLOS → In-between, as interference is reduced but probe link gets worse**
  - **LOS and NLOS → More realistic: we can model it with stochastic geometry**

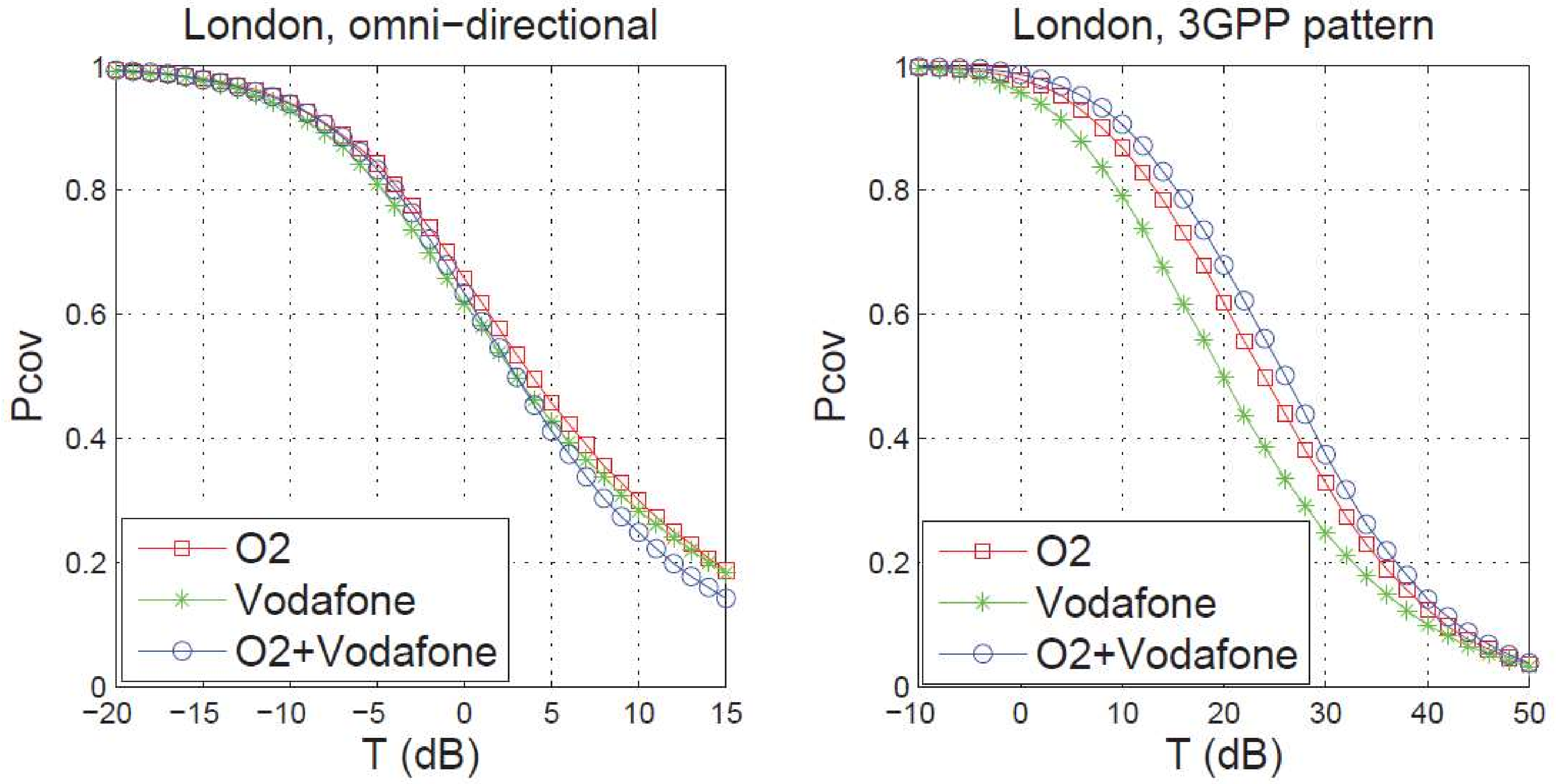

Coverage probability in London(O2 and Vodafone)


Coverage probability in London(O2)


Coverage probability in London(Vodafone)

The London Case Study (7/7)

Omni-Directional vs. 3GPP Radiation Patterns



Why Is This Modeling Approach So Accurate?

	O2 + Vodafone	O2	Vodafone
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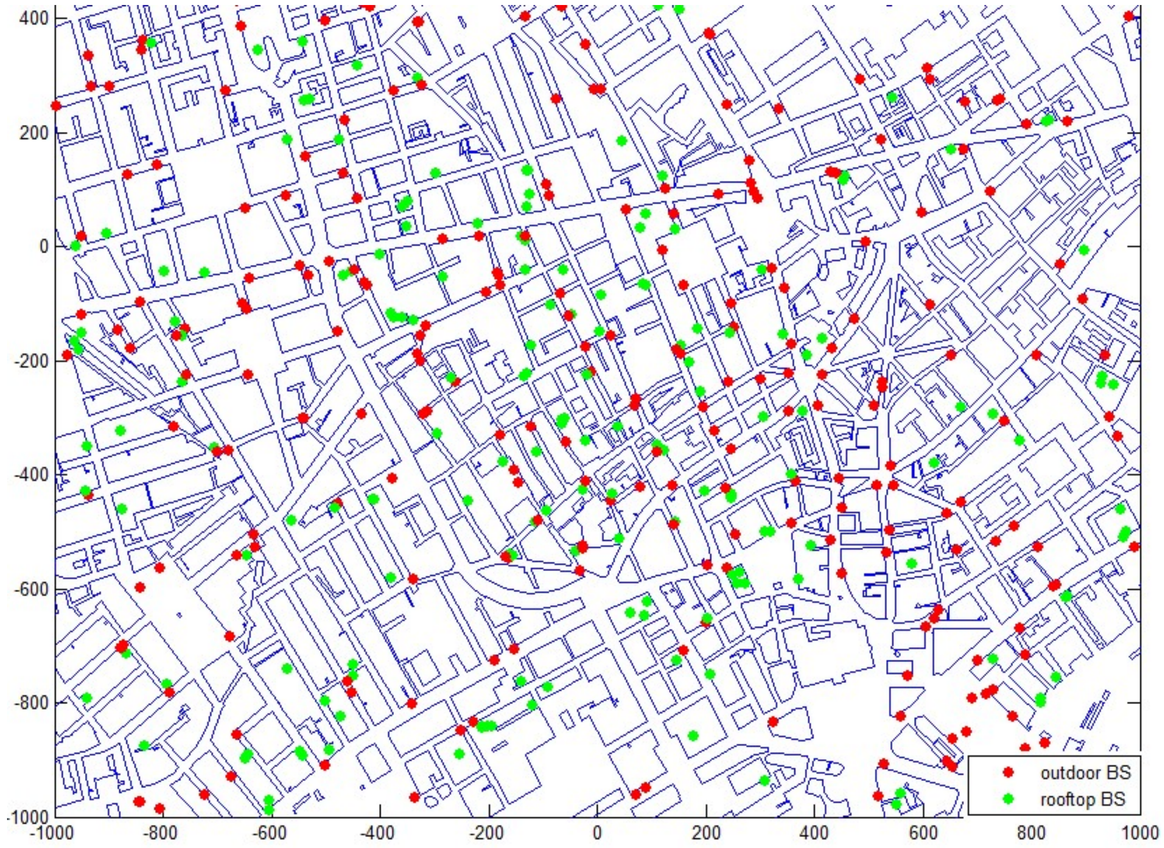
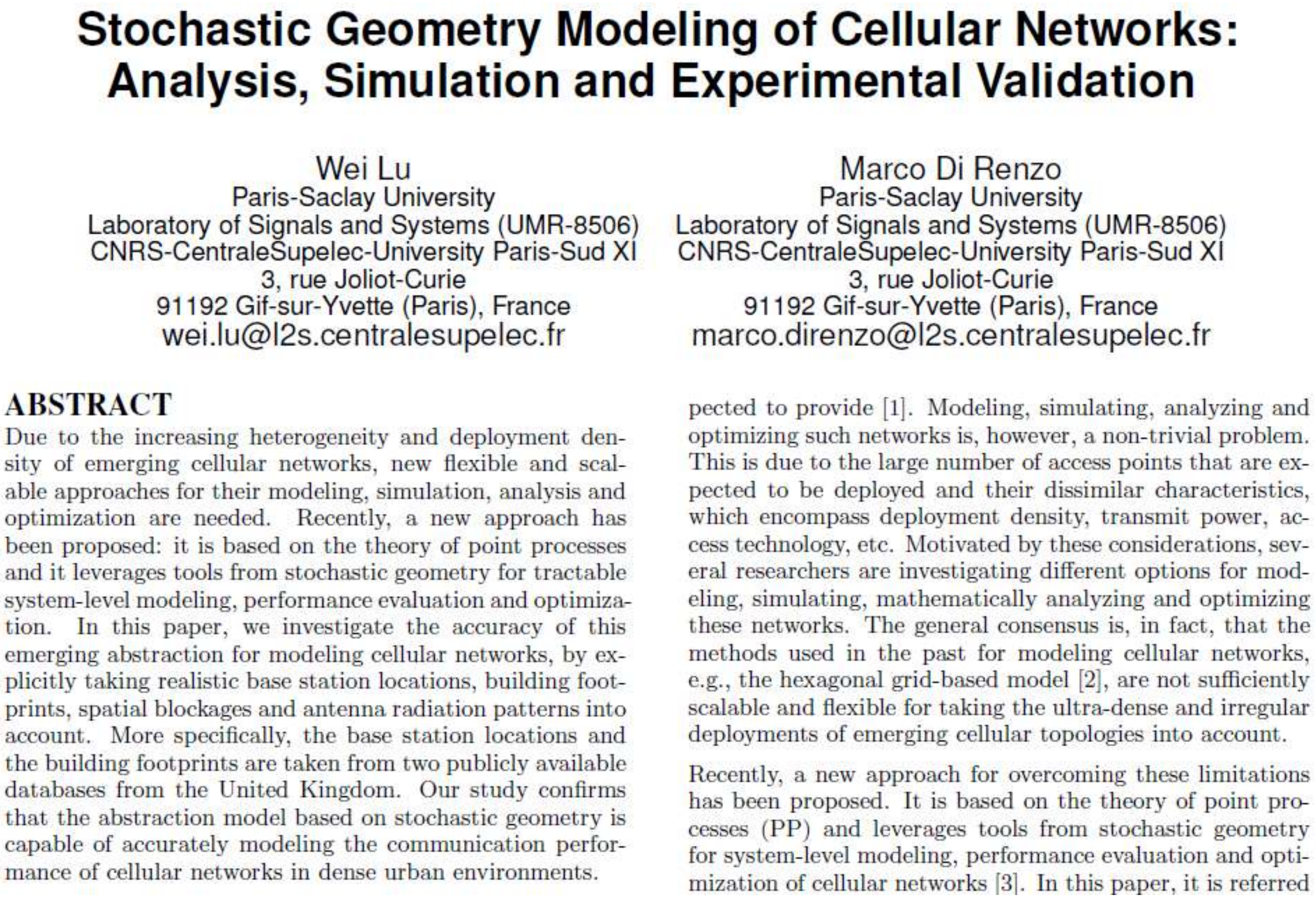


Intrigued Enough?

… Further Information and Case Studies …



Stochastic Geometry Modeling of Cellular Networks: Analysis, Simulation and Experimental Validation

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ABSTRACT

Due to the increasing heterogeneity and deployment density of emerging cellular networks, new flexible and scalable approaches for their modeling, simulation, analysis and optimization are needed. Recently, a new approach has been proposed: it is based on the theory of point processes and it leverages tools from stochastic geometry for tractable system-level modeling, performance evaluation and optimization. In this paper, we investigate the accuracy of this emerging abstraction for modeling cellular networks, by explicitly taking realistic base station locations, building footprints, spatial blockages and antenna radiation patterns into account. More specifically, the base station locations and the building footprints are taken from two publicly available databases from the United Kingdom. Our study confirms that the abstraction model based on stochastic geometry is capable of accurately modeling the communication performance of cellular networks in dense urban environments.

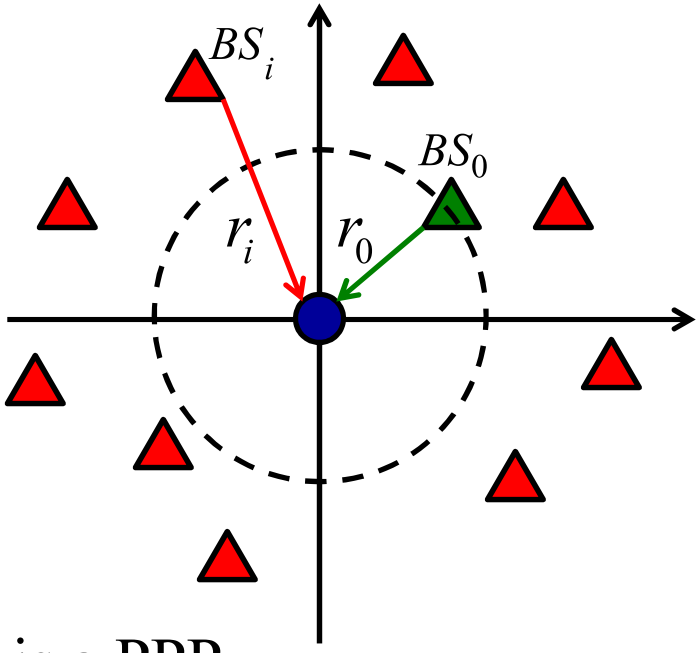
pected to provide [1]. Modeling, simulating, analyzing and optimizing such networks is, however, a non-trivial problem. This is due to the large number of access points that are expected to be deployed and their dissimilar characteristics, which encompass deployment density, transmit power, access technology, etc. Motivated by these considerations, several researchers are investigating different options for modeling, simulating, mathematically analyzing and optimizing these networks. The general consensus is, in fact, that the methods used in the past for modeling cellular networks, e.g., the hexagonal grid-based model [2], are not sufficiently scalable and flexible for taking the ultra-dense and irregular deployments of emerging cellular topologies into account.

Recently, a new approach for overcoming these limitations has been proposed. It is based on the theory of point processes (PP) and leverages tools from stochastic geometry for system-level modeling, performance evaluation and optimization of cellular networks [3]. In this paper, it is referred

W. Lu and M. Di Renzo, "Stochastic Geometry Modeling of Cellular Networks: Analysis, Simulation and Experimental Validation", ACM Int. Conf. Modeling, Analysis and Simulation of Wireless and Mobile Systems, Nov. 2015. [Online]. Available: http://arxiv.org/pdf/1506.03857.pdf.

How It Works: The Magic of Stochastic Geometry (1/5)

… understanding the basic math …



Φ is a PPP

$$\mathrm{P}_{\mathrm{cov}} = \Pr\left\{ \mathrm{SINR} > T \right\}$$

$$\mathrm{SINR} = \frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)}$$

$$I_{agg}\left(r_0\right) = \sum_{i\in\Phi\backslash BS_0} P\left|h_i\right|^2 r_i^{-\alpha}$$

$$\mathrm{P}_{\mathrm{cov}} = \Pr\left\{ \frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)} > T \right\} = \ldots$$

How It Works: The Magic of Stochastic Geometry (2/5)

… understanding the basic math …

$$\mathrm{P_{cov}} = \Pr\left\{ \frac{P\left|h_o\right|^2 r_o^{-\alpha}}{\sigma^2 + I_{agg}\left(r_0\right)} > T \right\}$$

$$= \Pr\left\{ \left|h_o\right|^2 > \left(\sigma^2 + I_{agg}\left(r_0\right)\right) P^{-1} T r_o^{\alpha} \right\}$$

$$\left(\left|h_o\right|^2 \sim \exp \Rightarrow \right) = \mathrm{E}_{I_{agg}\left(r_0\right), r_0} \left\{ \exp\left(-\left(\sigma^2 + I_{agg}\left(r_0\right)\right) P^{-1} T r_o^{\alpha} \right) \right\}$$

$$\begin{pmatrix} \mathrm{MGF}_X\left(s\right) = \\ \mathrm{E}_X\left\{e^{-sX}\right\} \Rightarrow \end{pmatrix} = \mathrm{E}_{r_0} \left\{ \exp\left(-\sigma^2 P^{-1} T r_o^{\alpha} \right) \mathrm{MGF}_{I_{agg}\left(r_0\right)}\left(P^{-1} T r_o^{\alpha} \right) \right\}$$

How It Works: The Magic of Stochastic Geometry (3/5)

… understanding the basic math …

$$\mathrm{P}_{\mathrm{cov}} = \mathrm{E}_{r_0}\left\{\exp\left(-T\sigma^2 P^{-1} r_o^{\alpha}\right)\mathrm{MGF}_{I_{agg}(r_0)}\left(P^{-1}T r_o^{\alpha}\right)\right\}$$

$$= \int_0^{+\infty} \exp\left(-T\sigma^2 P^{-1}\xi^{\alpha}\right)\mathrm{MGF}_{I_{agg}(r_0)}\left(P^{-1}T\xi^{\alpha}\right)\mathrm{PDF}_{r_0}\left(\xi\right)d\xi$$

How It Works: The Magic of Stochastic Geometry (3/5)

… understanding the basic math …

$$\mathrm{P}_{\mathrm{cov}} = \mathrm{E}_{r_0}\left\{\exp\left(-T\sigma^2 P^{-1} r_o^{\alpha}\right)\mathrm{MGF}_{I_{agg}(r_0)}\left(P^{-1} T r_o^{\alpha}\right)\right\}$$

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Trivial so far… where is the magic?

How It Works: The Magic of Stochastic Geometry (3/5)

… understanding the basic math …

$$\mathrm{P}_{\mathrm{cov}} = \mathrm{E}_{r_0}\left\{\exp\left(-T\sigma^2 P^{-1} r_o^{\alpha}\right)\mathrm{MGF}_{I_{agg}(r_0)}\left(P^{-1} T r_o^{\alpha}\right)\right\}$$

$$= \int_0^{+\infty} \exp\left(-T\sigma^2 P^{-1}\xi^{\alpha}\right)\mathrm{MGF}_{I_{agg}(r_0)}\left(P^{-1} T \xi^{\alpha}\right)\mathrm{PDF}_{r_0}\left(\xi\right)d\xi$$

Trivial so far… where is the magic?

Stochastic Geometry provides us with the mathematical tools for computing, in closed-form, the MGF and the PDF of the equation above

How It Works: The Magic of Stochastic Geometry (4/5)

… understanding the basic math …

$$I_{agg}\left(r_0\right) = \sum_{i \in \Phi \backslash BS_0} P\left|h_i\right|^2 r_i^{-\alpha}$$

The aggregate other-cell interference constitues a Marked PPP, where the marks are the channel power gains

$$\mathrm{PDF}_{r_0}\left(\xi\right) = 2\pi\lambda\xi \exp\left(-\pi\lambda\xi^2\right)$$

The PDF of the closest-distance follows from the null probability of spatial PPPs

$$\mathrm{MGF}_{I_{agg}\left(r_0\right)}\left(s\right) = \ldots$$

The MGF of the aggregate other-cell interference follows from the Probability Generating Functional (PGFL) of Marked PPPs

How It Works: The Magic of Stochastic Geometry (5/5)

… understanding the basic math …

$$\mathrm{MGF}_{I_{agg}(r_0)}(s) = \mathrm{E}_{\Phi,\{|h_i|^2\}}\left\{\exp\left(-s\sum_{i\in\Phi\backslash BS_0} P|h_i|^2 r_i^{-\alpha}\right)\right\}$$

$$= \mathrm{E}_{\Phi}\left\{\prod_{i\in\Phi\backslash BS_0} \mathrm{E}_{\{|h_i|^2\}}\left\{\exp\left(-sP|h_i|^2 r_i^{-\alpha}\right)\right\}\right\}$$

$$\left(\mathrm{PGFL}\Rightarrow\right) = \exp\left(-2\pi\lambda\int_{r_0}^{+\infty}\underbrace{\left(1-\mathrm{E}_{|h_i|^2}\left\{\exp\left(-sP|h_i|^2\xi_i^{-\alpha}\right)\right\}\right)\xi_i d\xi_i}_{\text{available in closed-form in papers}}\right)$$

So Powerful and Just Two Lemmas Need to be Used…

Sums over PPP

Lemma (Campbells theorem)

Let Φ *be a PPP of density* λ *and* $f(x) : \mathbb{R}^2 \to \mathbb{R}^+$.

$$\mathbb{E}[\sum_{x\in\Phi} f(x)] = \lambda \int_{\mathbb{R}^2} f(x)\mathrm{d}x$$

Products over PPP

Lemma (Probability generating functional (PGFL))

Let Φ *be a PPP of density* λ *and* $f(x) : \mathbb{R}^2 \to [0,1]$ *be a real valued function. Then*

$$\mathbb{E}\left[\prod_{x\in\Phi} f(x)\right] = \exp\left(-\lambda \int_{\mathbb{R}^2} (1 - f(x))\mathrm{d}x\right).$$

Error Probability: From Link-Level…

$$\text{BEP} = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{2E_b/N_0}^{+\infty}\exp\left(-\frac{t^2}{2}\right)dt = \frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{\left(2E_b/N_0\right)^2}{2\sin^2\theta}\right)d\theta$$

Error Probability: …to System-Level Analysis

$$\mathcal{I}^{(\infty)}\left(\alpha,\beta,\gamma\right)=\frac{\alpha}{\pi}-\frac{m_0}{\pi}\int_0^{+\infty}\frac{\mathcal{P}_{\mathrm{IAI}}\left(z/(\kappa\Omega_0)\right)\mathcal{T}\left(z;m_0,\alpha\right)}{\tilde{\mathcal{Q}}\left(z/(\kappa\Omega_0)\right)}dz$$

MIMO Setup	m_0	Ω_0	σ_{IAI}^2	$\mathcal{Q}\left(\xi\right)$	ASEP/AFEP	E/A
Sec. V-A	m	Ω/m	0	$\mathbb{E}_{\eta_0}\left\{{}_2F_2\left(-\frac{1}{b},m;1-\frac{1}{b};1;-\lvert\eta_0\rvert^2\frac{\Omega}{m}\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-B	N_r	$\Omega\left\Vert\tilde{\boldsymbol{\eta}}_0-\boldsymbol{\eta}_0\right\Vert^2$	0	$\mathbb{E}_{\eta_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\left\Vert\boldsymbol{\eta}_0\right\Vert^2\Omega\xi\right)\right\}-1$	Eq. (21)	E
Sec. V-C	N_r	Ω	0	$\mathbb{E}_{\eta_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\lvert\eta_0\rvert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-D	N_rN_t	$\bar{p}\Omega$	0	$\mathbb{E}_{\eta_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\left\Vert\boldsymbol{\eta}_0\right\Vert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E/A
Sec. V-E	N_r	Ω	$\Omega\sum_{m\neq\bar{m}=1}^{M}\left\lvert\boldsymbol{\eta}_0^{(m)}\right\rvert^2$	$\mathbb{E}_{\eta_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\left\Vert\boldsymbol{\eta}_0\right\Vert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-F	N_r-N_t+1	Ω	0	$\mathbb{E}_{\eta_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\left\Vert\boldsymbol{\eta}_0\right\Vert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-G	N_t-N_u+1	Ω/N_u	0	${}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\Omega\xi\right)-1$	Eq. (17)	A

A. Single-Input-Single-Output Transmission over Nakagami-m Fading
B. Spatial Multiplexing MIMO Transmission over Rayleigh Fading – Optimal Demodulation
C. Single-Input-Multiple-Output (SIMO) Transmission over Rayleigh Fading
D. Orthogonal Space-Time Block Coding (OSTBC) Transmission over Rayleigh Fading
E. Spatial Multiplexing MIMO Transmission over Rayleigh Fading – Worst-Case
F. Zero-Forcing (ZF) MIMO Receiver over Rayleigh Fading
G. Zero-Forcing MIMO Precoding over Rayleigh Fading

Three New and General Mathematical Tools

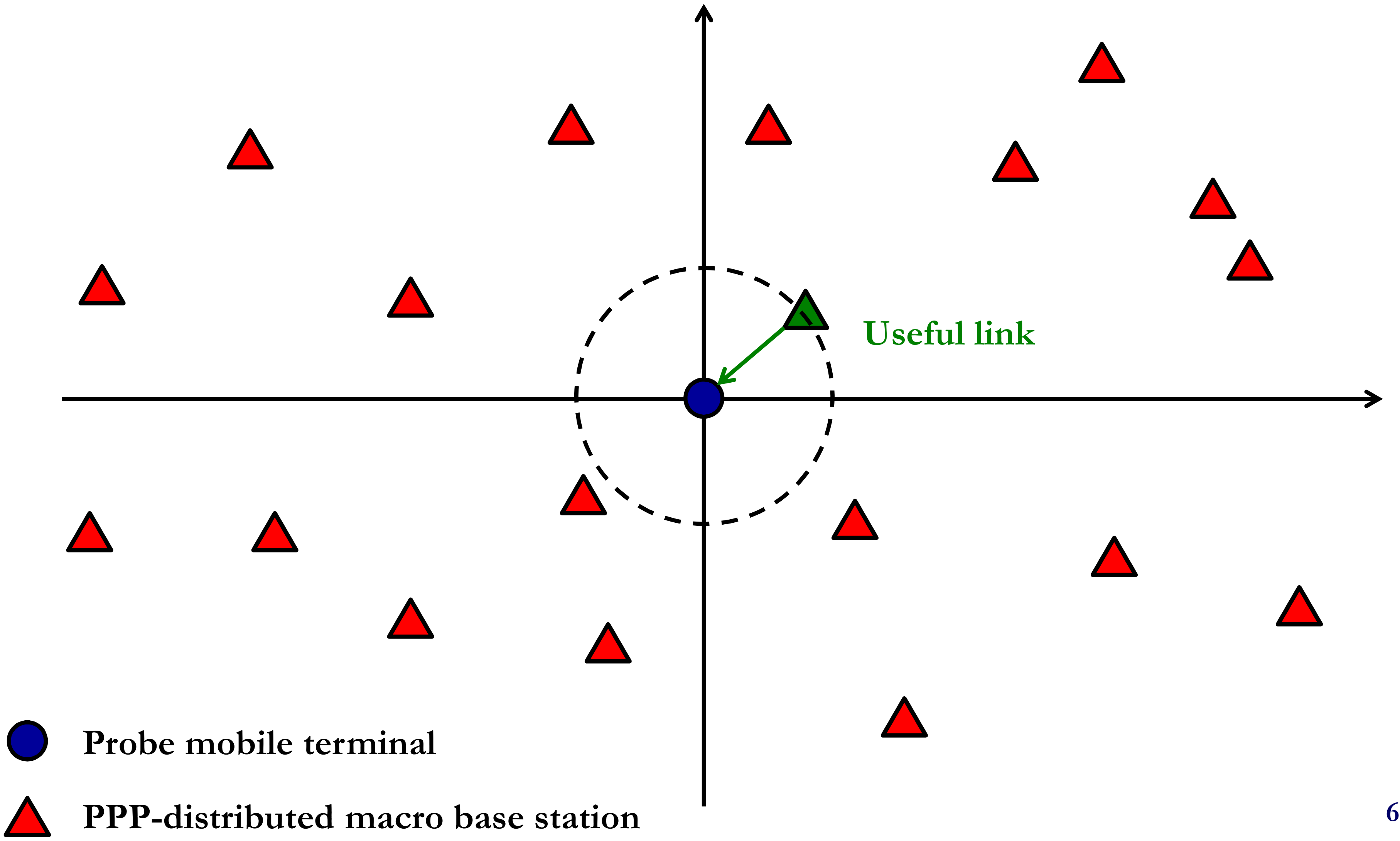
1. Average Rate: The MGF-Based Approach
   - M. Di Renzo, A. Guidotti, and G. E. Corazza, "Average Rate of Downlink Heterogeneous Cellular Networks over Generalized Fading Channels – A Stochastic Geometry Approach", IEEE Trans. Commun., vol. 61, no. 7, pp. 3050–3071, July 2013.

2. Average Error Probability: The EiD-Based Approach
   - M. Di Renzo and W. Lu, "The Equivalent–in–Distribution (EiD)–based Approach: On the Analysis of Cellular Networks Using Stochastic Geometry", IEEE Commun. Lett., vol. 18, no. 5, pp. 761-764, May 2014.
   - M. Di Renzo and W. Lu, "Stochastic Geometry Modeling and Performance Evaluation of MIMO Cellular Networks by Using the Equivalent-in-Distribution (EiD)-Based Approach", IEEE Trans. Commun., vol. 63, no. 3, pp. 977-996, March 2015.

3. Coverage Probability: The Gil-Pelaez-Based Approach
   - M. Di Renzo and P. Guan, "Stochastic Geometry Modeling of Coverage and Rate of Cellular Networks Using the Gil-Pelaez Inversion Theorem", IEEE Commun. Lett., vol. 18, no. 9, pp. 1575–1578, September 2014.

Average Rate of Cellular Networks: The Scenario

Downlink – 1-tier (the paper deals with HetNets)



Average Rate of Cellular Networks: Problem Statement

$$\begin{cases} R\left(\xi\right) = \mathrm{E}\left\{\ln\left(1+\mathrm{SINR}\left(\xi\right)\right)\right\} = \mathrm{E}\left\{\ln\left(1+\dfrac{Pg_0\xi^{-\alpha}}{\sigma_N^2+I_{\mathrm{agg}}\left(\xi\right)}\right)\right\} \\ I_{\mathrm{agg}}\left(\xi\right) = \displaystyle\sum_{b\in\Phi\left\{\backslash \mathrm{BS}_0\left(\xi\right)\right\}}\left(Pg_b d_b^{-\alpha}\right) \end{cases}$$

$$\boxed{R = 2\pi\lambda\int_0^{+\infty}\xi\exp\left\{-\pi\lambda\xi^2\right\}\mathrm{E}\left\{\ln\left(1+\frac{Pg_0\xi^{-\alpha}}{\sigma_N^2+I_{\mathrm{agg}}\left(\xi\right)}\right)\right\}d\xi}$$

The Rate in Terms of the Coverage: Sketch of the Proof

$$R = \mathrm{E}\left\{ \ln\left(1 + \mathrm{SINR}\right) \right\} = \int_0^{+\infty} \ln\left(1+x\right) f_{\mathrm{SINR}}\left(x\right) dx$$

$$\Rightarrow \text{Integration by parts } \left(\bar{F}_{\mathrm{SINR}}\left(x\right) = P_{\mathrm{cov}}\left(x\right),\ F_{\mathrm{SINR}}\left(x\right) = 1 - \bar{F}_{\mathrm{SINR}}\left(x\right) \right)$$

$$= -\ln\left(1+x\right)\left(1 - F_{\mathrm{SINR}}\left(x\right)\right)\Big|_0^{+\infty} + \int_0^{+\infty} \frac{1}{1+x}\left(1 - F_{\mathrm{SINR}}\left(x\right)\right) dx$$

$$= \int_0^{+\infty} \frac{1}{1+x}\left(1 - F_{\mathrm{SINR}}\left(x\right)\right) dx = \int_0^{+\infty} \frac{\bar{F}_{\mathrm{SINR}}\left(x\right)}{1+x} dx$$

$$\Rightarrow y = \ln\left(1+x\right)$$

$$= \int_0^{+\infty} \bar{F}_{\mathrm{SINR}}\left(e^{y} - 1\right) dy$$

Pcov-based Approach: State-of-the-Art

$$\begin{cases} \mathcal{R} \overset{(a)}{=} 2\pi\lambda \int_0^{+\infty} \int_0^{+\infty} r \exp\left\{-\pi\lambda r^2\right\} \bar{T}_1\left(r,t\right) drdt \\ \bar{T}_1\left(r,t\right) \overset{(b)}{=} \int_{-\infty}^{+\infty} \exp\left\{-2\pi\sigma_N^2 js\right\} \left(2\pi js\right)^{-1} \left[\mathcal{M}_0\left(-2\pi r^{-\alpha}\left(e^t-1\right)^{-1} js\right)-1\right] \bar{T}_2\left(r,s\right) ds \\ \bar{T}_2\left(r,s\right) \overset{(c)}{=} \exp\left\{\pi\lambda r^2 - 2\pi\lambda\alpha^{-1}\left(2\pi js\right)^{2/\alpha} \int_0^{+\infty} x^{2/\alpha}\left[\Gamma\left(-2/\alpha, 2\pi jsr^{-\alpha}x\right)-\Gamma\left(-2/\alpha\right)\right] f_I\left(x\right) dx\right\} \end{cases}$$

- **Bottom Line:**
  - **Mathematically tractable and insightful for Rayleigh fading**
  - **Closed-form expressions for special cellular setups (always for Rayleigh fading)**
  - **Intractable multi-fold integrals for fading channels different from Rayleigh**

J. G. Andrews, F. Baccelli, and R. K. Ganti, "A Tractable Approach to Coverage and Rate in Cellular Networks", IEEE Trans. Commun., vol. 59, no. 11, pp. 3122–3134, Nov. 2011.

The Enabling Result

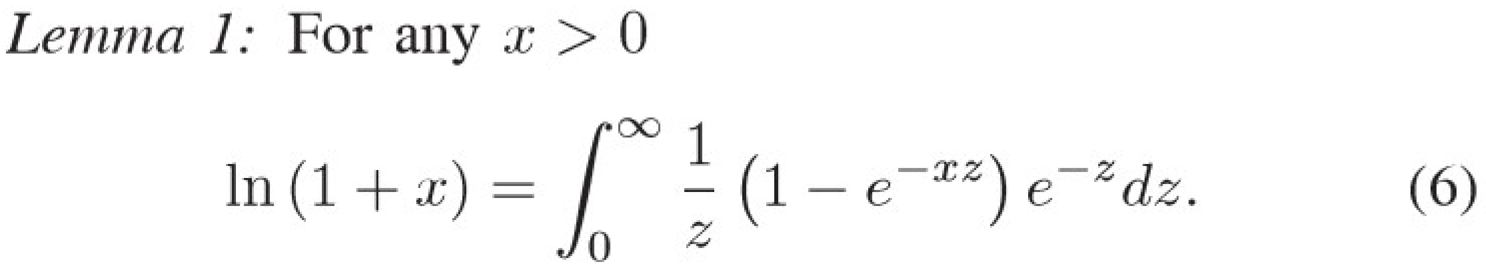
$$
\begin{aligned}
&\mathbb{E}\left\{\ln\left(1+\frac{X}{Y+1}\right)\right\} \\
&\quad = \int_{0}^{+\infty} \frac{\mathcal{M}_{Y}(z)-\mathcal{M}_{X,Y}(z)}{z} \exp\{-z\}\, dz \\
&\quad \overset{(a)}{=} \int_{0}^{+\infty} \frac{\mathcal{M}_{Y}(z)\left[1-\mathcal{M}_{X}(z)\right]}{z} \exp\{-z\}\, dz
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
&\mathbb{E}\left\{\ln\left(1+\frac{P g_0 \xi^{-\alpha}}{\sigma_N^2 + I_{\text{agg}}(\xi)}\right)\right\} \\
&= \int_{0}^{+\infty} \frac{\exp\{-z\}}{z} \mathcal{M}_{I_{\text{agg}}}(z;\xi)\left[1-\mathcal{M}_{0}\left(\text{SNR}\xi^{-\alpha} z\right)\right] dz
\end{aligned}
$$

K. Hamdi, "A Useful Lemma for Capacity Analysis of Fading Interference Channels", IEEE Trans. Commun., vol. 58, no. 2, pp. 411–416, Feb. 2010.

The Enabling Result: Sketch of Proof

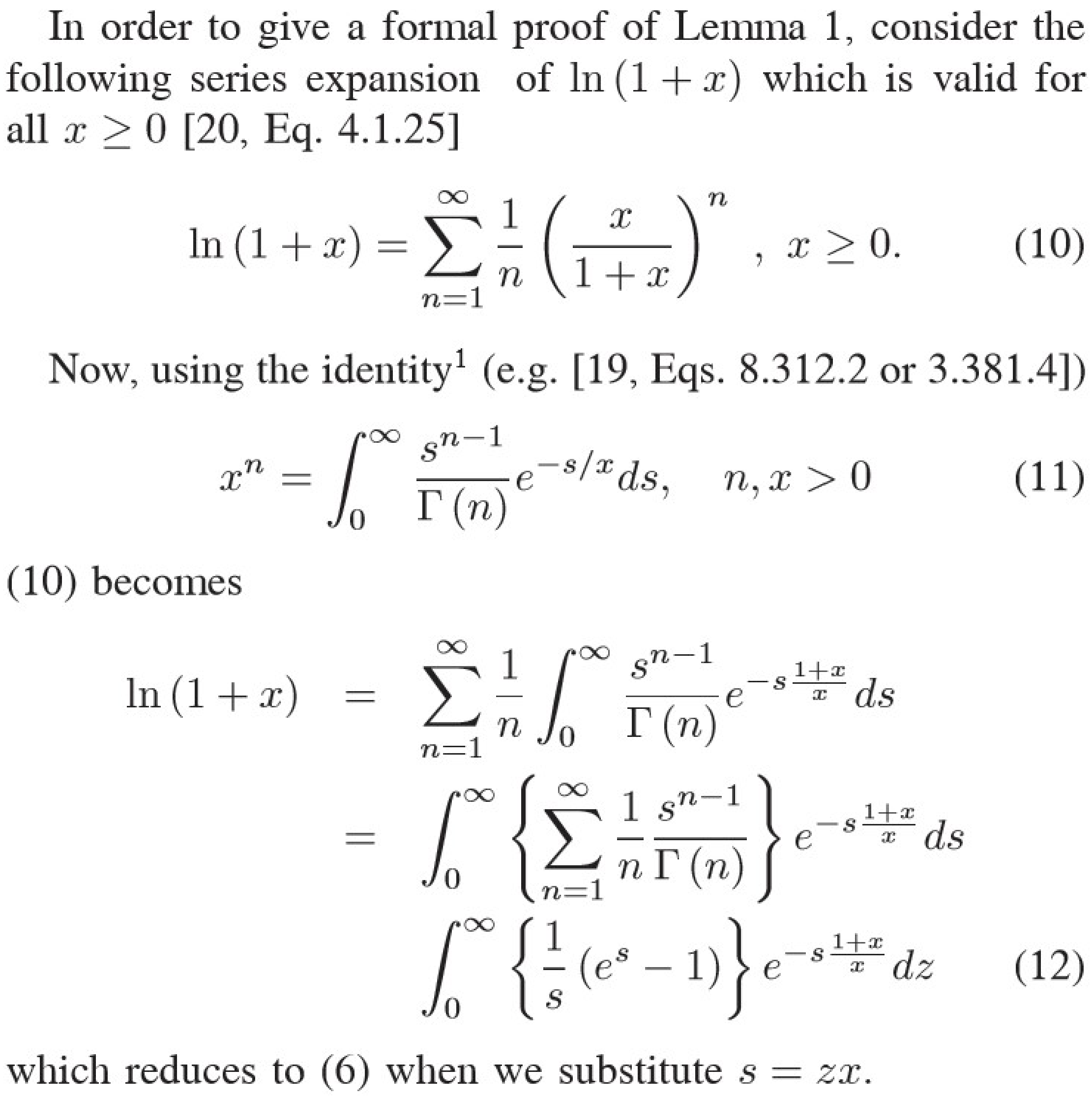


Lemma 1: For any $x > 0$

$$\ln(1+x) = \int_0^\infty \frac{1}{z}\left(1 - e^{-xz}\right) e^{-z} dz. \qquad (6)$$

K. Hamdi, "A Useful Lemma for Capacity Analysis of Fading Interference Channels", IEEE Trans. Commun., vol. 58, no. 2, pp. 411–416, Feb. 2010.

The Enabling Result: Sketch of Proof



In order to give a formal proof of Lemma 1, consider the following series expansion of $\ln(1+x)$ which is valid for all $x \geq 0$ [20, Eq. 4.1.25]

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{x}{1+x} \right)^{n}, \quad x \geq 0. \tag{10}$$

Now, using the identity[1] (e.g. [19, Eqs. 8.312.2 or 3.381.4])

$$x^{n} = \int_{0}^{\infty} \frac{s^{n-1}}{\Gamma(n)} e^{-s/x} ds, \quad n, x > 0 \tag{11}$$

(10) becomes

$$\begin{aligned} \ln(1+x) &= \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} \frac{s^{n-1}}{\Gamma(n)} e^{-s\frac{1+x}{x}} ds \\ &= \int_{0}^{\infty} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} \frac{s^{n-1}}{\Gamma(n)} \right\} e^{-s\frac{1+x}{x}} ds \\ & \int_{0}^{\infty} \left\{ \frac{1}{s} \left(e^{s} - 1 \right) \right\} e^{-s\frac{1+x}{x}} dz \end{aligned} \tag{12}$$

which reduces to (6) when we substitute $s = zx$.

K. Hamdi, "A Useful Lemma for Capacity Analysis of Fading Interference Channels", *IEEE Trans. Commun.*, vol. 58, no. 2, pp. 411–416, Feb. 2010.

MGF-based Approach: The Main Theorem

$$\begin{cases} \mathcal{R} = \int_0^{+\infty} \left[1-\mathcal{M}_0\left(\mathrm{SNR}y\right)\right] \frac{\mathcal{G}_I\left(y\right)}{y} dy \\ \mathcal{G}_I\left(y\right) = \frac{1}{\mathcal{Z}_I\left(\mathrm{SNR}y\right)} - \frac{\alpha}{2}\frac{y}{\mathcal{Z}_I\left(\mathrm{SNR}y\right)} \boxed{\int_0^{+\infty} \xi^{\frac{\alpha}{2}-1} \exp\left\{-\pi\lambda \mathcal{Z}_I\left(\mathrm{SNR}y\right)\xi\right\} \exp\left\{-y\xi^{\frac{\alpha}{2}}\right\} d\xi} \end{cases}$$

$$\begin{cases} \mathcal{Z}_I\left(y\right) = \mathcal{M}_I\left(y\right) + \mathcal{T}_I\left(y\right) \\ \mathcal{T}_I\left(y\right) = \Gamma\left(1-\frac{2}{\alpha}\right) \sum_{k=0}^{+\infty} y^{k+1} \mathcal{M}_I^{(k)}\left(y\right) \left[\Gamma\left(2-\frac{2}{\alpha}+k\right)\right]^{-1} \\ \mathcal{M}_I^{(k)}\left(y\right) = \mathbb{E}\left\{g_b^{k+1}\exp\left\{-yg_b\right\}\right\} \end{cases}$$

- **Note**: **The "inner integral" can be solved in closed-form with the aid of the Meijer G-function and the Mellin-Barnes theorem. Proposed efficient numerical methods for this computation**

M. Di Renzo, A. Guidotti, and G. E. Corazza, "Average Rate of Downlink Heterogeneous Cellular Networks over Generalized Fading Channels – A Stochastic Geometry Approach", IEEE Trans. Commun., vol. 61, no. 7, pp. 3050-3071, July 2013.

On the Computation of $T_I(\cdot)$

- **It can be computed in closed-form for the most common fading channel models**
- **No need of computing an infinite summation and the derivatives of the MGF of the interference**
- **An example: Composite Nakagami-m fast-fading with Log-Normal shadowing**

$$
\begin{aligned}
\mathcal{T}_I\left(y\right) \approx{}& m^{m+1}\left(1-\frac{2}{\alpha}\right)^{-1} y\frac{1}{\sqrt{\pi}} \\
&\times \sum_{n=1}^{N_{\mathrm{GHQ}}} \tilde{w}_n \tilde{\omega}_n^m \left(y+m\tilde{\omega}_n\right)^{-(m+1)} \\
&\times {}_2F_1\left(m+1,1,2-\frac{2}{\alpha},y\left(y+m\tilde{\omega}_n\right)^{-1}\right)
\end{aligned}
$$

M. Di Renzo, A. Guidotti, and G. E. Corazza, "Average Rate of Downlink Heterogeneous Cellular Networks over Generalized Fading Channels – A Stochastic Geometry Approach", IEEE Trans. Commun., vol. 61, no. 7, pp. 3050-3071, July 2013.

Simple Mathematical Expressions for Special Setups

- **Dense cellular networks:**

$$\mathcal{R} \leq \lim_{\lambda \to +\infty} \mathcal{R}(\lambda) = \mathcal{R}^{(\lambda_\infty)} = \int_0^{+\infty} \frac{1 - \mathcal{M}_0(z)}{\mathcal{M}_I(z) + \mathcal{T}_I(z)} \frac{dz}{z}$$

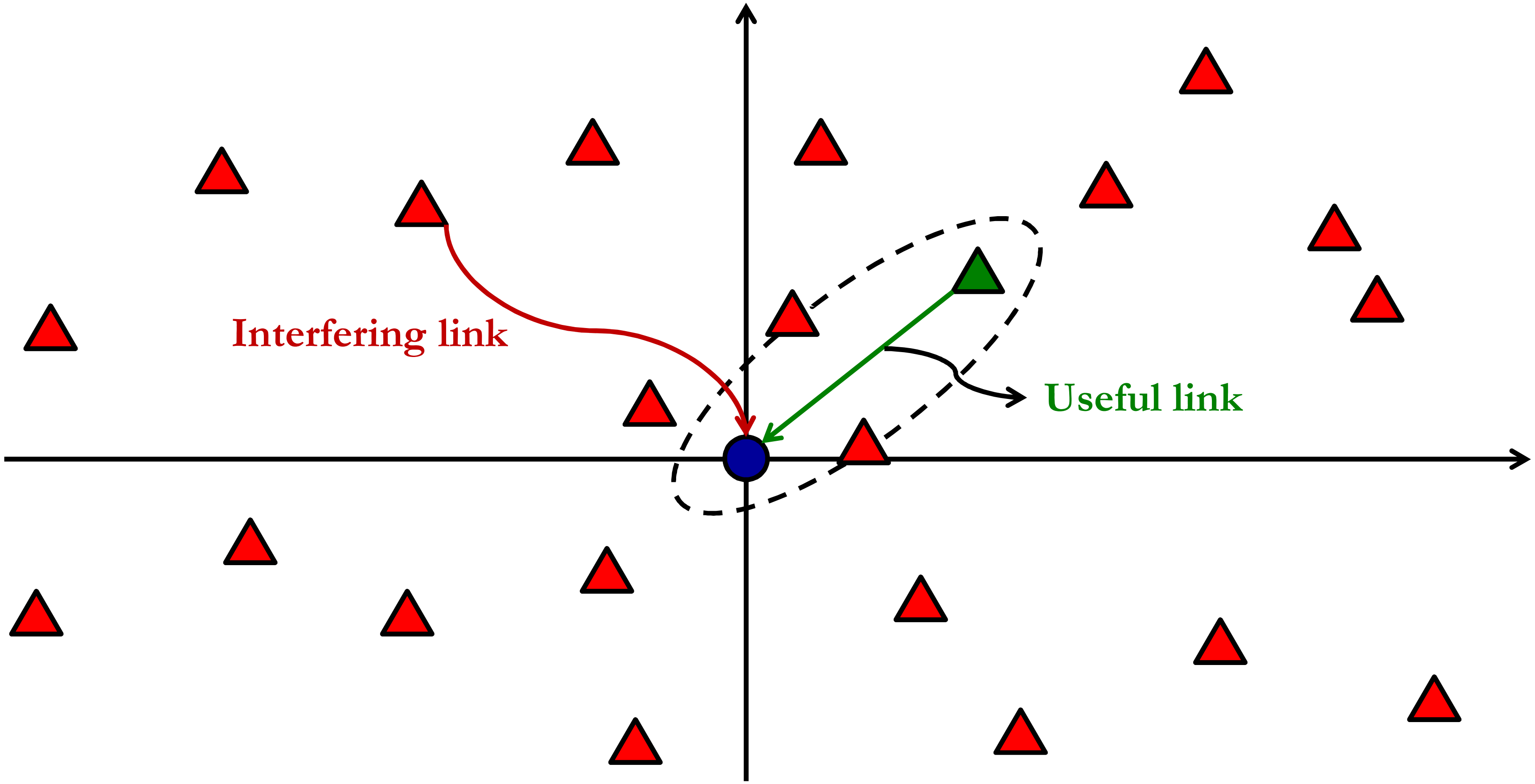
- **Interference-limited cellular networks:**

$$\mathcal{R}|_{\sigma_N^2 = 0} = \mathcal{R}^{(\mathrm{SNR}_\infty)} = \int_0^{+\infty} \frac{1 - \mathcal{M}_0(z)}{\mathcal{M}_I(z) + \mathcal{T}_I(z)} \frac{dz}{z}$$

- **High-SNR regime:**

$$\begin{aligned} \mathcal{R}^{(\mathrm{SNR} \gg 1)} &= \mathcal{R}^{(\mathrm{SNR}_\infty)} - (\pi\lambda)^{-\alpha/2} \Gamma\left(1 + \frac{\alpha}{2}\right) \frac{1}{\mathrm{SNR}} \\ &\times \int_0^{+\infty} \frac{1 - \mathcal{M}_0(z)}{\left[\mathcal{M}_I(z) + \mathcal{T}_I(z)\right]^{1+(\alpha/2)}} dz \\ &\leq \mathcal{R}(\mathrm{SNR}) \leq \mathcal{R}^{(\mathrm{SNR}_\infty)} \end{aligned}$$

ASEP of Ad-Hoc Networks: The Scenario



● **Probe/intended receiver**

▲ **PPP-distributed interferers**

▲ **Useful transmitter at a fixed distance → cell association is neglected**

ASEP of Ad-Hoc Networks: Problem Statement

$$\underbrace{\Lambda}_{\substack{\text{Decision}\\ \text{Metric}}} \propto \underbrace{\left|\Delta_0\right|^2 \mathrm{U}}_{\substack{\text{Useful}\\ \text{Signal}}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \bar{\mathrm{N}}\right\}}_{\text{AWGN}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \bar{\mathrm{I}}_{\mathrm{AGG}}\right\}}_{\substack{\text{Aggregate}\\ \text{Interference}}}$$

⇩

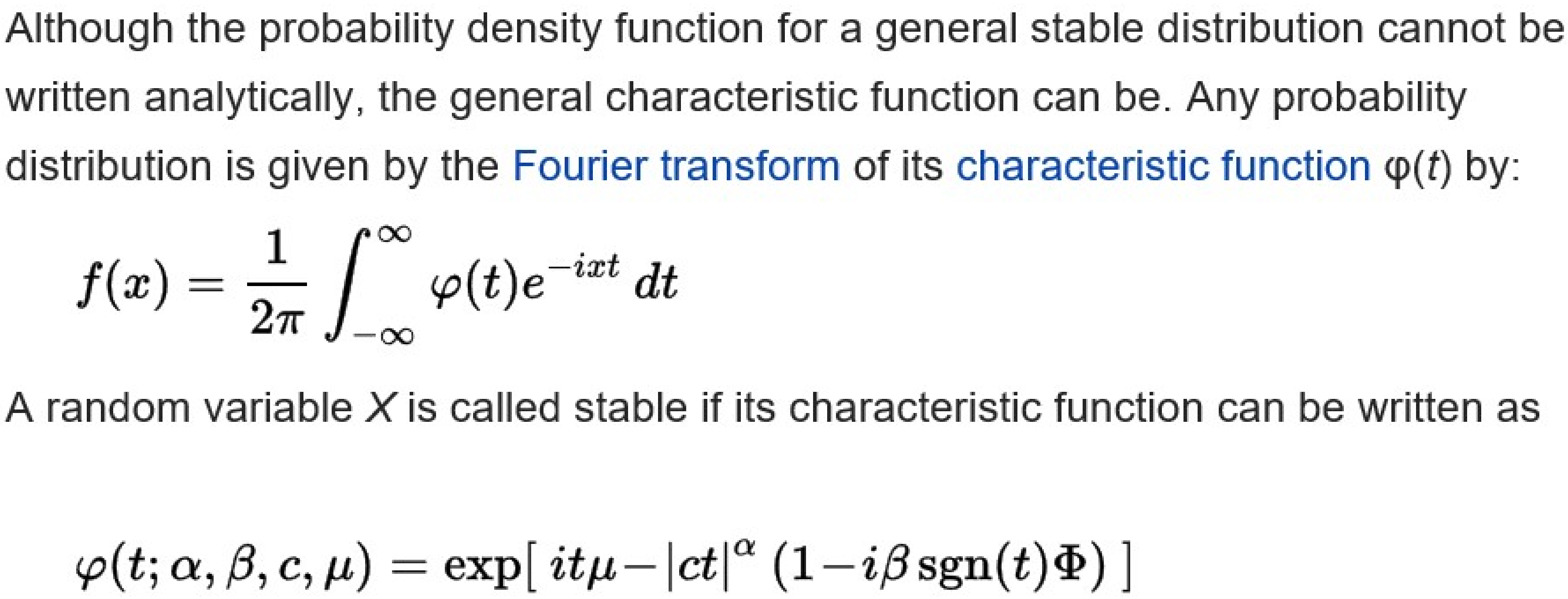
$$\boxed{\bar{\mathrm{I}}_{\mathrm{AGG}} = \sum_{i\in\Phi_{\mathrm{PPP}}} \left(\frac{\bar{Z}_i}{d_i^{b_I}}\right) \Rightarrow \bar{\mathrm{I}}_{\mathrm{AGG}} \overset{d}{=} B_I^{1/2}\bar{G}_I \sim S\alpha S\left(\alpha_I = 2/b_I, \gamma_I\right)}$$

⇧

$$\begin{cases} B_I \sim S\left(1/b_I, 1, \cos^{b_I}\left(\pi/2b_I\right)\right) \\ M_{B_I}(s) = \mathrm{E}_{B_I}\left\{\exp\left(-sB_I\right)\right\} = \exp\left(-s^{1/b_I}\right) \\ \bar{G}_I \sim CN\left(0, 4\gamma_I^{b_I}\right) \end{cases}$$

M. Di Renzo, C. Merola, A. Guidotti, F. Santucci, G. E. Corazza, "Error Performance of Multi–Antenna Receivers in a Poisson Field of Interferers – A Stochastic Geometry Approach", IEEE Trans. Commun., vol. 61, no. 5, pp. 2025–2047, May 2013.

Stable Distribution (α=2 → Gaussian, β=0 → SαS)



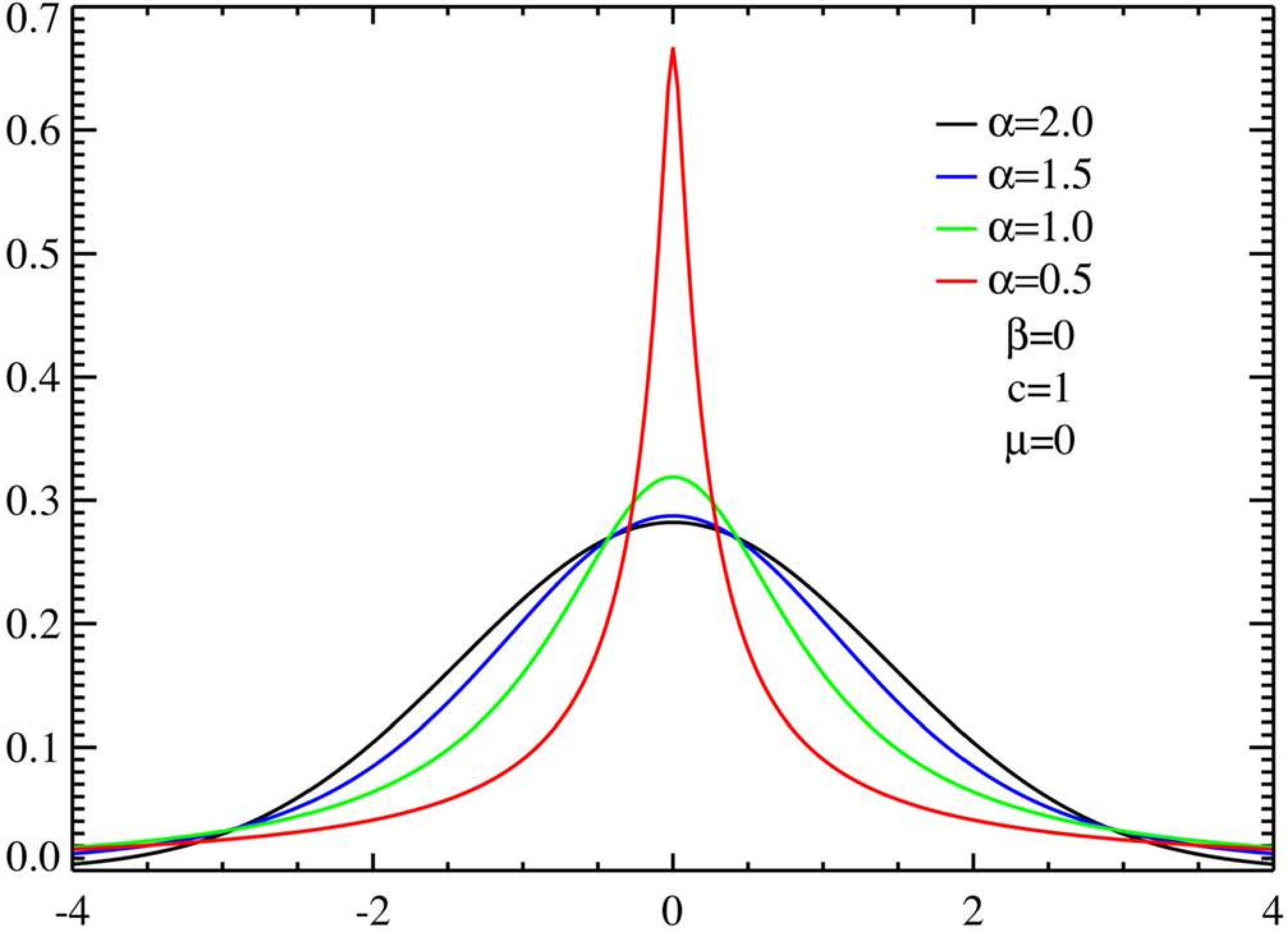
Although the probability density function for a general stable distribution cannot be written analytically, the general characteristic function can be. Any probability distribution is given by the Fourier transform of its characteristic function φ(*t*) by:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(t)e^{-ixt}\,dt$$

A random variable *X* is called stable if its characteristic function can be written as

$$\varphi(t;\alpha,\beta,c,\mu) = \exp[\, it\mu - |ct|^{\alpha}\,(1 - i\beta\,\mathrm{sgn}(t)\Phi)\,]$$

Stable Distribution (α=2 → Gaussian, β=0 → SαS)



ASEP of Ad-Hoc Networks: Methodology

$$\underbrace{\Lambda}_{\text{Decision Metric}} \propto \underbrace{\left|\Delta_0\right|^2 \mathrm{U}}_{\text{Useful Signal}} + \underbrace{\underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \overline{\mathrm{N}}\right\}}_{\text{AWGN}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \left(B_I^{1/2} \overline{G}_I\right)\right\}}_{\text{Aggregate Interference}}}_{\text{Equivalent AWGN conditioning upon } B_I}$$

STEP 1: The frameworks developed without interference can be applied by conditioning upon B_I

STEP 2: The conditioning can be removed either numerically or analytically (we did it analytically)

M. Di Renzo, C. Merola, A. Guidotti, F. Santucci, G. E. Corazza, "Error Performance of Multi–Antenna Receivers in a Poisson Field of Interferers – A Stochastic Geometry Approach", IEEE Trans. Commun., vol. 61, no. 5, pp. 2025–2047, May 2013.

ASEP of Ad-Hoc Networks: The Result

$$\underbrace{\Lambda}_{\text{Decision Metric}} \propto \underbrace{\left|\Delta_0\right|^2 \mathrm{U}}_{\text{Useful Signal}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \overline{\mathrm{N}}\right\}}_{\text{AWGN}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \overline{\mathrm{I}}_{\mathrm{AGG}}\right\}}_{\text{Aggregate Interference}}$$

$$\Downarrow$$

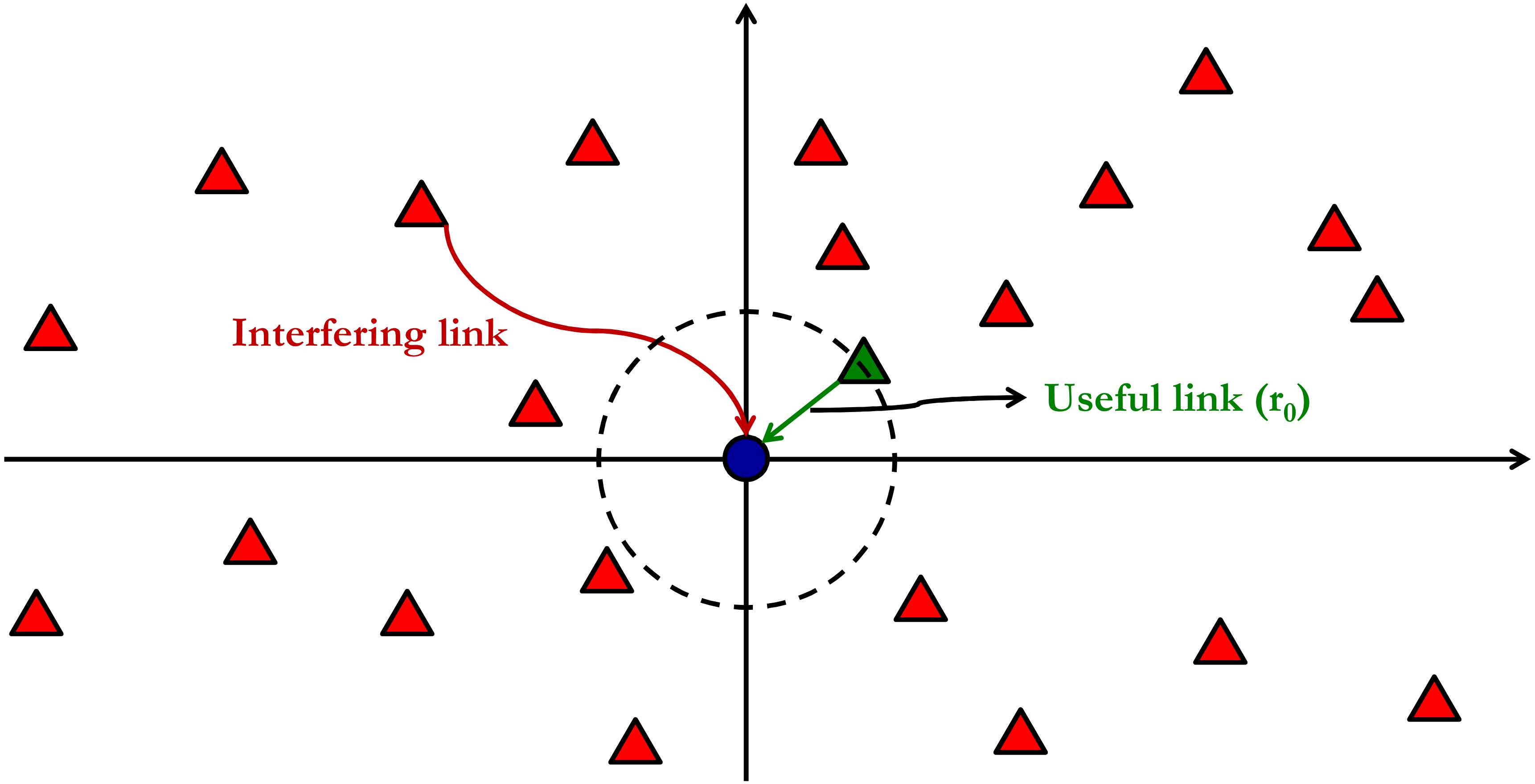
$$\boxed{\overline{\mathrm{I}}_{\mathrm{AGG}} = \sum_{i \in \Phi_{\mathrm{PPP}}} \left(\frac{\overline{Z}_i}{d_i^{b_I}} \right) \Rightarrow \overline{\mathrm{I}}_{\mathrm{AGG}} \overset{d}{=} B_I^{1/2} \overline{G}_I \sim S\alpha S\left(\alpha_I = 2/b_I, \gamma_I\right)}$$

$$\Downarrow$$

$$\boxed{\mathrm{ASEP} = \int_0^{\mu_p} \mathrm{E}_{B_I} \left\{ \left(1 + \frac{K_1}{\sin^2(\omega)} \frac{1}{K_2 N_0 + K_3 B_I} \right)^{-m_0 N_r} \right\} d\omega}$$

M. Di Renzo, C. Merola, A. Guidotti, F. Santucci, G. E. Corazza, "Error Performance of Multi–Antenna Receivers in a Poisson Field of Interferers – A Stochastic Geometry Approach", IEEE Trans. Commun., vol. 61, no. 5, pp. 2025–2047, May 2013.

ASEP of Cellular Networks: The Scenario



- ● **Probe mobile terminal**
- 🔺 **PPP-distributed interfering macro base stations**
- ▲ **Tagged macro base station at a random distance → cell association is <u>NOT</u> neglected**

ASEP of Cellular Networks: Problem Statement

$$\underbrace{\Lambda\left(r_0\right)}_{\text{Decision Metric}} \propto \underbrace{\left|\Delta_0\right|^2 \mathrm{U}\left(r_0\right)}_{\text{Useful Signal}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \overline{\mathrm{N}}\right\}}_{\text{AWGN}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \overline{\mathrm{I}}_{\mathrm{AGG}}\left(r_0\right)\right\}}_{\text{Aggregate Interference}}$$

⇓

$$\boxed{\overline{\mathrm{I}}_{\mathrm{AGG}}\left(r_0\right) = \sum_{i \in \Phi_{\mathrm{PPP}}} \left(\frac{\overline{Z}_i}{d_i^b}\right)\Bigg|_{d_i > r_0} \quad \Rightarrow \overline{\mathrm{I}}_{\mathrm{AGG}}\left(r_0\right) \overset{d}{=} ???}$$

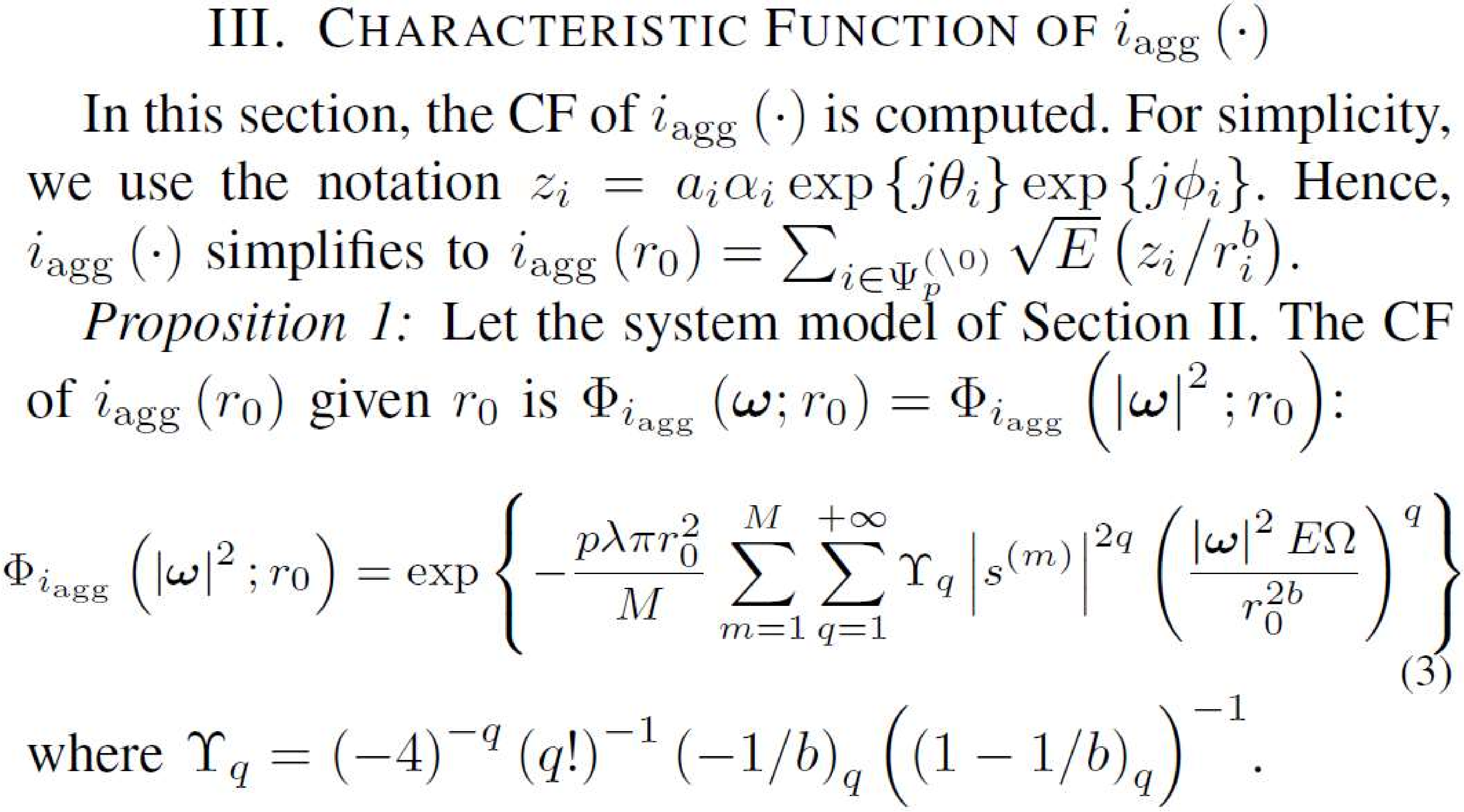
⇓

Q1: What is the distribution of $\mathrm{I}_{\mathrm{AGG}}$?

Q2: Can we develop an Equivalent-in-Distribution (EiD) representation of $\mathrm{I}_{\mathrm{AGG}}$ for arbitrary r_0>0 ?

M. Di Renzo and W. Lu, "The Equivalent-in-Distribution (EiD)-based Approach: On the Analysis of Cellular Networks Using Stochastic Geometry", IEEE Commun. Lett., vol. 18, no. 5, pp. 761-764, May 2014.

EiD-based Approach: Main Results (1/2)



III. CHARACTERISTIC FUNCTION OF $i_{\text{agg}}(\cdot)$

In this section, the CF of $i_{\text{agg}}(\cdot)$ is computed. For simplicity, we use the notation $z_i = a_i \alpha_i \exp\{j\theta_i\} \exp\{j\phi_i\}$. Hence, $i_{\text{agg}}(\cdot)$ simplifies to $i_{\text{agg}}(r_0) = \sum_{i \in \Psi_p^{(\backslash 0)}} \sqrt{E}\left(z_i / r_i^b\right)$.

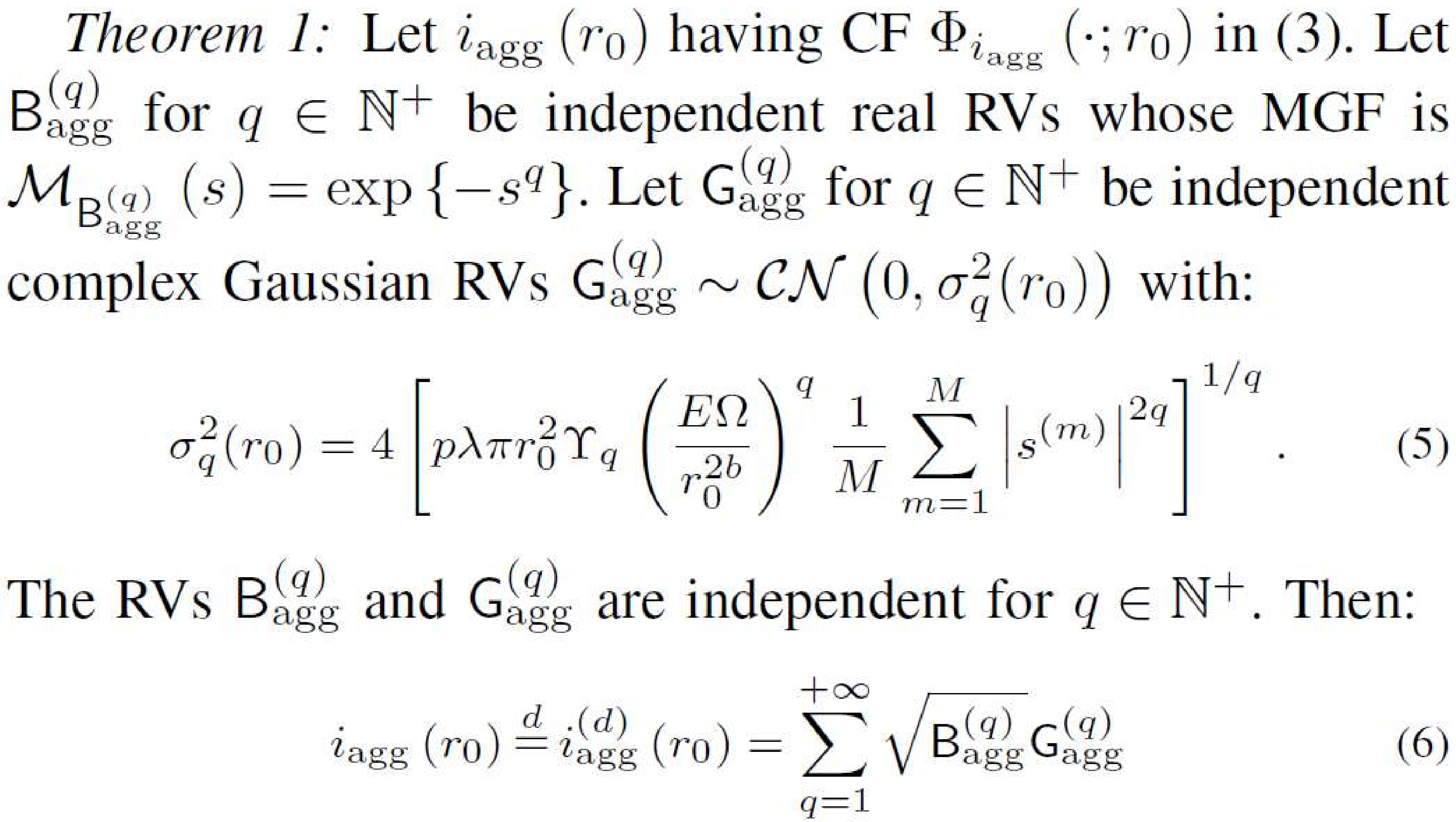
Proposition 1: Let the system model of Section II. The CF of $i_{\text{agg}}(r_0)$ given r_0 is $\Phi_{i_{\text{agg}}}(\boldsymbol{\omega}; r_0) = \Phi_{i_{\text{agg}}}\left(|\boldsymbol{\omega}|^2; r_0\right)$:

$$\Phi_{i_{\text{agg}}}\left(|\boldsymbol{\omega}|^2; r_0\right) = \exp\left\{-\frac{p\lambda\pi r_0^2}{M} \sum_{m=1}^{M} \sum_{q=1}^{+\infty} \Upsilon_q \left|s^{(m)}\right|^{2q} \left(\frac{|\boldsymbol{\omega}|^2 E\Omega}{r_0^{2b}}\right)^q\right\} \tag{3}$$

where $\Upsilon_q = (-4)^{-q} (q!)^{-1} (-1/b)_q \left((1-1/b)_q\right)^{-1}$.

M. Di Renzo and W. Lu, "The Equivalent-in-Distribution (EiD)-based Approach: On the Analysis of Cellular Networks Using Stochastic Geometry", IEEE Commun. Lett., vol. 18, no. 5, pp. 761-764, May 2014.

EiD-based Approach: Main Results (2/2)



Theorem 1: Let $i_{\mathrm{agg}}(r_0)$ having CF $\Phi_{i_{\mathrm{agg}}}(\cdot; r_0)$ in (3). Let $\mathsf{B}_{\mathrm{agg}}^{(q)}$ for $q \in \mathbb{N}^+$ be independent real RVs whose MGF is $\mathcal{M}_{\mathsf{B}_{\mathrm{agg}}^{(q)}}(s) = \exp\{-s^q\}$. Let $\mathsf{G}_{\mathrm{agg}}^{(q)}$ for $q \in \mathbb{N}^+$ be independent complex Gaussian RVs $\mathsf{G}_{\mathrm{agg}}^{(q)} \sim \mathcal{CN}\left(0, \sigma_q^2(r_0)\right)$ with:

$$\sigma_q^2(r_0) = 4\left[p\lambda\pi r_0^2 \Upsilon_q \left(\frac{E\Omega}{r_0^{2b}}\right)^q \frac{1}{M}\sum_{m=1}^{M}\left|s^{(m)}\right|^{2q}\right]^{1/q}. \tag{5}$$

The RVs $\mathsf{B}_{\mathrm{agg}}^{(q)}$ and $\mathsf{G}_{\mathrm{agg}}^{(q)}$ are independent for $q \in \mathbb{N}^+$. Then:

$$i_{\mathrm{agg}}(r_0) \overset{d}{=} i_{\mathrm{agg}}^{(d)}(r_0) = \sum_{q=1}^{+\infty} \sqrt{\mathsf{B}_{\mathrm{agg}}^{(q)}}\,\mathsf{G}_{\mathrm{agg}}^{(q)} \tag{6}$$

M. Di Renzo and W. Lu, "The Equivalent-in-Distribution (EiD)-based Approach: On the Analysis of Cellular Networks Using Stochastic Geometry", IEEE Commun. Lett., vol. 18, no. 5, pp. 761-764, May 2014.

ASEP of Cellular Networks: Methodology

$$\underbrace{\Lambda}_{\substack{\text{Decision}\\ \text{Metric}}} \propto \underbrace{\left|\Delta_0\right|^2 \mathrm{U}}_{\substack{\text{Useful}\\ \text{Signal}}} + \underbrace{\underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \overline{\mathrm{N}}\right\}}_{\text{AWGN}} + \underbrace{2\,\mathrm{Re}\left\{\Delta_0^* \left(\sum_{q=1}^{+\infty} B_I^{1/2}(q)\,\bar{G}_I(q)\right)\right\}}_{\substack{\text{Aggregate}\\ \text{Interference}}}}_{\substack{\text{Equivalent AWGN}\\ \text{conditioning upon } B_I(1), B_I(2), \cdots, B_I(\infty)}}$$

STEP 1: The frameworks developed without interference can be applied by conditioning upon $B_I(1)$, $B_I(2)$, …, $B_I(\infty)$

STEP 2: The conditioning can be removed either numerically or analytically (we did it analytically)

M. Di Renzo and W. Lu., "Stochastic Geometry Modeling and Performance Evaluation of MIMO Cellular Networks by Using the Equivalent-in-Distribution (EiD)-Based Approach", IEEE Trans. Commun., vol. 63, no. 3, pp. 977-996, March 2015.

EiD-based Approach: The MIMO Case

$$\mathcal{I}^{(\infty)}(\alpha,\beta,\gamma)=\frac{\alpha}{\pi}-\frac{m_0}{\pi}\int_0^{+\infty}\frac{\mathcal{P}_{\mathrm{IAI}}(z/(\kappa\Omega_0))\,\mathcal{T}(z;m_0,\alpha)}{\tilde{\mathcal{Q}}(z/(\kappa\Omega_0))}dz$$

MIMO Setup	m_0	Ω_0	σ_{IAI}^2	$\mathcal{Q}(\xi)$	ASEP/AFEP	E/A
Sec. V-A	m	Ω/m	0	$\mathbb{E}_{\eta_0}\left\{{}_2F_2\left(-\frac{1}{b},m;1-\frac{1}{b};1;-\lvert\eta_0\rvert^2\frac{\Omega}{m}\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-B	N_r	$\Omega\lVert\tilde{\boldsymbol{\eta}}_0-\boldsymbol{\eta}_0\rVert^2$	0	$\mathbb{E}_{\boldsymbol{\eta}_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\lVert\boldsymbol{\eta}_0\rVert^2\Omega\xi\right)\right\}-1$	Eq. (21)	E
Sec. V-C	N_r	Ω	0	$\mathbb{E}_{\eta_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\lvert\eta_0\rvert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-D	N_rN_t	$\bar{p}\Omega$	0	$\mathbb{E}_{\boldsymbol{\eta}_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\lVert\boldsymbol{\eta}_0\rVert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E/A
Sec. V-E	N_r	Ω	$\Omega\sum\limits_{m\neq\bar{m}=1}^{M}\left\lvert\eta_0^{(m)}\right\rvert^2$	$\mathbb{E}_{\boldsymbol{\eta}_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\lVert\boldsymbol{\eta}_0\rVert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-F	N_r-N_t+1	Ω	0	$\mathbb{E}_{\boldsymbol{\eta}_0}\left\{{}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\lVert\boldsymbol{\eta}_0\rVert^2\Omega\xi\right)\right\}-1$	Eq. (17)	E
Sec. V-G	N_t-N_u+1	Ω/N_u	0	${}_1F_1\left(-\frac{1}{b};1-\frac{1}{b};-\Omega\xi\right)-1$	Eq. (17)	A

A. Single-Input-Single-Output Transmission over Nakagami-m Fading
B. Spatial Multiplexing MIMO Transmission over Rayleigh Fading – Optimal Demodulation
C. Single-Input-Multiple-Output (SIMO) Transmission over Rayleigh Fading
D. Orthogonal Space-Time Block Coding (OSTBC) Transmission over Rayleigh Fading
E. Spatial Multiplexing MIMO Transmission over Rayleigh Fading – Worst-Case
F. Zero-Forcing (ZF) MIMO Receiver over Rayleigh Fading
G. Zero-Forcing MIMO Precoding over Rayleigh Fading

EiD-based Approach: The MIMO Case (Approx.)

Proposition 3: Let $\mathrm{ASEP}_{\mathrm{PSK}}$ in (32). The following approximation holds:

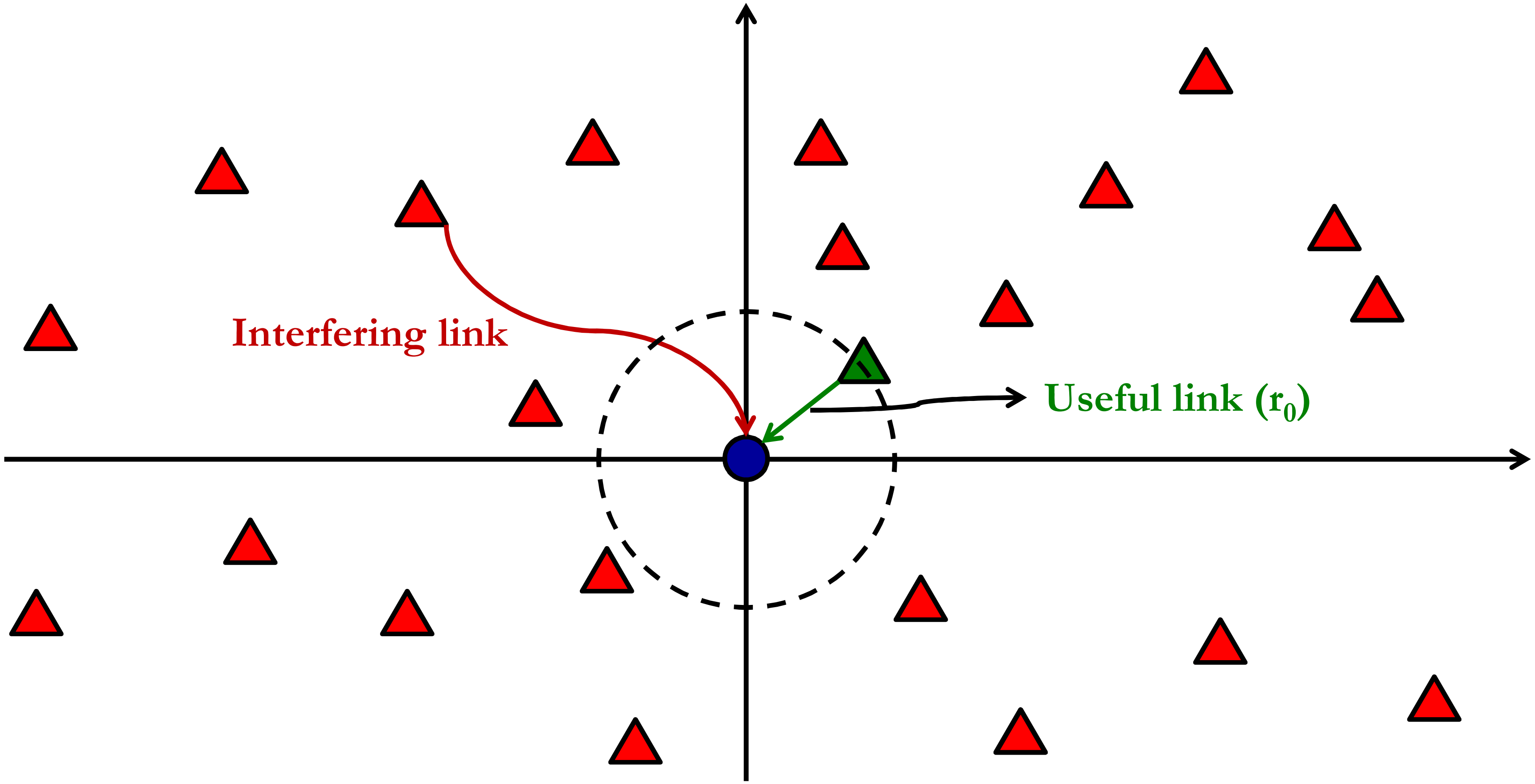
$$\mathrm{ASEP}_{\mathrm{PSK}} \approx \mathcal{K}_{\mathrm{PSK}}^{(0)} - \frac{\mathcal{K}_{\mathrm{PSK}}^{(0)}}{\sqrt{\pi}\Gamma\left(\mathcal{K}_{\mathrm{PSK}}^{(1)}\right)} G_{4,3}^{2,3}\left(\mathcal{K}_{\mathrm{PSK}}^{(2)} \middle| \begin{array}{cccc} 1/2 & 0 & 0 & 1 \\ 0 & \mathcal{K}_{\mathrm{PSK}}^{(1)} & 0 & \end{array}\right) \tag{33}$$

where $\mathcal{K}_{\mathrm{PSK}}^{(0)} = 1/2$ if $\mathsf{M} = 2$ and $\mathcal{K}_{\mathrm{PSK}}^{(0)} = 1$ if $\mathsf{M} \geq 4$, $\mathcal{K}_{\mathrm{PSK}}^{(1)} = m_0$ and $\mathcal{K}_{\mathrm{PSK}}^{(2)} = p\,(b-1)^{-1} \sin^{-2}(\pi/\mathsf{M})\boxed{\kappa_{\mathrm{MIMO}}}$.

Proof: See Appendix IV. ☐

In spite of being an approximation, (33) is surprisingly simple and insightful. In particular, it depends on two main parameters, *i.e.*, $\mathcal{K}_{\mathrm{PSK}}^{(1)}$ and $\mathcal{K}_{\mathrm{PSK}}^{(2)}$. By direct inspection of the Meijer G-function in (33), it follows that $\mathrm{ASEP}_{\mathrm{PSK}}$ monotonically decreases as $\mathcal{K}_{\mathrm{PSK}}^{(1)}$ increases and that it monotonically increases as $\mathcal{K}_{\mathrm{PSK}}^{(2)}$ increases.

Coverage/Rate of Cellular Networks: The Scenario



● Probe mobile terminal

🔺 PPP-distributed interfering macro base stations

▲ Tagged macro base station at a random distance

Coverage/Rate of Cellular Networks: Problem Statement

$$\mathrm{SINR} = \frac{\mathsf{P}\gamma_0 r_0^{-\alpha}}{\sigma_N^2 + \mathsf{P}\mathcal{I}_{agg}\left(r_0\right)} \qquad \mathcal{I}_{agg}\left(r_0\right) = \sum\nolimits_{i \in \Psi(\backslash 0)} \gamma_i r_i^{-\alpha}$$

$$\Downarrow$$

$$\mathrm{P_{cov}}\left(\mathrm{T}\right) = \Pr\left\{\mathrm{SINR} \ge \mathrm{T}\right\}$$

$$\mathcal{R} = \mathbb{E}\left\{\ln\left(1 + \mathrm{SINR}\right)\right\} \overset{(a)}{=} \int_0^{+\infty} \mathrm{P_{cov}}\left(\exp\left(t\right) - 1\right) dt$$

$$\overset{(b)}{=} -\int_0^{+\infty} \ln\left(1 + y\right) \mathrm{P}_{\mathrm{cov}}^{(1)}\left(y\right) dy$$

The Rate in Terms of the Coverage: Sketch of the Proof

$$R = \mathrm{E}\left\{ \ln\left(1 + \mathrm{SINR}\right) \right\} = \int_0^{+\infty} \ln\left(1 + x\right) f_{\mathrm{SINR}}\left(x\right) dx$$

$\Rightarrow$ Integration by parts

$$= -\ln\left(1 + x\right)\left(1 - F_{\mathrm{SINR}}\left(x\right)\right)\Big|_0^{+\infty} + \int_0^{+\infty} \frac{1}{1 + x}\left(1 - F_{\mathrm{SINR}}\left(x\right)\right) dx$$

$$= \int_0^{+\infty} \frac{1}{1 + x}\left(1 - F_{\mathrm{SINR}}\left(x\right)\right) dx = \int_0^{+\infty} \frac{\bar{F}_{\mathrm{SINR}}\left(x\right)}{1 + x} dx$$

$\Rightarrow y = \ln\left(1 + x\right)$

$$= \int_0^{+\infty} \bar{F}_{\mathrm{SINR}}\left(e^y - 1\right) dy$$

$\Rightarrow$ Integration by parts...again...

Gil-Pelaez Based Approach

$$\begin{aligned}
\mathrm{P}_{\mathrm{cov}}\left(\mathrm{T}\right) &= \int_0^{+\infty} \Pr\left\{ \mathcal{I}_{agg}\left(\xi\right) \le \frac{\gamma_0 \xi^{-\alpha}}{\mathrm{T}} - \frac{\sigma_N^2}{\mathsf{P}} \middle| \xi \right\} f_{r_0}\left(\xi\right) d\xi \\
&= \int_0^{+\infty} \mathbb{E}_{\gamma_0}\left\{ F_{\mathcal{I}_{agg}}\left(\frac{\gamma_0 \xi^{-\alpha}}{\mathrm{T}} - \frac{\sigma_N^2}{\mathsf{P}} \right) \right\} f_{r_0}\left(\xi\right) d\xi
\end{aligned}$$

⇧

$$F_X\left(x\right) = \mathrm{CDF}_X\left(x\right) = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} \mathrm{Im}\left\{\exp\left(-j\omega x\right)\mathrm{CF}_X\left(\omega\right)\right\}\frac{d\omega}{\omega}$$

M. Di Renzo and P. Guan, "Stochastic Geometry Modeling of Coverage and Rate of Cellular Networks Using the Gil-Pelaez Inversion Theorem", IEEE Commun. Lett., vol. 18, no. 9, pp. 1575–1578, Sep. 2014.

Gil-Pelaez Approach: Coverage Probability

$$\mathrm{P}_{\mathrm{cov}}\left(\mathrm{T}\right) = \Pr\left\{\mathrm{SINR} \geq \mathrm{T}\right\}$$

⇩

Theorem 1: Let $\mathrm{P}_{\mathrm{cov}}\left(\cdot\right)$ in (2). It can be formulated as:

$$\mathrm{P}_{\mathrm{cov}}\left(\mathrm{T}\right) = \frac{1}{2} - 2\lambda \int_{0}^{+\infty} \mathrm{Im}\left\{\mathcal{M}_{\gamma_0}\left(j\frac{x}{\mathrm{T}}\right)\mathcal{F}_{\mathrm{NI}}\left(x\right)\right\}\frac{dx}{x} \qquad (4)$$

where the following functions are introduced:

$$\mathcal{F}_{\mathrm{NI}}\left(x\right) = \int_{0}^{+\infty} y \exp\left(jy^{\alpha}x\frac{\sigma_N^2}{\mathsf{P}}\right)\exp\left(-\pi\lambda y^2 \Upsilon_{\mathrm{I}}\left(jx\right)\right)dy \qquad (5)$$

$$\Upsilon_{\mathrm{I}}\left(z\right) = \mathbb{E}_{\gamma_i}\left\{{}_1F_1\left(-\frac{2}{\alpha};1-\frac{2}{\alpha};z\gamma_i\right)\right\} \qquad (6)$$

Gil-Pelaez Approach: Average Rate

$$\mathcal{R} \overset{(b)}{=} -\int_{0}^{+\infty} \ln\left(1+y\right) \mathrm{P}_{\mathrm{cov}}^{(1)}\left(y\right) dy$$

⇩

Theorem 2: Let $\mathcal{R}$ in (3). It can be formulated as:

$$\mathcal{R} = -2\lambda \int_{0}^{+\infty} \mathrm{Im}\left\{ j\mathcal{F}_{0}\left(jx\right) \mathcal{F}_{\mathrm{NI}}\left(x\right) \right\} dx \tag{9}$$

where $\mathcal{F}_{\mathrm{NI}}\left(\cdot\right)$ is defined in (5) and $\mathcal{F}_{0}\left(\cdot\right)$ is given as follows:

$$\mathcal{F}_{0}\left(z\right) = \int_{0}^{+\infty} \frac{\ln\left(1+y\right)}{y^{2}} \mathcal{M}_{\gamma_{0}}^{(1)}\left(\frac{z}{y}\right) dy \tag{10}$$

Gil-Pelaez Approach: Interference-Limited Scenario

Corollary 1: Let $\mathrm{P}_{\mathrm{cov}}(\cdot)$ in (2) with $\sigma_N^2 = 0$, *i.e.*, $\mathrm{P}_{\mathrm{cov}}^{[\infty]}(\cdot)$. It can be formulated as:

$$\mathrm{P}_{\mathrm{cov}}^{[\infty]}(\mathrm{T}) = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} \mathrm{Im}\left\{ \frac{\mathcal{M}_{\gamma_0}\left(j\frac{x}{\mathrm{T}}\right)}{\Upsilon_{\mathrm{I}}(jx)} \right\} \frac{dx}{x} \qquad (7)$$

Corollary 2: Let $\mathcal{R}$ in (3) with $\sigma_N^2 = 0$, *i.e.*, $\mathcal{R}^{[\infty]}$. It can be formulated as:

$$\mathcal{R}^{[\infty]} = -\frac{1}{\pi}\int_0^{+\infty} \mathrm{Im}\left\{ j\frac{\mathcal{F}_0(jx)}{\Upsilon_{\mathrm{I}}(jx)} \right\} dx \qquad (11)$$

Gil-Pelaez Approach: Gamma-Distributed Power-Gains (1/3)

Relevant for studying MIMO cellular networks over Rayleigh fading

By assuming $\gamma_0 \sim \mathcal{G}(m_0, \Omega_0)$ and $\gamma_I \sim \mathcal{G}(m_I, \Omega_I)$, *Corollary 1* and *Corollary 2* can be simplified as follows:

$$\mathrm{P}_{\mathrm{cov}}^{[\infty]}(\mathrm{T}) \overset{(a)}{=} \frac{1}{2} - \frac{1}{\pi} \int_0^{+\infty} \mathrm{Im}\left\{ \frac{(1 + j\tilde{\kappa}_0 x)^{-m_0}}{{}_2F_1\left(-\frac{2}{\alpha}, m_I; 1 - \frac{2}{\alpha}; j\kappa_I x\right)} \right\} \frac{dx}{x} \quad (12)$$

$$\mathcal{R}^{[\infty]} \overset{(b)}{=} \frac{1}{\pi\Gamma(m_0)} \int_0^{+\infty} \mathrm{Im}\left\{ \frac{G_{3,3}^{2,3}\left(j\kappa_0 x \left| \begin{matrix} 1 & 1 & 1 - m_0 \\ 1 & 1 & 0 \end{matrix} \right. \right)}{{}_2F_1\left(-\frac{2}{\alpha}, m_I; 1 - \frac{2}{\alpha}; j\kappa_I x\right)} \right\} \frac{dx}{x} \quad (13)$$

Gil-Pelaez Approach: Gamma-Distributed Power-Gains (2/3)

Closed-form approximations

Proposition 1: Let $\mathrm{P}_{\mathrm{cov}}^{[\infty]}(\cdot)$ in (12). The following holds:

$$\mathrm{P}_{\mathrm{cov}}^{[\infty]}(\mathrm{T}) \approx 1 - \left(1 + \frac{1}{2}\frac{1}{m_0}\frac{\Omega_0}{\Omega_I}(\alpha - 2)\frac{1}{\mathrm{T}}\right)^{-m_0} \tag{14}$$

Proposition 2: Let $\mathcal{R}^{[\infty]}$ in (13). The following holds:

$$\mathcal{R}^{[\infty]} \approx \frac{1}{\Gamma(m_0)} G_{3,3}^{3,2}\left(\left(\frac{1}{2}\frac{1}{m_0}\frac{\Omega_0}{\Omega_I}(\alpha - 2)\right)^{-1} \middle| \begin{array}{ccc} 0 & 0 & 1 \\ m_0 & 0 & 0 \end{array}\right) \tag{15}$$

Gil-Pelaez Approach: Gamma-Distributed Power-Gains (3/3)

Closed-form approximations: Performance trends

Remark 3: By direct inspection of (14), the coverage probability has the following performance trends: i) it increases as $\eta = \Omega_0 (\alpha - 2)/(\mathrm{T}\Omega_I)$ increases; ii) it increases as m_0 increases; and iii) it is independent of m_I. $\square$

Remark 5: By plotting the Meijer G-function in (15) as a function of $\tilde{\eta} = (\Omega_0/\Omega_I)(\alpha - 2)$ and m_0, the same performance trends as for the coverage (*Remark 3*) hold. $\square$

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- ❑ M. Di Renzo, W. Lu, and P. Guan, "The Intensity Matching Approach: A Tractable Stochastic Geometry Approximation to System-Level Analysis of Cellular Networks", IEEE Trans. Wireless Commun., IEEE Early Access.

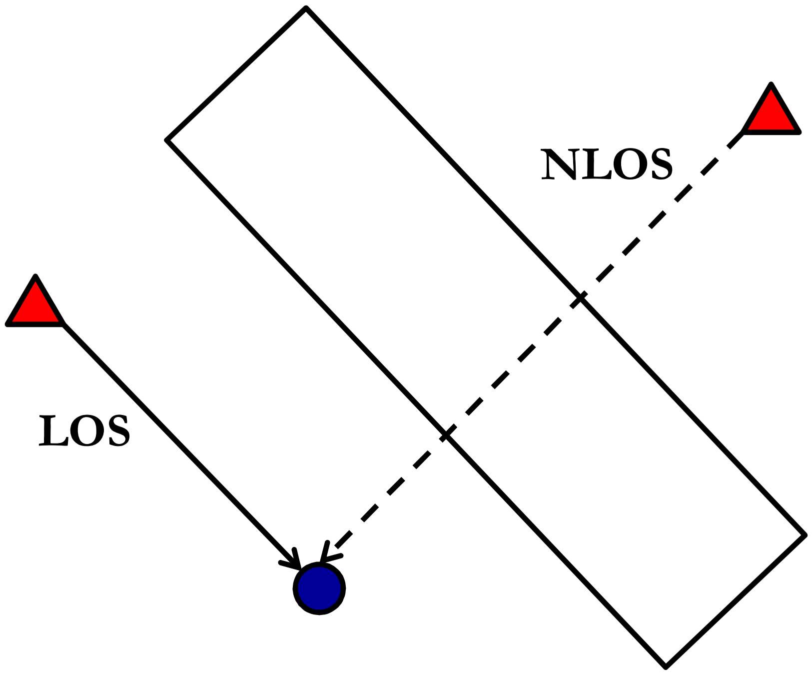
The Intensity Matching (IM) Approach

A Complete Mathematical Framework for System-Level Analysis

- ❑ **M. Di Renzo, W. Lu, and P. Guan, "The Intensity Matching Approach: A Tractable Stochastic Geometry Approximation to System-Level Analysis of Cellular Networks", IEEE Trans. Wireless Commun., IEEE Early Access.**
  - ➢ **Realistic path-loss model with LOS/NLOS conditions**
  - ➢ **Arbitrary shadowing and fading**
  - ➢ **General antenna-array radiation pattern**
  - ➢ **Multi-tier topology with practical cell association**
  - ➢ **Realistic traffic load models as a function of the densities of BSs and MTs**
  - ➢ **…**

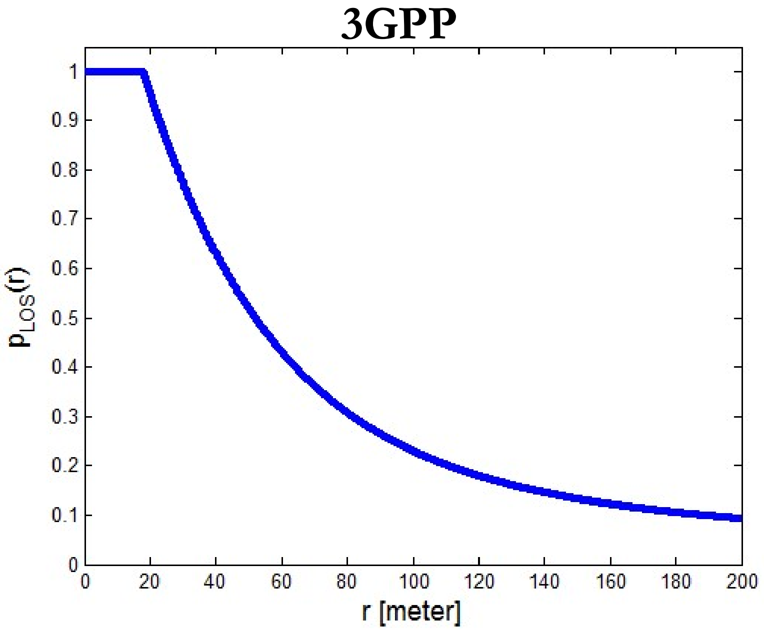
IM Approach: Why So Many Details are Needed?

… Impact of LOS/NLOS …



● **Mobile terminal**

▲ **Base station**



$$p_{\mathrm{LOS}}(r) = \min\left\{\frac{18}{r},1\right\}\left(1-e^{-\frac{r}{36}}\right)+e^{-\frac{r}{36}}$$

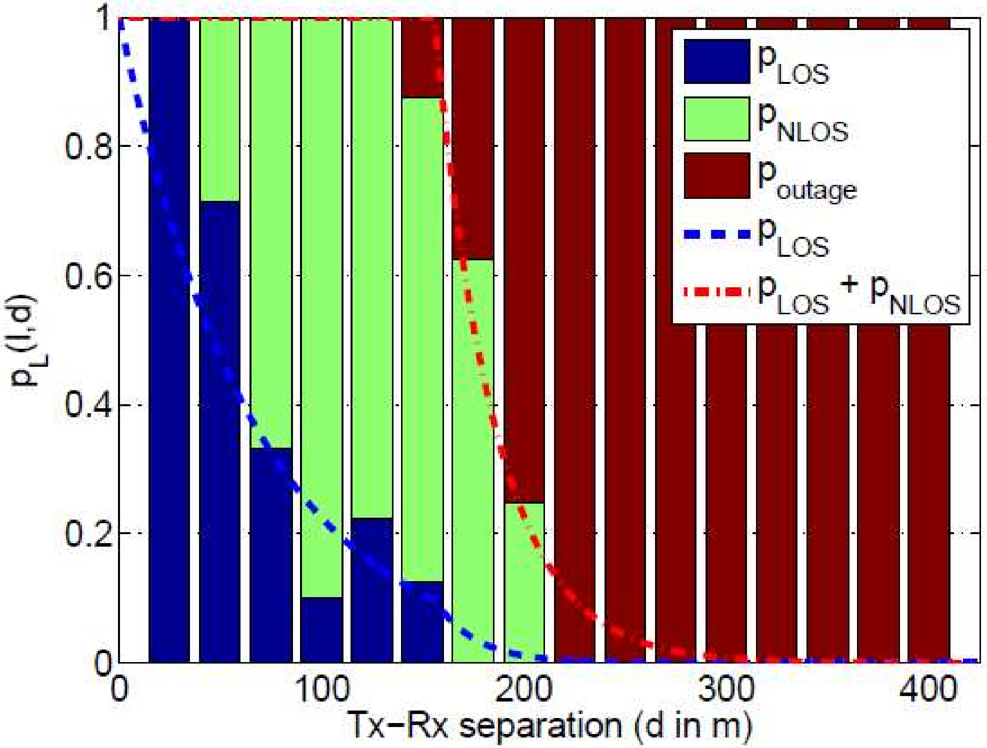
Modeling Blockages: A Practical Example

… the mmWave case study …

T. S. Rappaport et al. "Millimeter wave channel modeling and cellular capacity evaluation", IEEE JSAC, June 2014.

Accurate statistical characterization of the mmWave channel based on the measurements conducted by researchers at NYU-WIRELESS:

1. **Path-loss model**
2. **Shodowing model**
3. **LOS/NLOS/outage link state**
4. **Directional beamforming**
5. **Beamsteering errors**
6. **Multi-tier topology**
7. **Different cell associations**



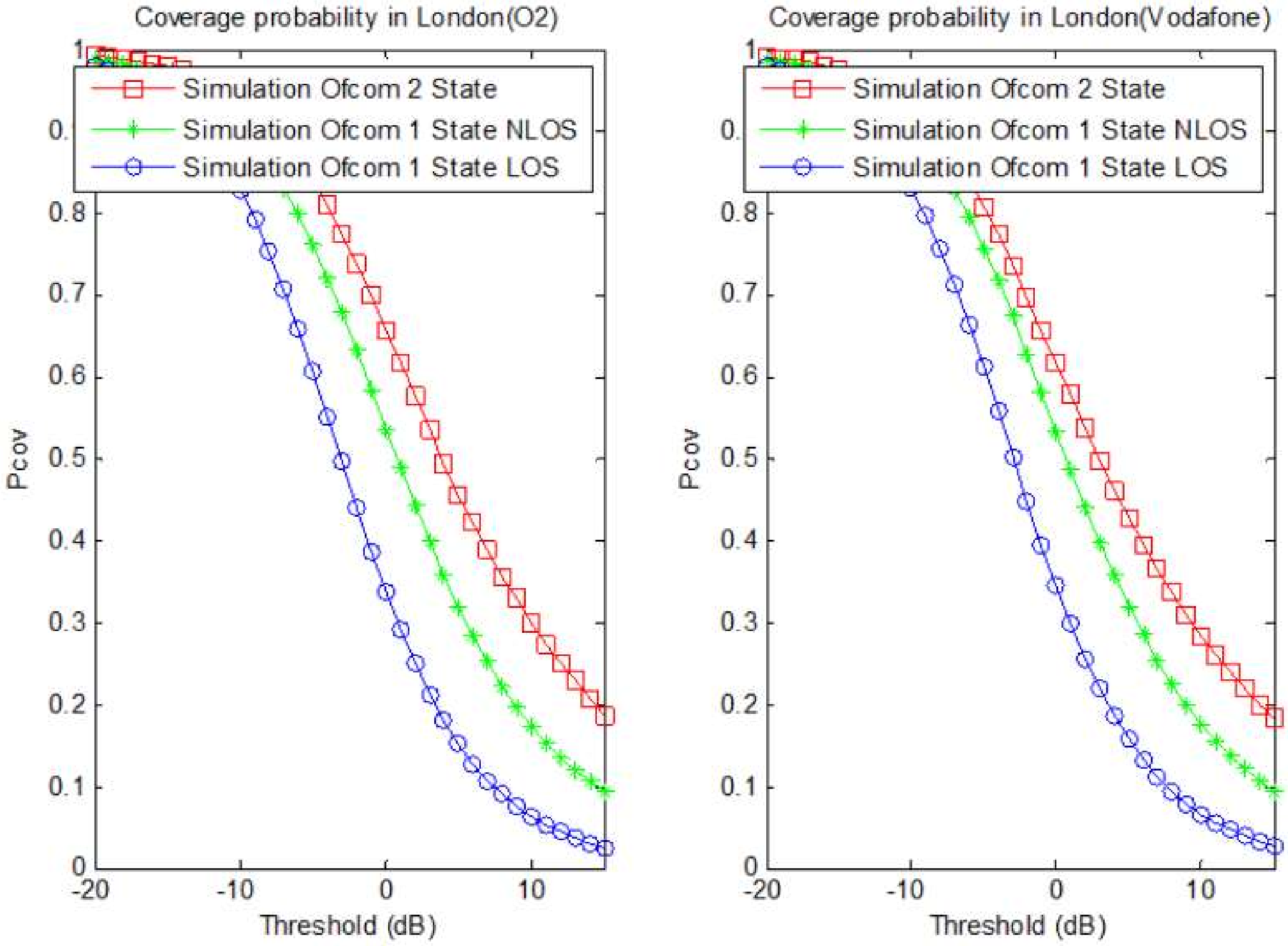
$$p_{\mathrm{OUT}}(r) = \max\left\{0, 1 - \gamma_{\mathrm{OUT}} e^{-\delta_{\mathrm{OUT}} r}\right\}$$

$$p_{\mathrm{LOS}}(r) = (1 - p_{\mathrm{OUT}}(r))\, \gamma_{\mathrm{LOS}} e^{-\delta_{\mathrm{LOS}} r}$$

$$p_{\mathrm{NLOS}}(r) = (1 - p_{\mathrm{OUT}}(r))\left(1 - \gamma_{\mathrm{LOS}} e^{-\delta_{\mathrm{LOS}} r}\right)$$

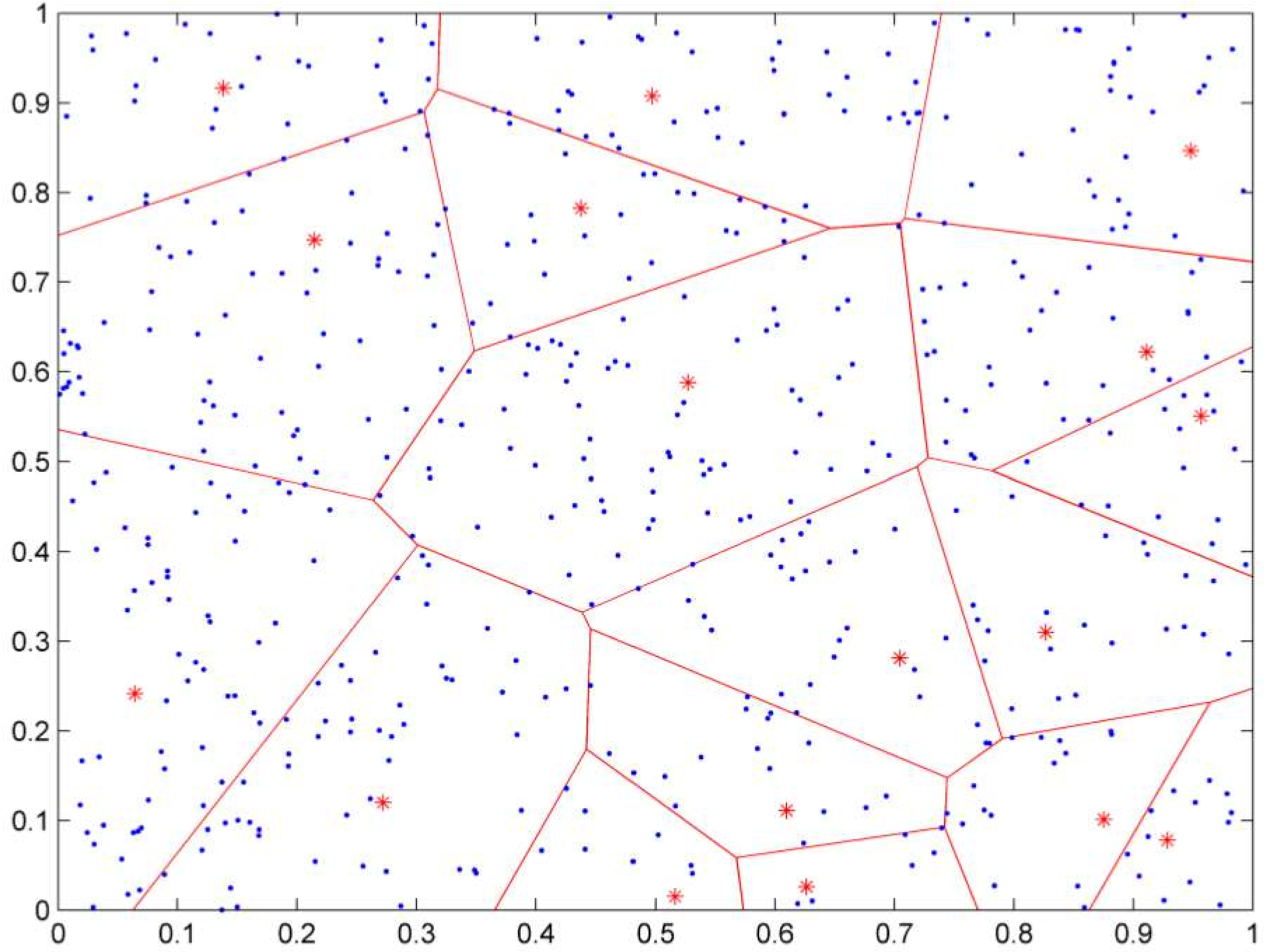
IM Approach: Why So Many Details are Needed?

… Impact of LOS/NLOS …



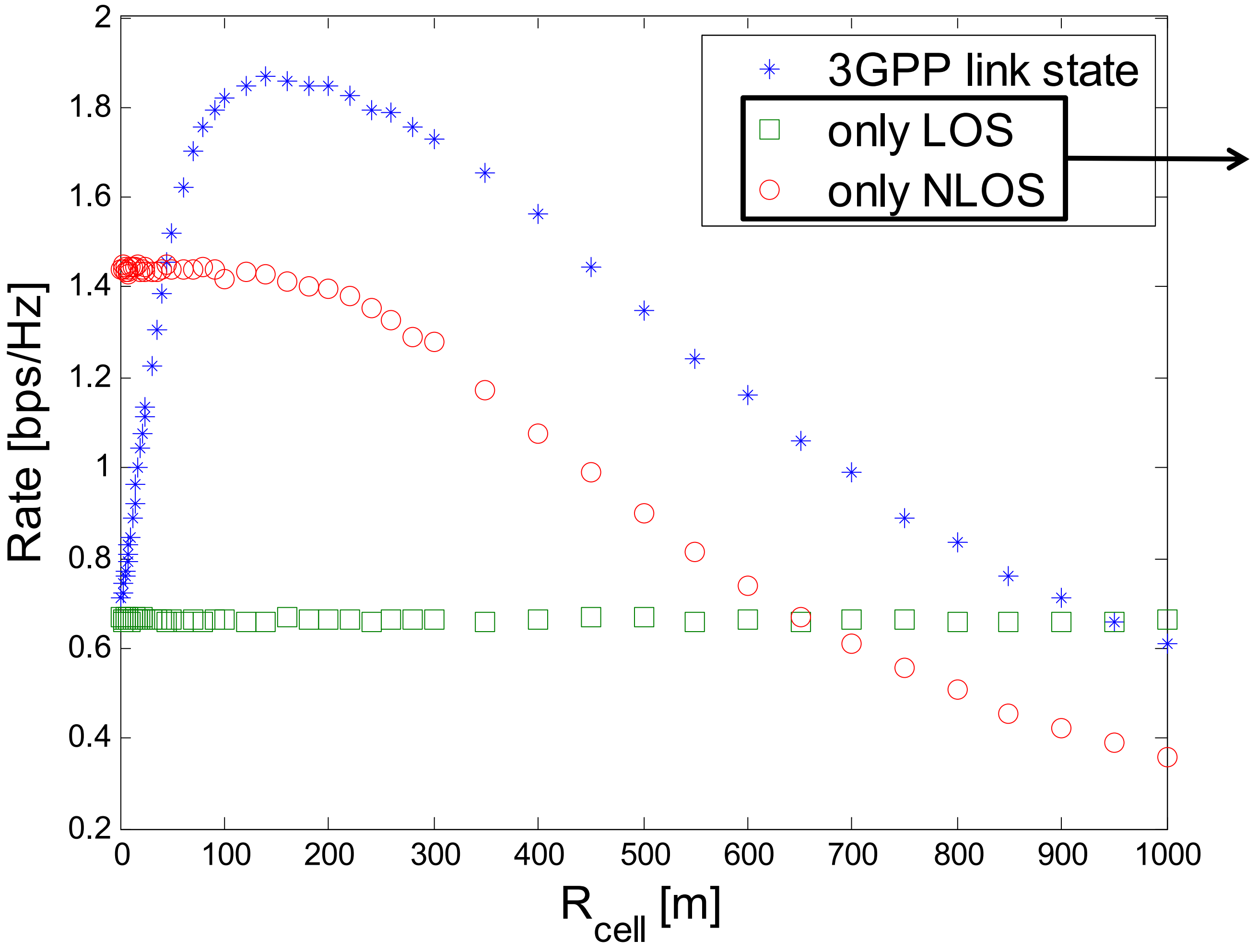
IM Approach: Why So Many Details are Needed?

... Impact of LOS/NLOS on network densification (fully-loaded) ...



IM Approach: Why So Many Details are Needed?

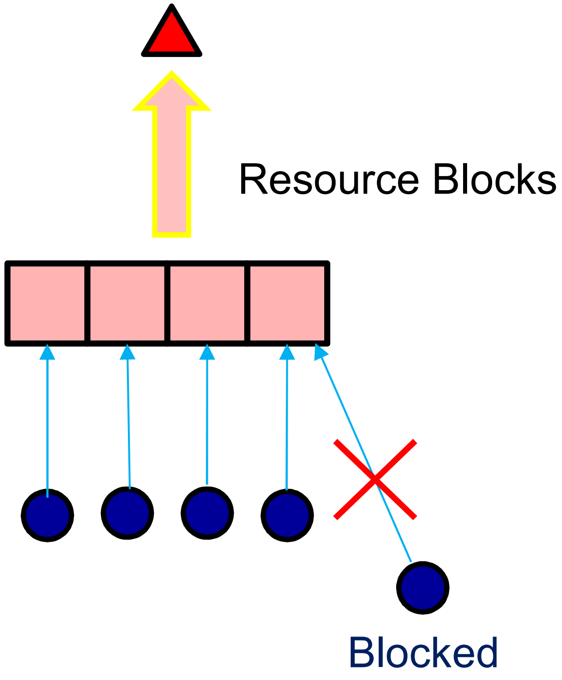
... Impact of LOS/NLOS (fully-loaded) …

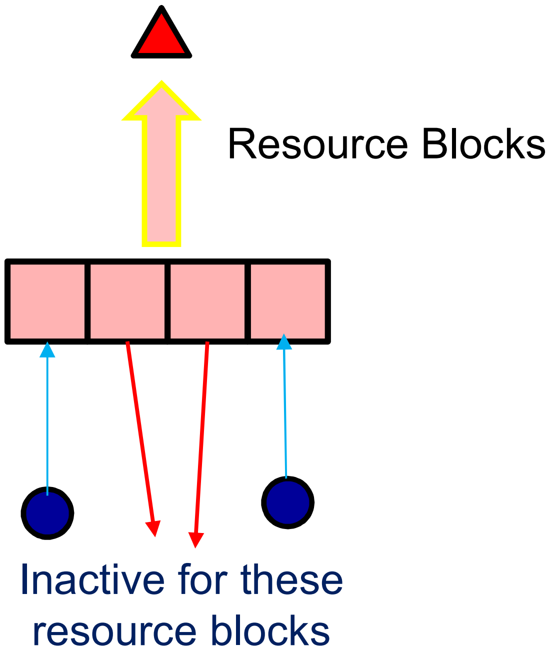


Current assumption (for tractability) in stochastic geometry modeling (99.99% of papers)

IM Approach: Why So Many Details are Needed?

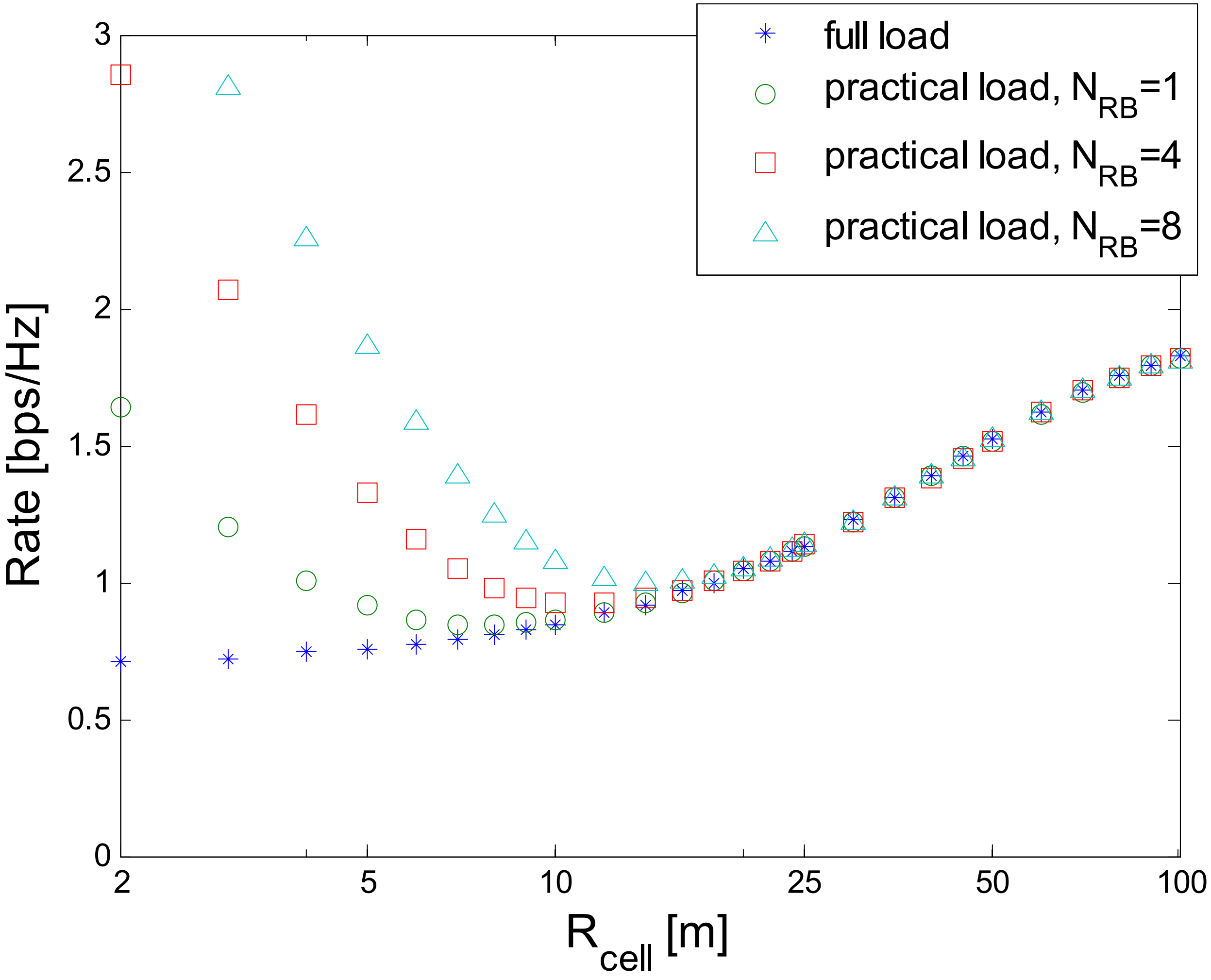
... Impact of Load of Base Stations …





IM Approach: Why So Many Details are Needed?

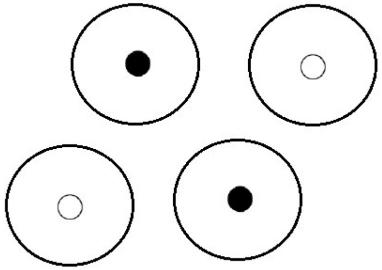
… Impact of Load of Base Stations …



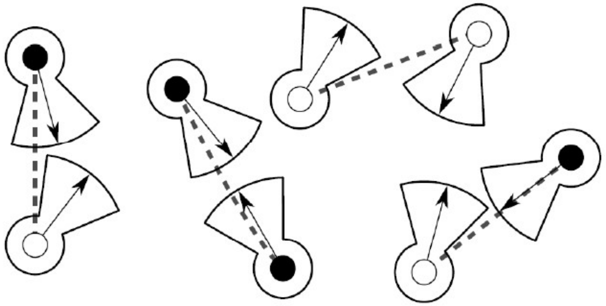
IM Approach: Why So Many Details are Needed?

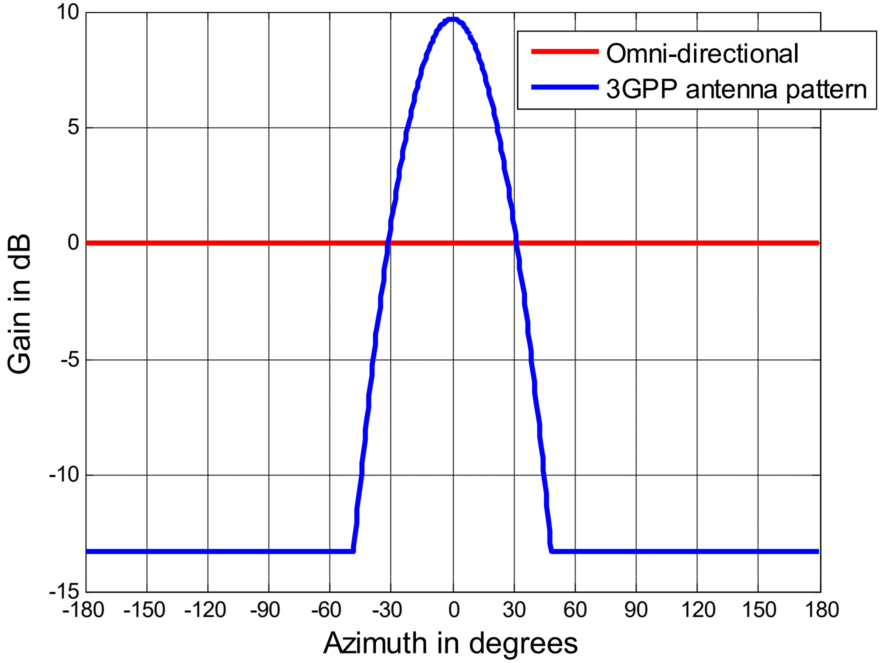
… Impact of Antenna Directionality …

Omni-directional antennas



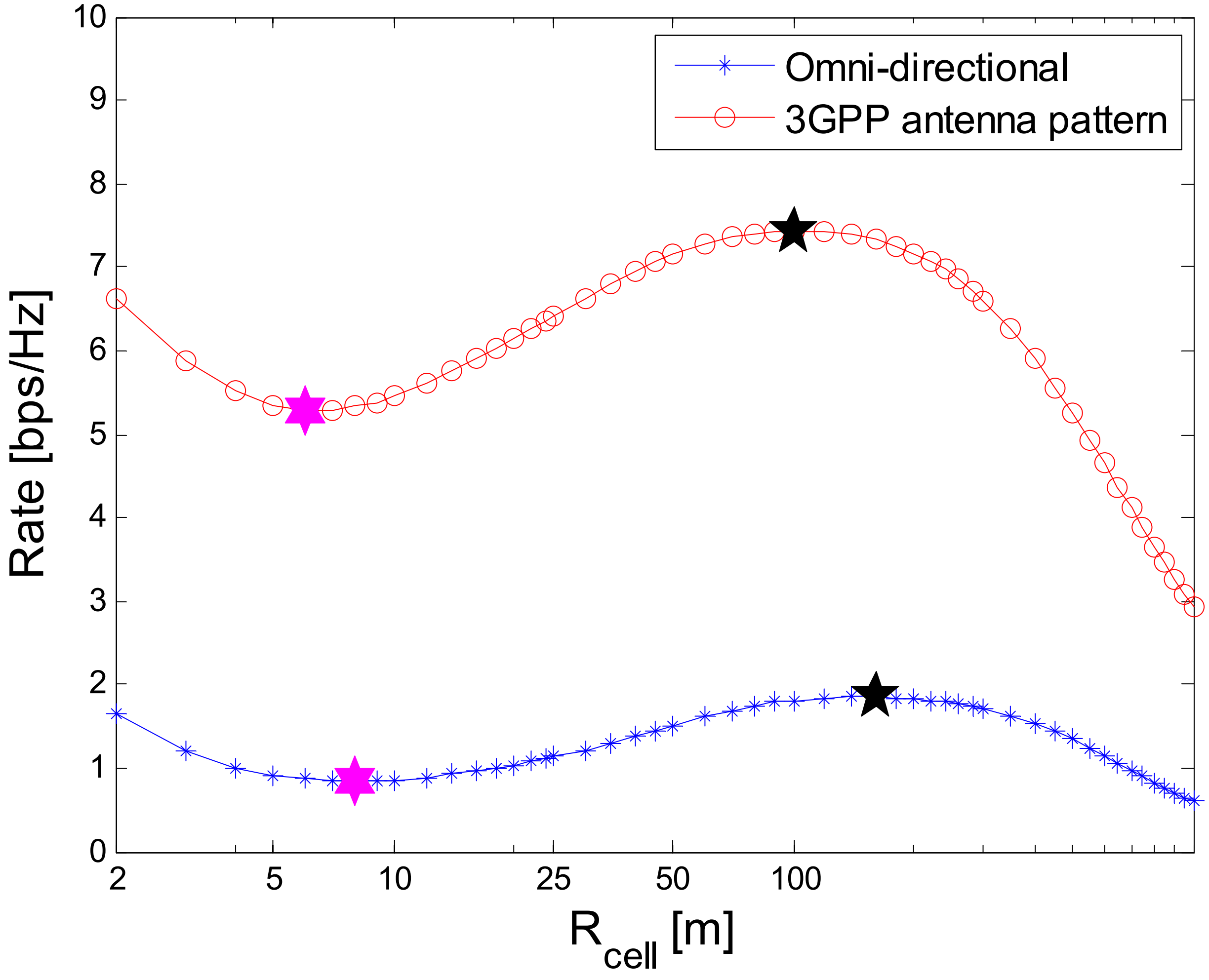
Directional antennas





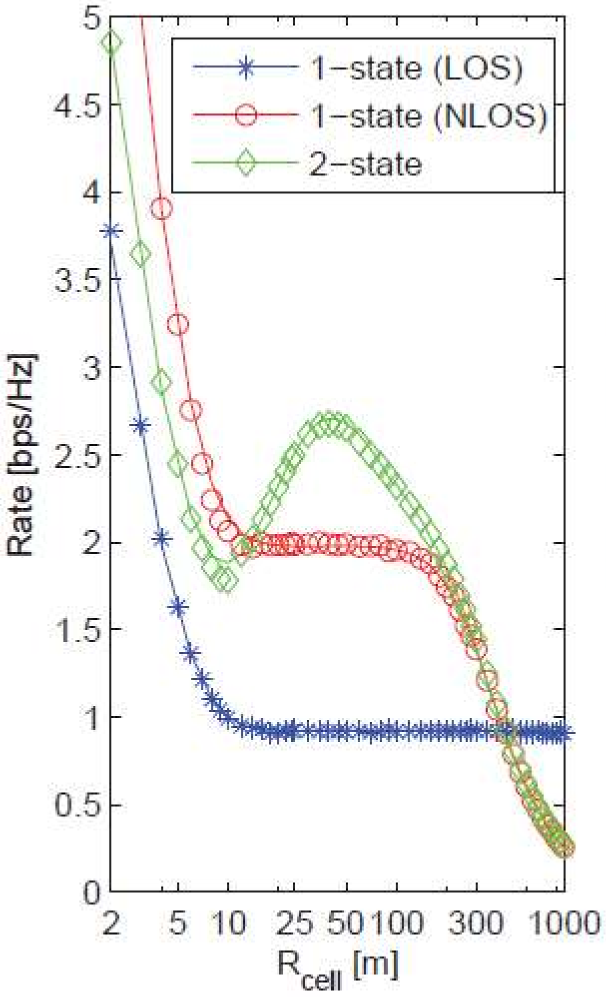
IM Approach: Why So Many Details are Needed?

… Impact of Antenna Directionality …

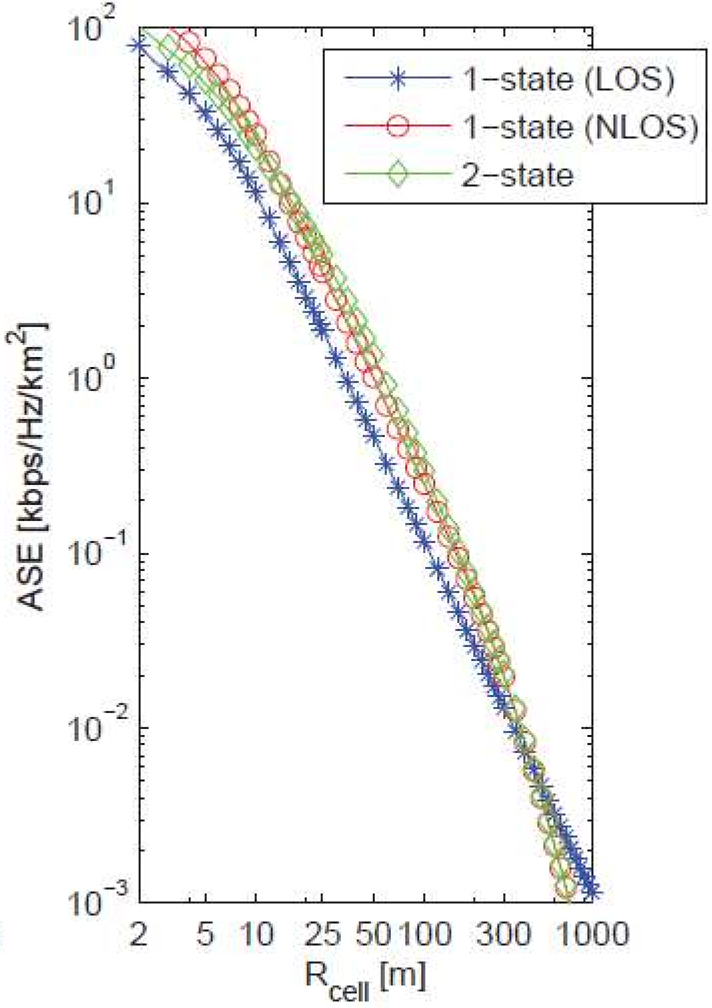


IM Approach: Why So Many Details are Needed?

… Sub-Linear Trend of the Area Spectral Efficiency = λ·Rate …

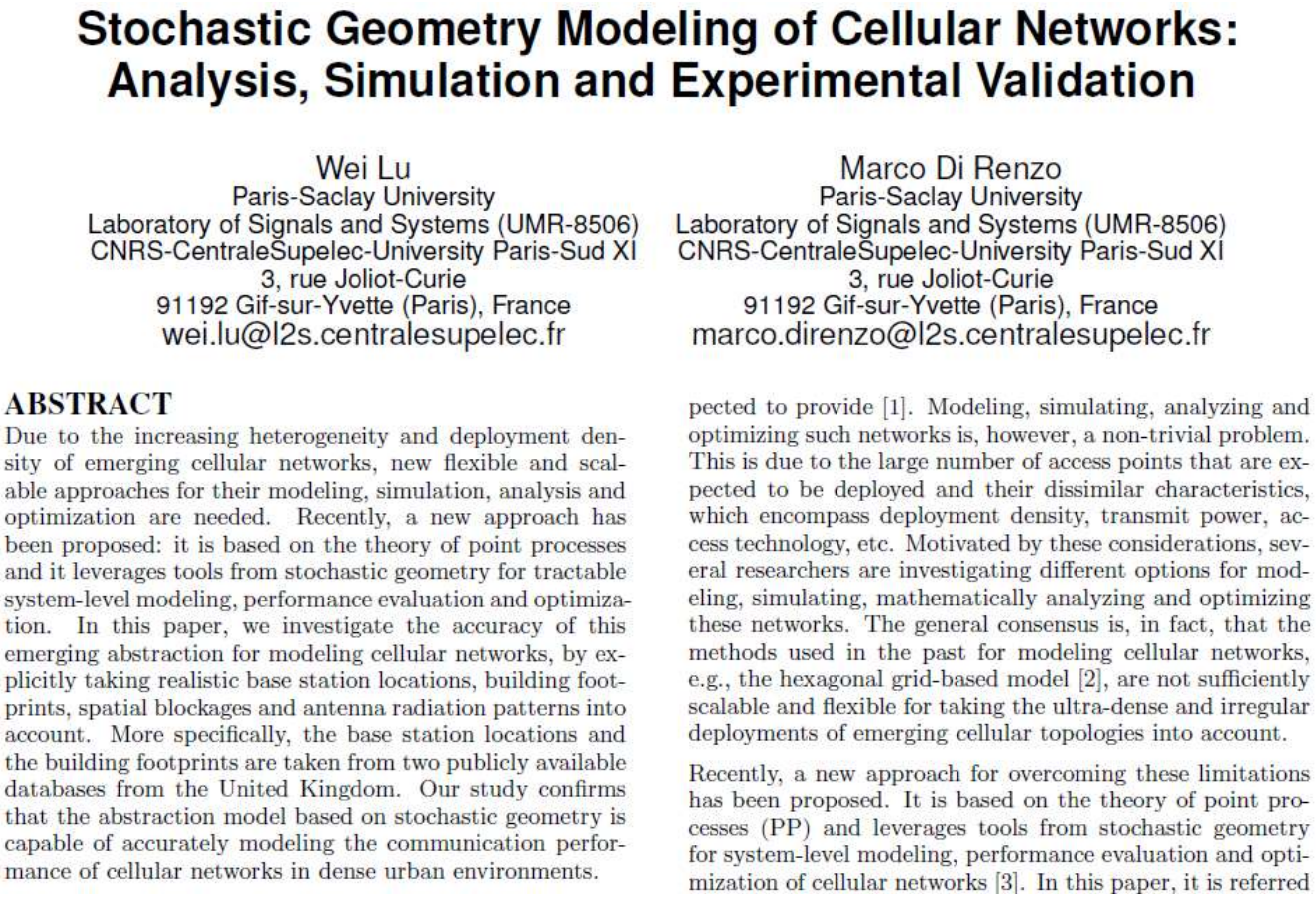


RATE



ASE

Intrigued Enough? On Experimental Validation…



Stochastic Geometry Modeling of Cellular Networks: Analysis, Simulation and Experimental Validation

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ABSTRACT

Due to the increasing heterogeneity and deployment density of emerging cellular networks, new flexible and scalable approaches for their modeling, simulation, analysis and optimization are needed. Recently, a new approach has been proposed: it is based on the theory of point processes and it leverages tools from stochastic geometry for tractable system-level modeling, performance evaluation and optimization. In this paper, we investigate the accuracy of this emerging abstraction for modeling cellular networks, by explicitly taking realistic base station locations, building footprints, spatial blockages and antenna radiation patterns into account. More specifically, the base station locations and the building footprints are taken from two publicly available databases from the United Kingdom. Our study confirms that the abstraction model based on stochastic geometry is capable of accurately modeling the communication performance of cellular networks in dense urban environments.

pected to provide [1]. Modeling, simulating, analyzing and optimizing such networks is, however, a non-trivial problem. This is due to the large number of access points that are expected to be deployed and their dissimilar characteristics, which encompass deployment density, transmit power, access technology, etc. Motivated by these considerations, several researchers are investigating different options for modeling, simulating, mathematically analyzing and optimizing these networks. The general consensus is, in fact, that the methods used in the past for modeling cellular networks, e.g., the hexagonal grid-based model [2], are not sufficiently scalable and flexible for taking the ultra-dense and irregular deployments of emerging cellular topologies into account.

Recently, a new approach for overcoming these limitations has been proposed. It is based on the theory of point processes (PP) and leverages tools from stochastic geometry for system-level modeling, performance evaluation and optimization of cellular networks [3]. In this paper, it is referred

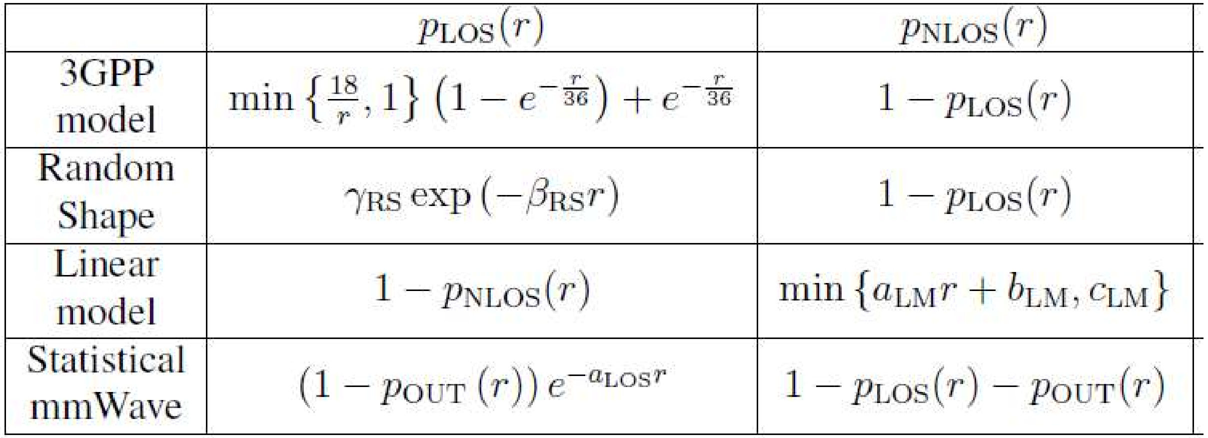
W. Lu and M. Di Renzo, "Stochastic Geometry Modeling of Cellular Networks: Analysis, Simulation and Experimental Validation", ACM Int. Conf. Modeling, Analysis and Simulation of Wireless and Mobile Systems, Nov. 2015. [Online]. Available: http://arxiv.org/pdf/1506.03857.pdf.

W. Lu and M. Di Renzo, "Stochastic Geometry Modeling of mmWave Cellular Networks: Analysis and Experimental Validation", IEEE Int. Workshop on Measurement and Networking (M&N) – Special Session on Advances in 5G Wireless Networks, Oct. 12-13, 2015.

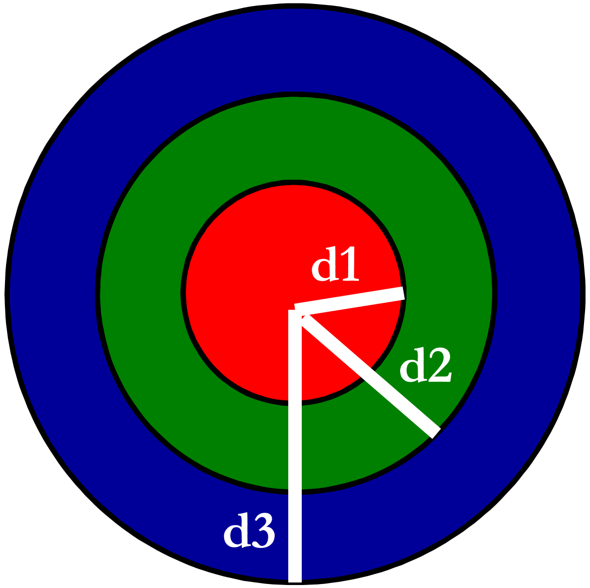
Rationale of the IM Approach: Multi-Ball Approximation

… the approach (e.g., 3-ball case) …

- ❑ **Practical link-state models are approximated using a multi-ball model**
- ❑ **The related parameters are computed using the “intensity matching” criterion**



	$p_{\mathrm{LOS}}(r)$	$p_{\mathrm{NLOS}}(r)$
3GPP model	$\min\left\{\frac{18}{r},1\right\}\left(1-e^{-\frac{r}{36}}\right)+e^{-\frac{r}{36}}$	$1-p_{\mathrm{LOS}}(r)$
Random Shape	$\gamma_{\mathrm{RS}}\exp\left(-\beta_{\mathrm{RS}}r\right)$	$1-p_{\mathrm{LOS}}(r)$
Linear model	$1-p_{\mathrm{NLOS}}(r)$	$\min\left\{a_{\mathrm{LM}}r+b_{\mathrm{LM}},c_{\mathrm{LM}}\right\}$
Statistical mmWave	$\left(1-p_{\mathrm{OUT}}\left(r\right)\right)e^{-a_{\mathrm{LOS}}r}$	$1-p_{\mathrm{LOS}}(r)-p_{\mathrm{OUT}}(r)$



$$p_S\left(r\right)=\sum_{n=1}^{N=3} q_S^{\left[d_{n-1},d_n\right]}\mathbf{1}_{\left[d_{n-1},d_n\right)}\left(r\right) \quad \text{with} \quad \sum_{S\in\{\text{LOS,NLOS,...}\}} q_S^{\left[d_{n-1},d_n\right]}=1 \quad \forall n=1,2,\ldots,N$$

$$\text{minimize}\left\{\left\|\ln\left(\sum_{S\in\{\text{LOS,NLOS,}\ldots\}}\Lambda_{\Psi_S}^{(\text{actual})}\left(\left[0,x_{\max}\right)\right)\right)-\ln\left(\sum_{S\in\{\text{LOS,NLOS,}\ldots\}}\Lambda_{\Psi_S}^{(\text{approx})}\left(\left[0,x_{\max}\right)\right)\right)\right\|_F^2\right\}$$

Rationale of the IM Approach: Multi-Ball Approximation

Why Matching the Intensity Measures ?

- ❑ **Consider the general association criterion as follows:**

$$\mathrm{BS}_0 \text{ is chosen as the minimum of the set } \Psi = \left\{ \frac{l(r_n)}{\Upsilon_n}, n \in \Phi \right\}$$

Φ is a (non-homogeneous) PPP of BSs with density λ(r) = λ*p(r)

l(r) denotes the path-loss function

ϒ is a random variable that accounts for all random variables that are taken into account for cell association except for the distance (e.g., shadowing)

- ❑ **Based on the displacement theorem of PPPs, the set Ψ is a PPP in R^+ whose intensity measure is the following:**

$$\Lambda_\Psi([0,x)) = 2\pi\lambda \int_0^{+\infty} \Pr\left\{ \frac{l(r)}{\Upsilon} \in [0,x) \right\} p(r) r dr$$

$$= 2\pi\lambda \mathrm{E}_\Upsilon \left\{ \int_0^{+\infty} \Pr\left\{ l(r) \in [0, x\xi) \middle| \Upsilon = \xi \right\} p(r) r dr \right\}$$

Rationale of the IM Approach: Multi-Ball Approximation

Why Matching the Intensity Measures ?

- **Since the intensity measure is now known and Ψ is still a PPP, the coverage probability can be formulated, after some algebra, as follows:**

$$\tilde{l}_{\mathrm{LOS}} = \min_{\Psi_{\mathrm{LOS}}} \left\{ l_{\mathrm{LOS}}(r) / \Upsilon_{\mathrm{LOS}} \right\} \qquad \tilde{l}_{\mathrm{NLOS}} = \min_{\Psi\,\mathrm{NLOS}} \left\{ l_{\mathrm{NLOS}}(r) / \Upsilon_{\mathrm{NLOS}} \right\}$$

$$\boxed{\begin{aligned}
\mathrm{P}_{\mathrm{cov}} &= \mathrm{E}_{\tilde{l}_{\mathrm{LOS}}} \left\{ \Pr\left\{ \frac{P\left|\tilde{h}_{o,\mathrm{LOS}}\right|^2 / \tilde{l}_{\mathrm{LOS}}}{\sigma^2 + I_{agg}\left(\tilde{l}_{\mathrm{LOS}}\right)} > T \,\middle|\, \tilde{l}_{\mathrm{LOS}} \right\} \Pr\left\{ \tilde{l}_{\mathrm{NLOS}} > \tilde{l}_{\mathrm{LOS}} \,\middle|\, \tilde{l}_{\mathrm{LOS}} \right\} \right\} \\
&\quad + \mathrm{E}_{\tilde{l}_{\mathrm{NLOS}}} \left\{ \Pr\left\{ \frac{P\left|\tilde{h}_{o,\mathrm{NLOS}}\right|^2 / \tilde{l}_{\mathrm{NLOS}}}{\sigma^2 + I_{agg}\left(\tilde{l}_{\mathrm{NLOS}}\right)} > T \,\middle|\, \tilde{l}_{\mathrm{LOS}} \right\} \Pr\left\{ \tilde{l}_{\mathrm{LOS}} > \tilde{l}_{\mathrm{NLOS}} \,\middle|\, \tilde{l}_{\mathrm{NLOS}} \right\} \right\} \\
&= \int_0^{+\infty} \Pr\left\{ \frac{P\left|\tilde{h}_{o,\mathrm{LOS}}\right|^2 / x}{\sigma^2 + I_{agg}(x)} > T \,\middle|\, x \right\} \mathrm{CCDF}_{\tilde{l}_{\mathrm{NLOS}}}(x)\, \mathrm{PDF}_{\tilde{l}_{\mathrm{LOS}}}(x)\, dx \\
&\quad + \int_0^{+\infty} \Pr\left\{ \frac{P\left|\tilde{h}_{o,\mathrm{NLOS}}\right|^2 / y}{\sigma^2 + I_{agg}(y)} > T \,\middle|\, y \right\} \mathrm{CCDF}_{\tilde{l}_{\mathrm{LOS}}}(y)\, \mathrm{PDF}_{\tilde{l}_{\mathrm{NLOS}}}(y)\, dy
\end{aligned}}$$

Rationale of the IM Approach: Multi-Ball Approximation

Why Matching the Intensity Measures ?

$$\mathrm{CCDF}_{\tilde{l}_S}(\xi) \overset{\text{void probability th.}}{=} \exp\left(-\Lambda_{\Psi_S}([0,\xi))\right) \Rightarrow \mathrm{PDF}_{\tilde{l}_S}(\xi) = -d\,\mathrm{CCDF}_{\tilde{l}_S}(\xi)/d\xi$$

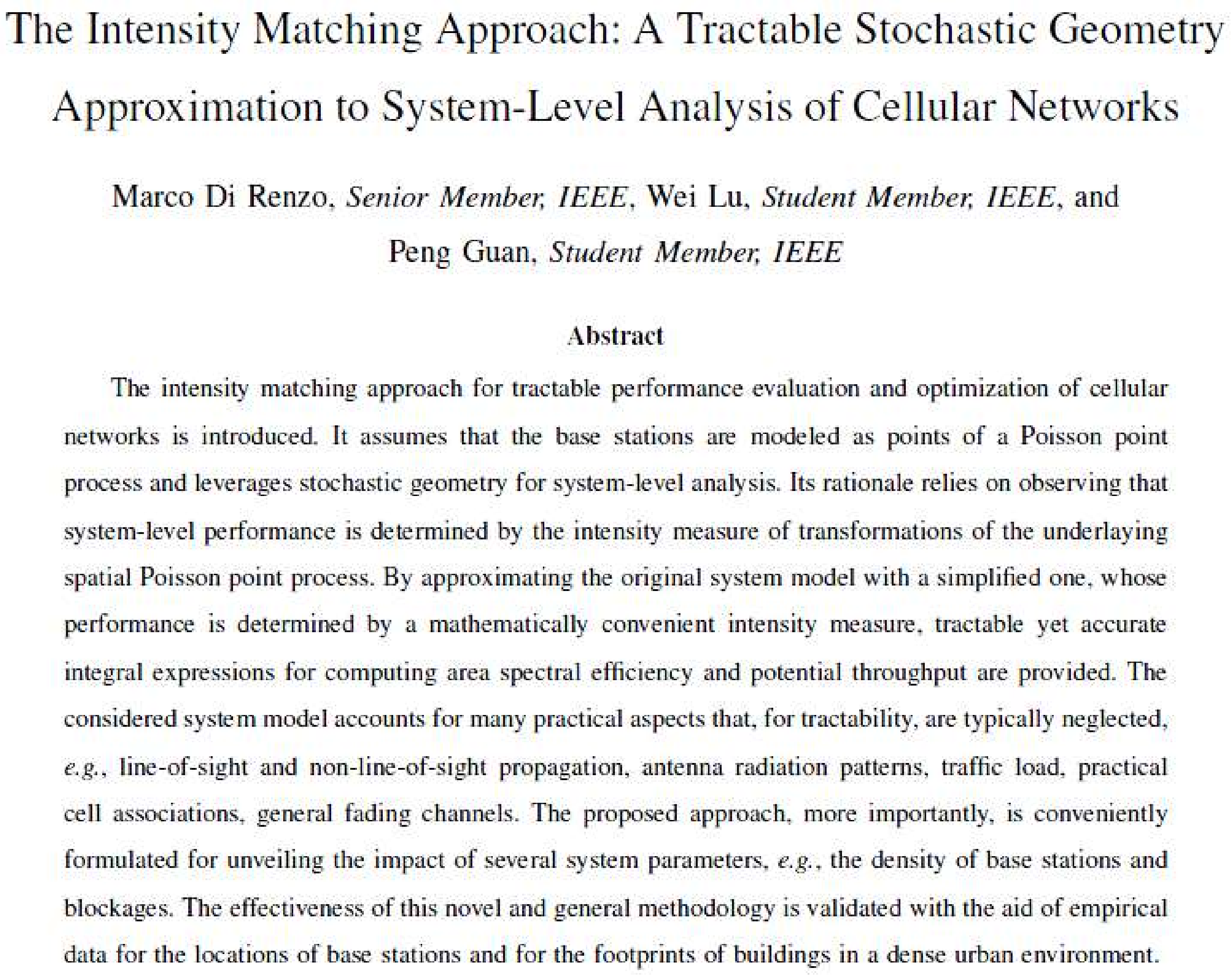
$$\mathrm{MGF}_{I_{agg},Q}\left(w;\tilde{l}_S\right)\left(\Leftarrow \mathrm{MGF}_{I_{agg}}\left(w;\tilde{l}_S\right) = \mathrm{MGF}_{I_{agg},\mathrm{LOS}}\left(w;\tilde{l}_S\right)\mathrm{MGF}_{I_{agg},\mathrm{NLOS}}\left(w;\tilde{l}_S\right)\right)$$

$$= \mathrm{E}_{\Phi_Q,\left\{\left|\tilde{h}_{k,Q}\right|^2\right\}}\left\{\exp\left(-w\sum\nolimits_{k\in\Phi_Q}\left(P\left|\tilde{h}_{k,Q}\right|^2/\tilde{l}_{k,Q}\right)\mathbf{1}\left(\tilde{l}_{k,Q}>\tilde{l}_S\right)\right)\right\}$$

$$= \mathrm{E}_{\Phi_Q}\left\{\prod\nolimits_{k\in\Phi_Q}\mathrm{E}_{\left\{\left|\tilde{h}_{k,Q}\right|^2\right\}}\left\{\exp\left(-w\left(P\left|\tilde{h}_{k,Q}\right|^2/\tilde{l}_{k,Q}\right)\mathbf{1}\left(\tilde{l}_{k,Q}>\tilde{l}_S\right)\right)\right\}\right\}$$

$$\overset{\text{PGFL}}{=} \exp\left(-\int_{\tilde{l}_S}^{+\infty}\left(1-\mathrm{E}_{\left|\tilde{h}_{k,Q}\right|^2}\left\{\exp\left(-w\left(P\left|\tilde{h}_{k,Q}\right|^2/l\right)\right)\right\}\right)\underbrace{\Lambda^{(1)}_{\Psi_Q}([0,l))}_{=d/dl\left(\Lambda_{\Psi_Q}([0,l))\right)}dl\right)$$

Intrigued Enough? On Mathematical Modeling…



The Intensity Matching Approach: A Tractable Stochastic Geometry Approximation to System-Level Analysis of Cellular Networks

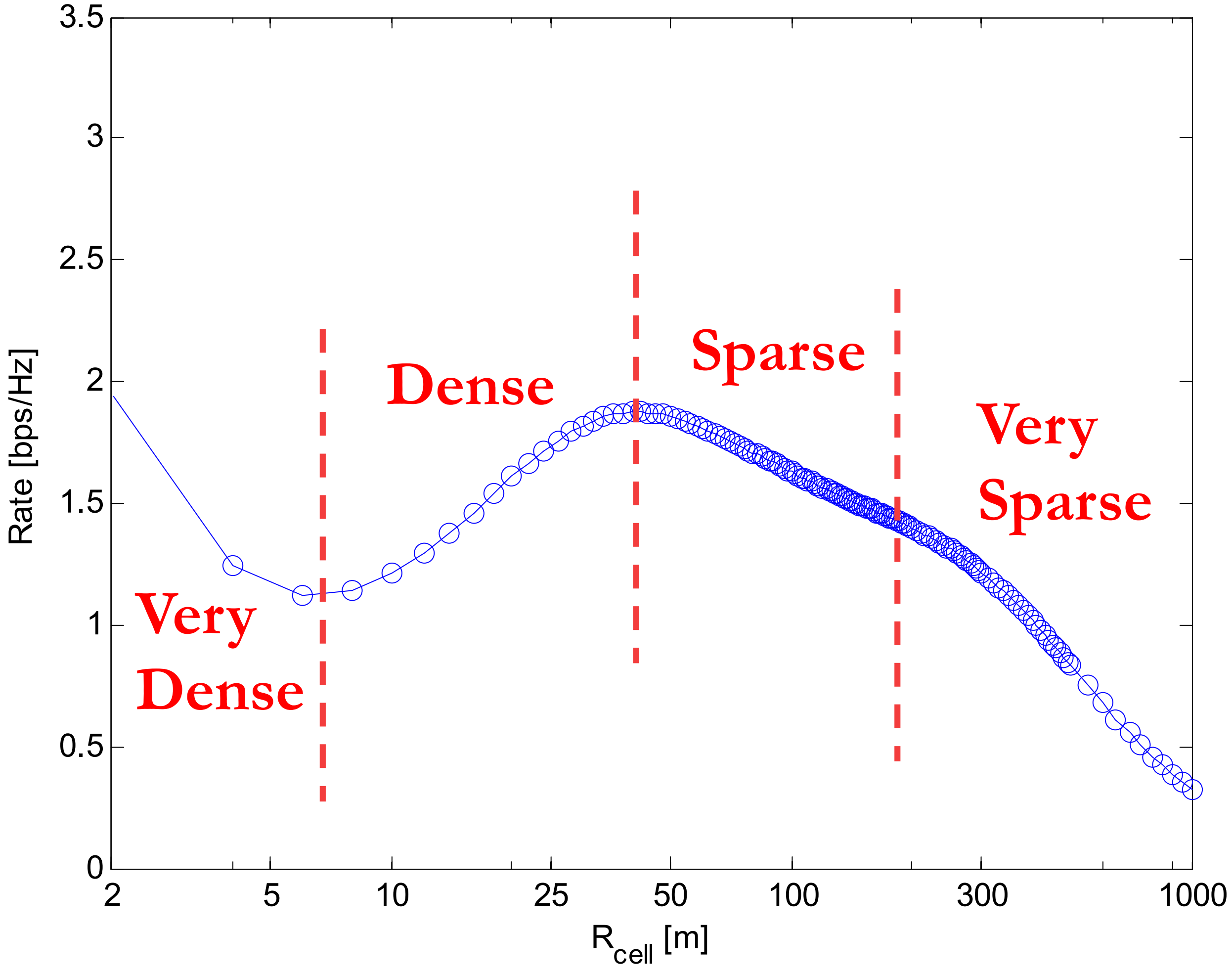
Marco Di Renzo, *Senior Member, IEEE*, Wei Lu, *Student Member, IEEE*, and
Peng Guan, *Student Member, IEEE*

Abstract

The intensity matching approach for tractable performance evaluation and optimization of cellular networks is introduced. It assumes that the base stations are modeled as points of a Poisson point process and leverages stochastic geometry for system-level analysis. Its rationale relies on observing that system-level performance is determined by the intensity measure of transformations of the underlaying spatial Poisson point process. By approximating the original system model with a simplified one, whose performance is determined by a mathematically convenient intensity measure, tractable yet accurate integral expressions for computing area spectral efficiency and potential throughput are provided. The considered system model accounts for many practical aspects that, for tractability, are typically neglected, *e.g.*, line-of-sight and non-line-of-sight propagation, antenna radiation patterns, traffic load, practical cell associations, general fading channels. The proposed approach, more importantly, is conveniently formulated for unveiling the impact of several system parameters, *e.g.*, the density of base stations and blockages. The effectiveness of this novel and general methodology is validated with the aid of empirical data for the locations of base stations and for the footprints of buildings in a dense urban environment.

M. Di Renzo, W. Lu, and P. Guan, "The Intensity Matching Approach: A Tractable Stochastic Geometry Approximation to System-Level Analysis of Cellular Networks", IEEE Trans. Wireless Commun., to appear.

The Intensity Matching Approach: Main Takes



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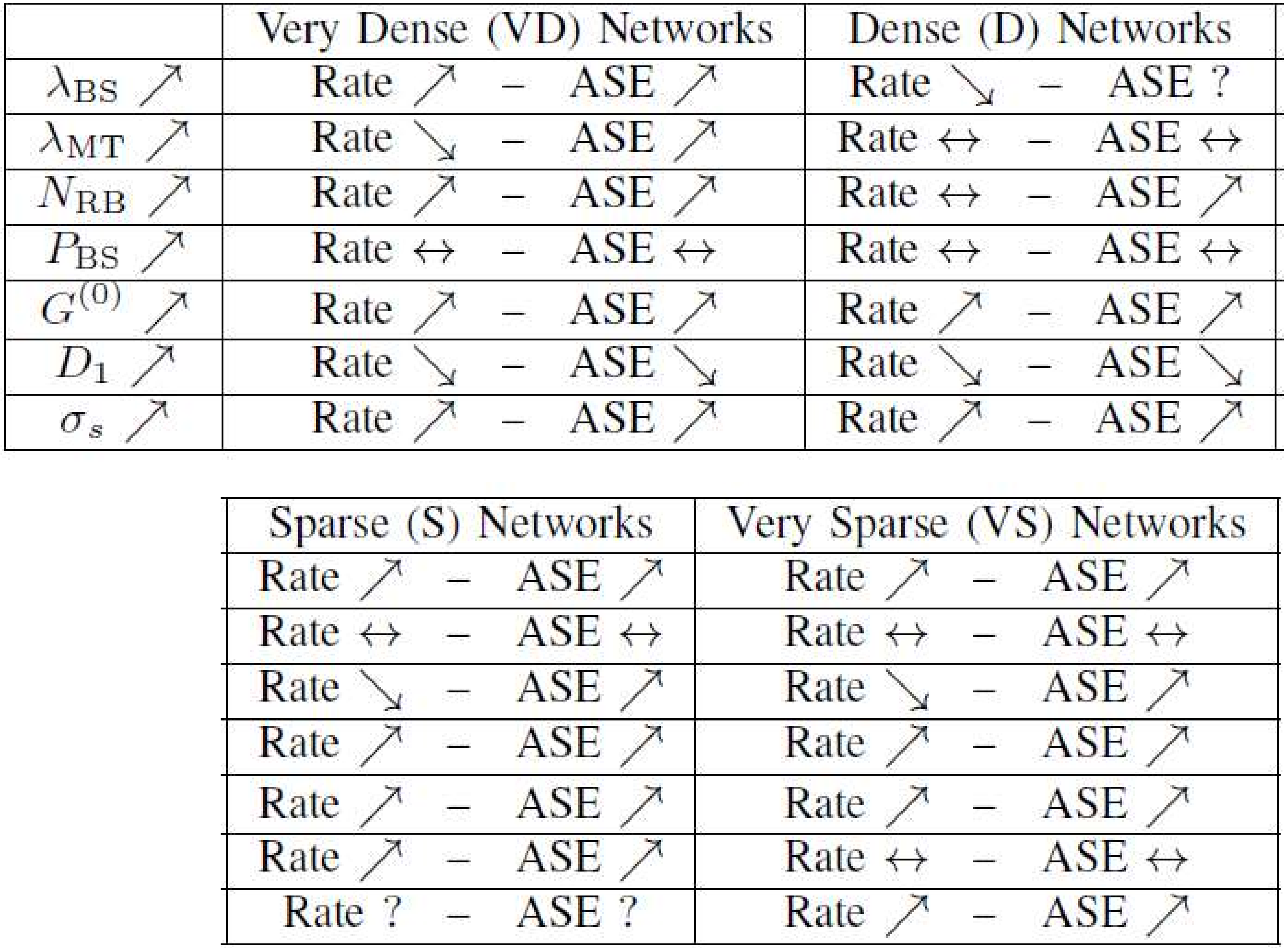

	Very Dense (VD) Networks	Dense (D) Networks	Sparse (S) Networks	Very Sparse (VS) Networks
$\lambda_{\mathrm{BS}} \nearrow$	Rate ↗ – ASE ↗	Rate ↘ – ASE ?	Rate ↗ – ASE ↗	Rate ↗ – ASE ↗
$\lambda_{\mathrm{MT}} \nearrow$	Rate ↘ – ASE ↗	Rate ↔ – ASE ↔	Rate ↔ – ASE ↔	Rate ↔ – ASE ↔
$N_{\mathrm{RB}} \nearrow$	Rate ↗ – ASE ↗	Rate ↔ – ASE ↗	Rate ↘ – ASE ↗	Rate ↘ – ASE ↗
$P_{\mathrm{BS}} \nearrow$	Rate ↔ – ASE ↔	Rate ↔ – ASE ↔	Rate ↗ – ASE ↗	Rate ↗ – ASE ↗
$G^{(0)} \nearrow$	Rate ↗ – ASE ↗	Rate ↗ – ASE ↗	Rate ↗ – ASE ↗	Rate ↗ – ASE ↗
$D_1 \nearrow$	Rate ↘ – ASE ↘	Rate ↘ – ASE ↘	Rate ↗ – ASE ↗	Rate ↔ – ASE ↔
$\sigma_s \nearrow$	Rate ↗ – ASE ↗	Rate ↗ – ASE ↗	Rate ? – ASE ?	Rate ↗ – ASE ↗

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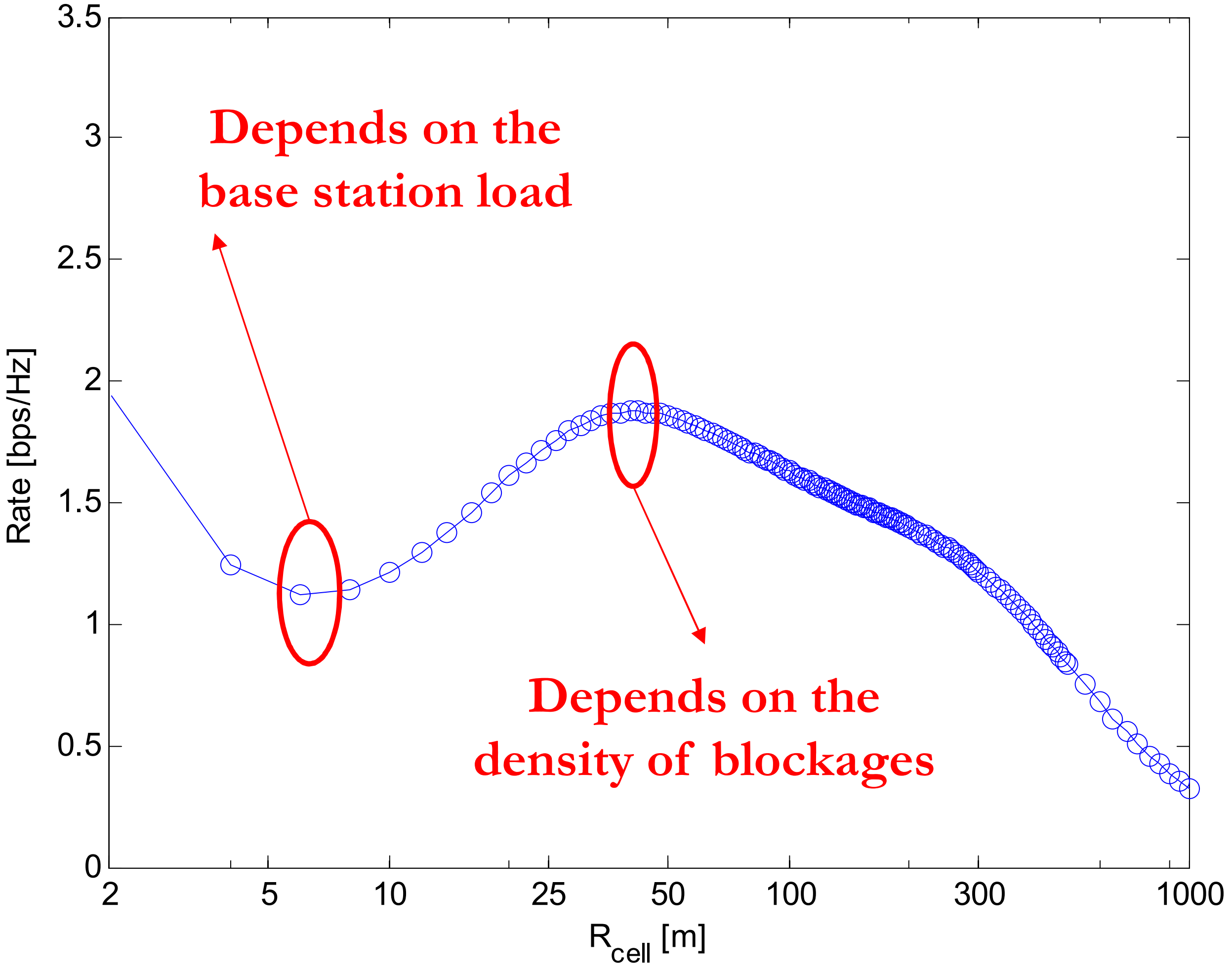
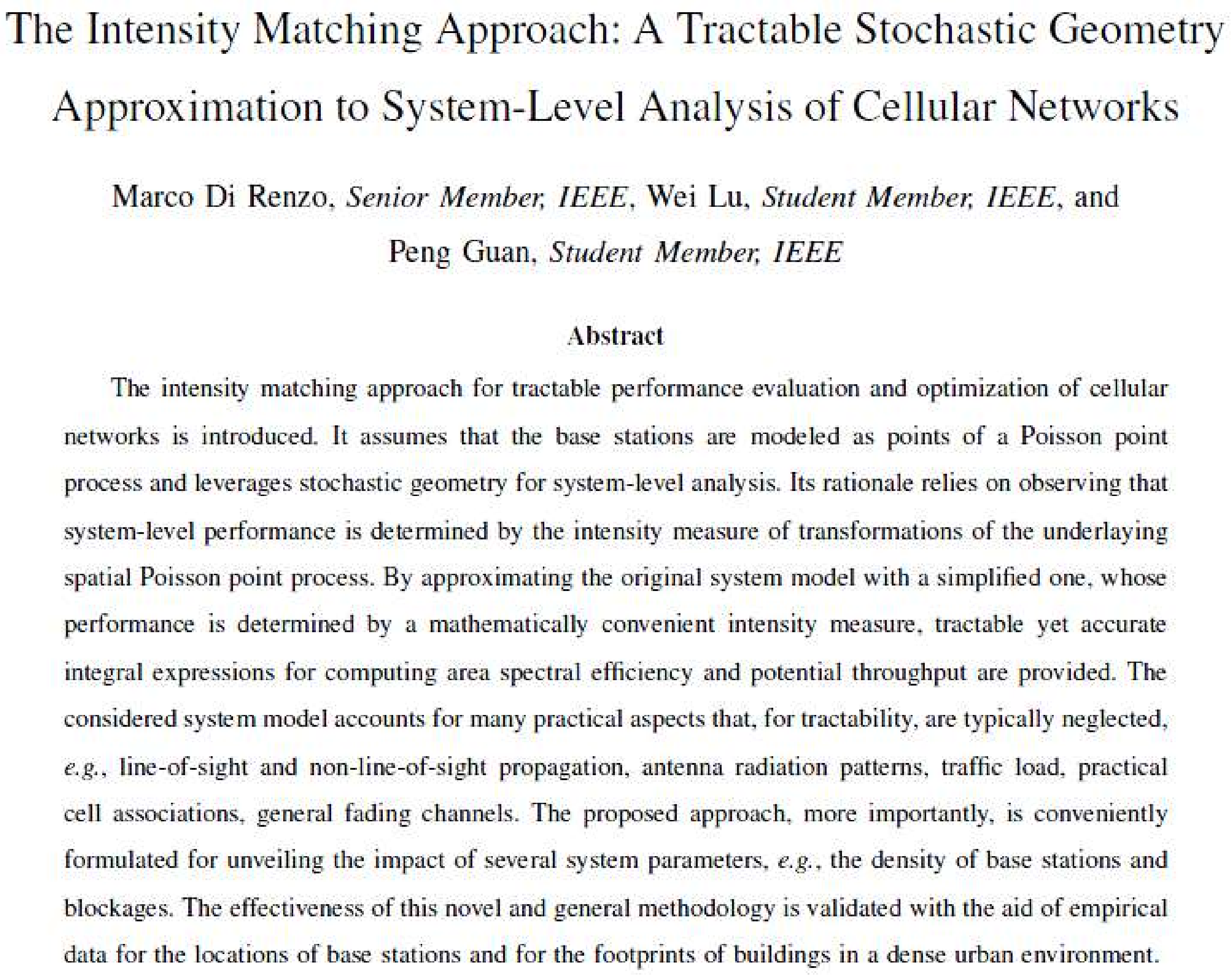


Depends on the base station load

Depends on the density of blockages

M. Di Renzo, W. Lu, and P. Guan, "The Intensity Matching Approach: A Tractable Stochastic Geometry Approximation to System-Level Analysis of Cellular Networks", IEEE Trans. Wireless Commun., **to appear.**

What the IM Approach Allows Us to Model…?



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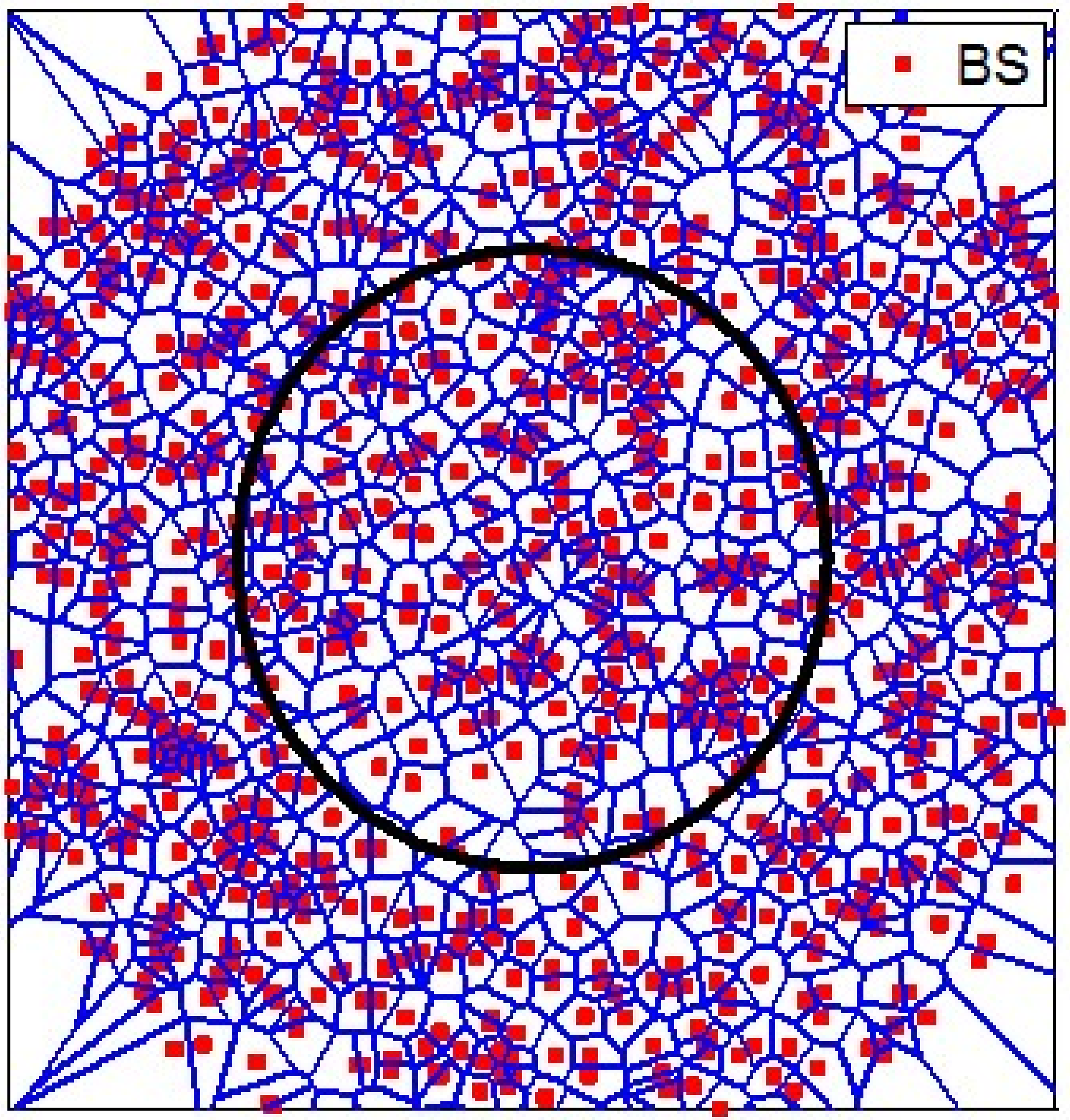
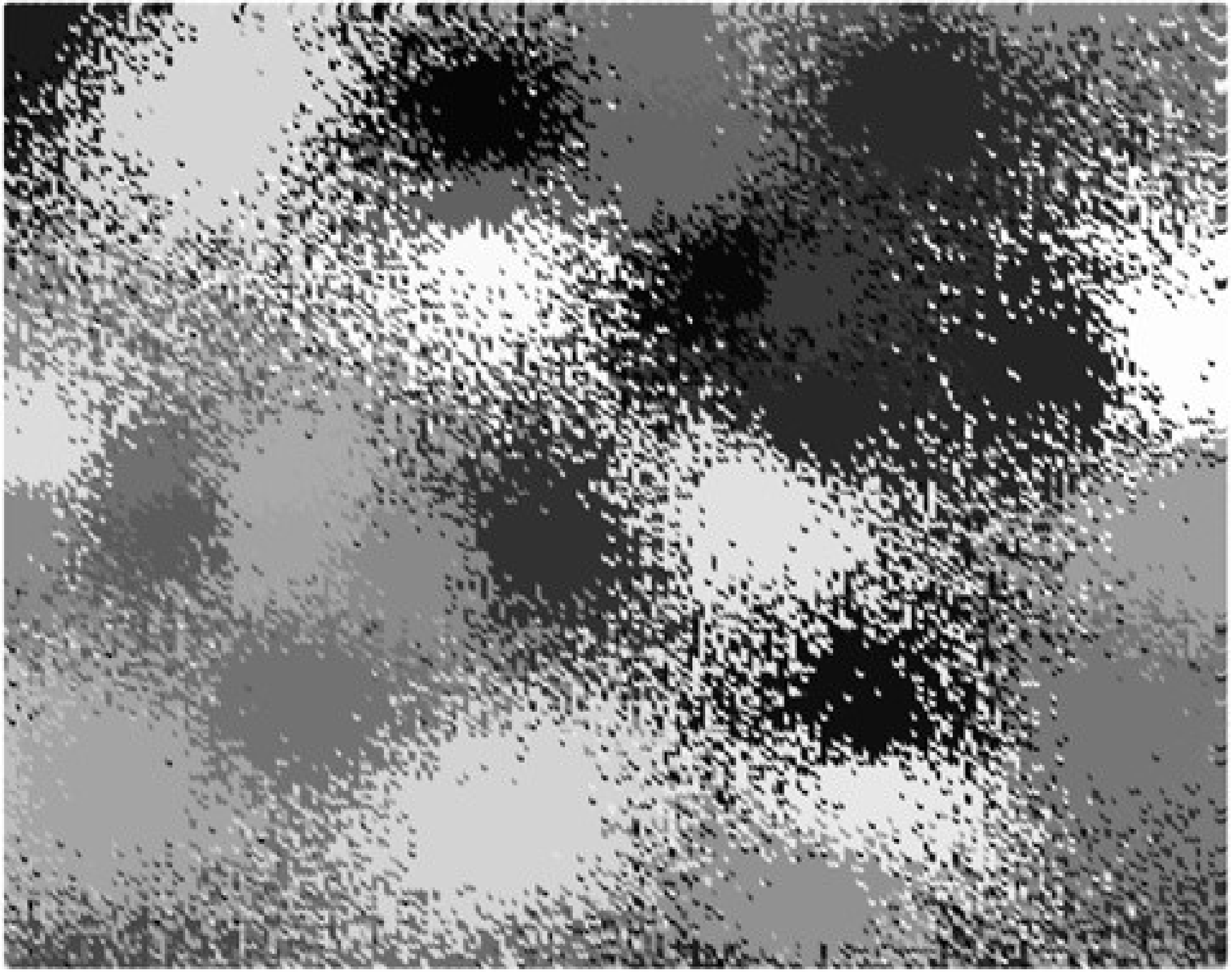
… General Cell Association Criteria

… shortest distance …



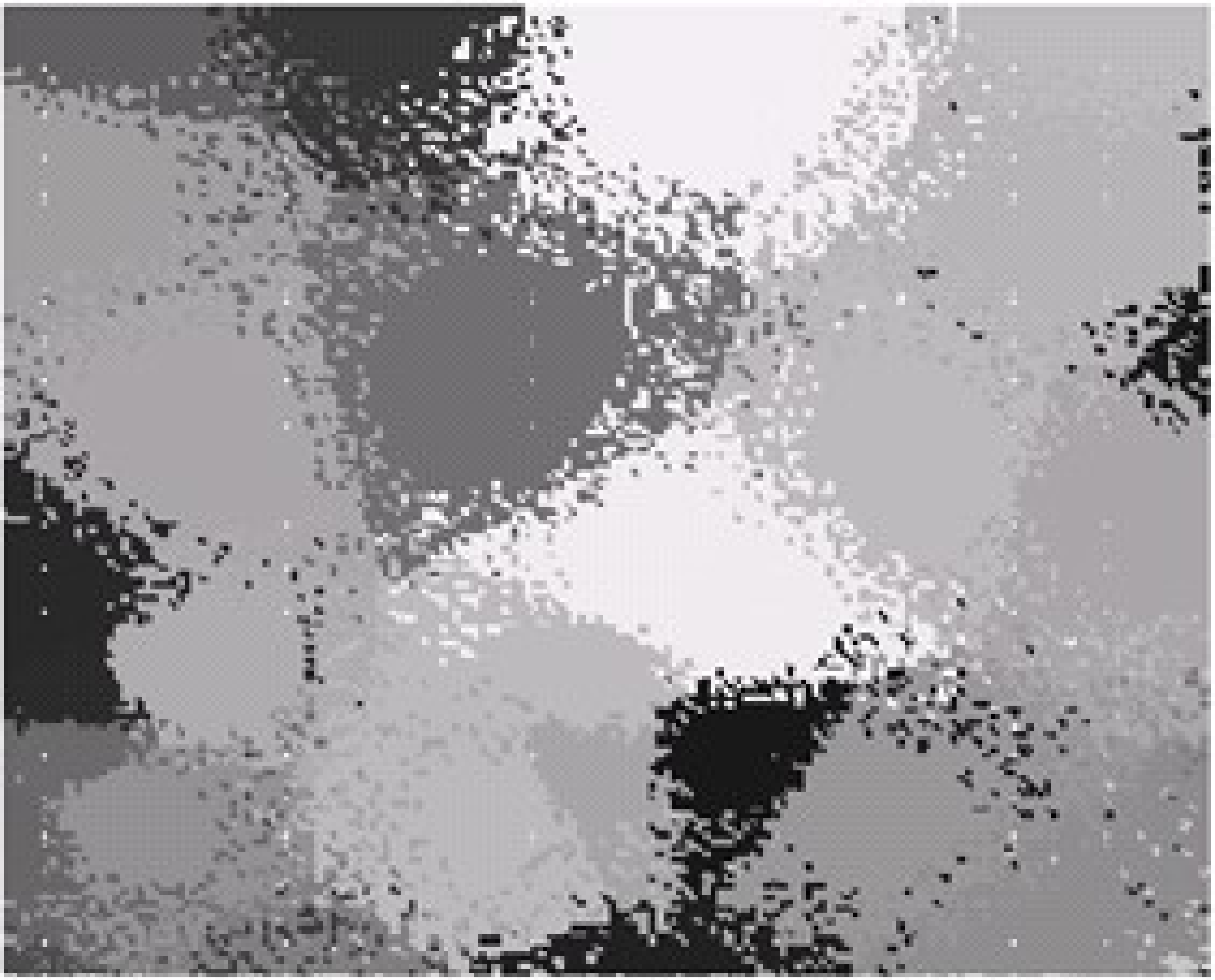
… General Cell Association Criteria

… highest average received power with shadowing …



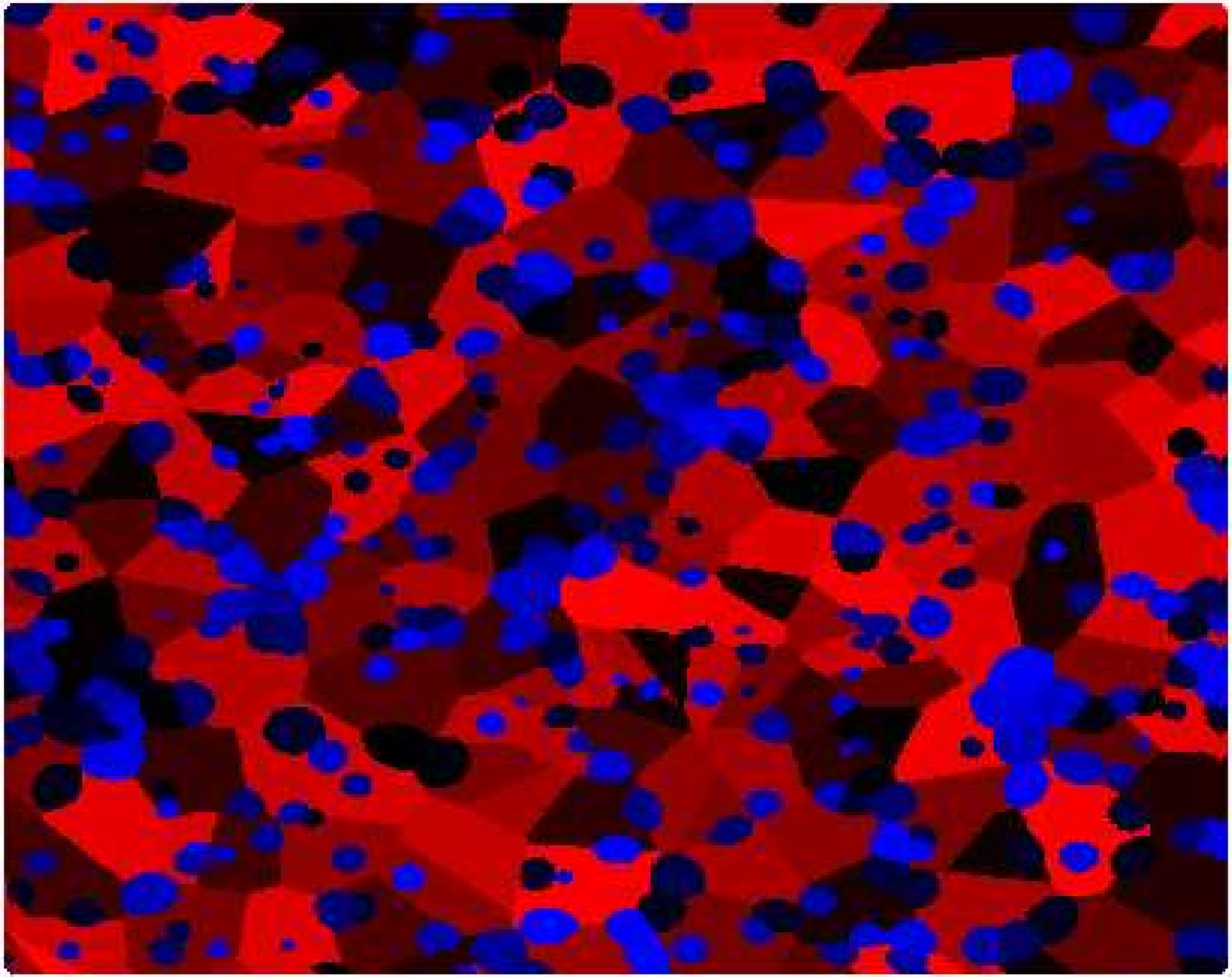
… General Cell Association Criteria

… smallest path-loss with LOS/NLOS links …



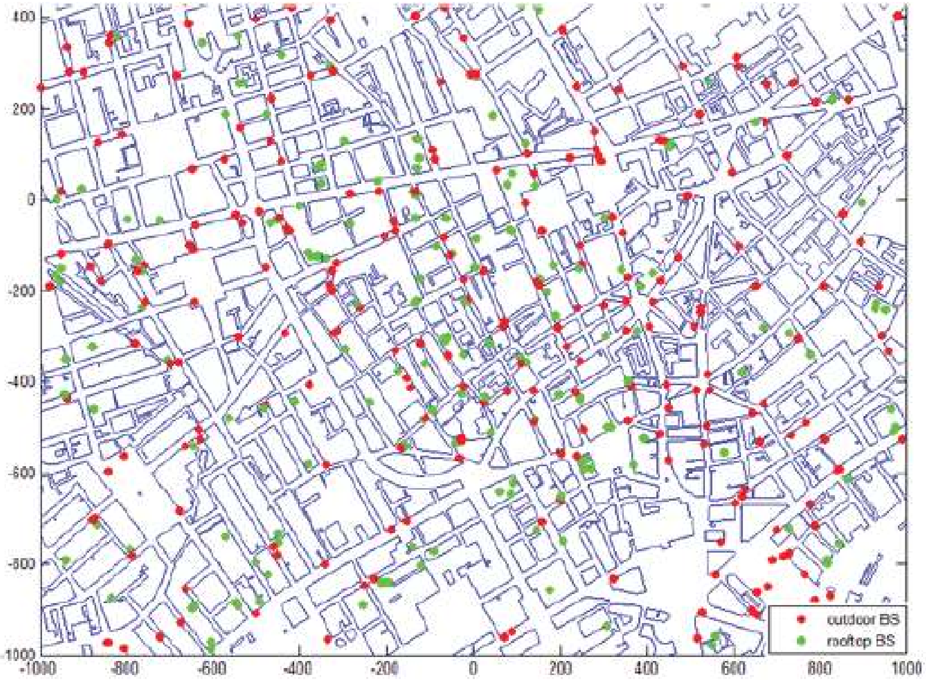
… General Cell Association Criteria

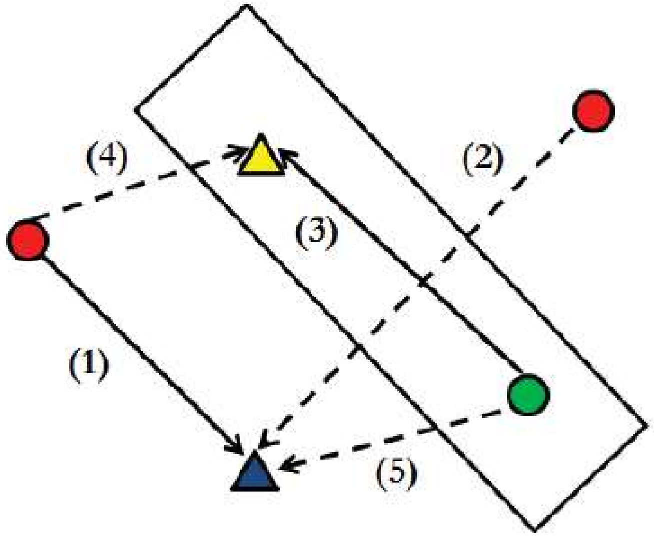
… two-tier biased smallest path-loss …



… Practical Blockage Models

	$p_{\mathrm{LOS}}(r)$	$p_{\mathrm{NLOS}}(r)$	$p_{\mathrm{OUT}}(r)$
3GPP [8]	$\min\left\{\frac{a_{3\mathrm{G}}}{r}, c_{3\mathrm{G}}\right\}\left(1-e^{-\frac{r}{b_{3\mathrm{G}}}}\right)+e^{-\frac{r}{b_{3\mathrm{G}}}}$	$1-p_{\mathrm{LOS}}(r)$	0
Random Shape [15]	$a_{\mathrm{RS}}\exp\left(-b_{\mathrm{RS}}r\right)$	$1-p_{\mathrm{LOS}}(r)$	0
Linear [12]	$1-p_{\mathrm{NLOS}}(r)$	$\min\left\{a_{\mathrm{L}}r+b_{\mathrm{L}}, c_{\mathrm{L}}\right\}$	0
Empirical mmWave [10]	$\left(1-p_{\mathrm{OUT}}\left(r\right)\right)e^{-a_{\mathrm{mm}}r}$	$1-p_{\mathrm{LOS}}(r)-p_{\mathrm{OUT}}(r)$	$\max\left\{0, 1-e^{-b_{\mathrm{mm}}r+c_{\mathrm{mm}}}\right\}$
Two-ball mmWave [10]	see (1) with $\mathcal{S}=3$, $s=\{\mathrm{LOS},\mathrm{NLOS},\mathrm{OUT}\}$, $\mathcal{B}=2$		





… Practical Load Models

Lemma 2: Consider the triplet $\{\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\}$. $p_{\mathrm{sel}}(\cdot)$ can be formulated as follows:

$$p_{\mathrm{sel}}\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\right) = 1 - f_{\mathrm{sel}}^{(a)}\left(f_{\mathrm{sel}}^{(b)} - f_{\mathrm{sel}}^{(c)}\right)$$

where ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$ is the Gauss hypergeometric function and:

$$\begin{aligned}
f_{\mathrm{sel}}^{(a)} &= f_{\mathrm{sel}}^{(a)}\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\right) = \frac{3.5^{4.5}\Gamma(4.5+N_{\mathrm{RB}})}{\Gamma(4.5)}\left(\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}\right)^{N_{\mathrm{RB}}}\left(\frac{1}{3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}\right)^{4.5+N_{\mathrm{RB}}} \\
f_{\mathrm{sel}}^{(b)} &= f_{\mathrm{sel}}^{(b)}\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\right) = \frac{1}{\Gamma(1+N_{\mathrm{RB}})}{}_2F_1\left(1, 4.5+N_{\mathrm{RB}}, 1+N_{\mathrm{RB}}, \frac{\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}{3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}\right) \\
f_{\mathrm{sel}}^{(c)} &= f_{\mathrm{sel}}^{(c)}\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\right) = \frac{N_{\mathrm{RB}}}{\Gamma(2+N_{\mathrm{RB}})}{}_2F_1\left(1, 4.5+N_{\mathrm{RB}}, 2+N_{\mathrm{RB}}, \frac{\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}{3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}\right)
\end{aligned}$$

… Practical Load Models

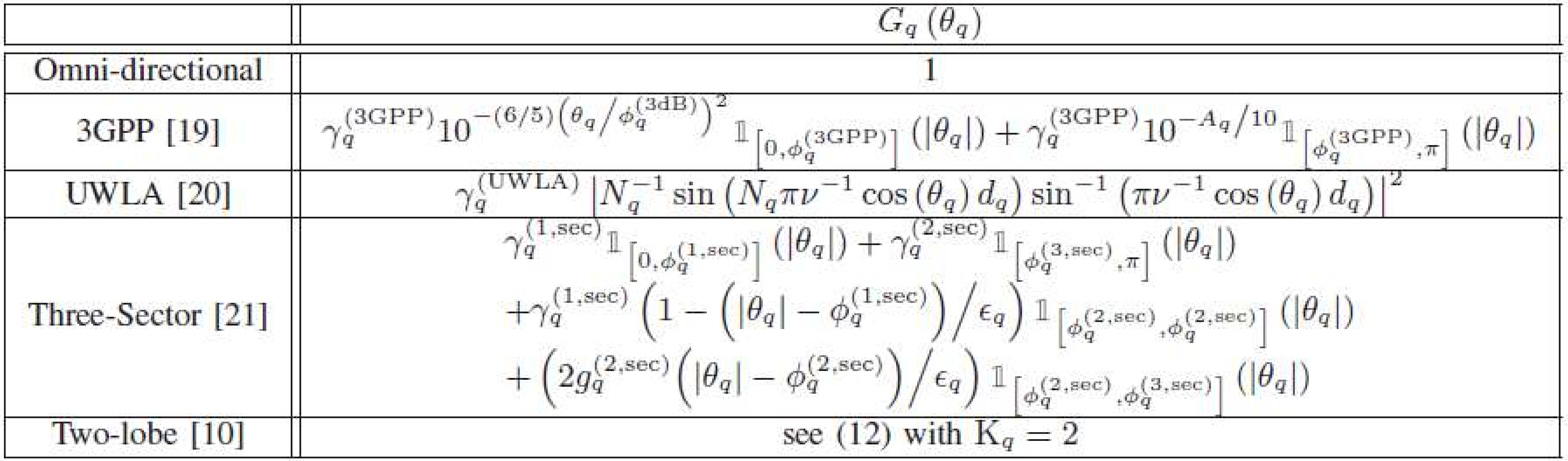
Lemma 3: Consider the triplet $\{\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\}$. $p_{\mathrm{off}}(\cdot)$ can be formulated as follows:

$$p_{\mathrm{off}}\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\right) = 1 - \lambda_{\mathrm{MT}}/(\lambda_{\mathrm{BS}}N_{\mathrm{RB}}) - p_{\mathrm{off}}^{(a)} + p_{\mathrm{off}}^{(b)} + p_{\mathrm{off}}^{(c)}$$

where $p_{\mathrm{off}}^{(x)} = p_{\mathrm{off}}^{(x)}\left(\lambda_{\mathrm{BS}}, \lambda_{\mathrm{MT}}, N_{\mathrm{RB}}\right)$ for $x = \{a, b, c, \}$ are as follows:

$$\begin{aligned}
p_{\mathrm{off}}^{(a)} &= \frac{3.5^{3.5}\Gamma(4.5+N_{\mathrm{RB}})}{\Gamma(3.5)\Gamma(2+N_{\mathrm{RB}})} \frac{(\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}})^{1+N_{\mathrm{RB}}}}{(3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}})^{4.5+N_{\mathrm{RB}}}} {}_2F_1\left(1, 4.5 + N_{\mathrm{RB}}, 2 + N_{\mathrm{RB}}, \frac{\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}{3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}\right) \\
p_{\mathrm{off}}^{(b)} &= \frac{3.5^{3.5}\Gamma(4.5+N_{\mathrm{RB}})}{\Gamma(3.5)N_{\mathrm{RB}}\Gamma(1+N_{\mathrm{RB}})} \frac{(\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}})^{1+N_{\mathrm{RB}}}}{(3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}})^{4.5+N_{\mathrm{RB}}}} {}_2F_1\left(1, 4.5 + N_{\mathrm{RB}}, 2 + N_{\mathrm{RB}}, \frac{\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}{3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}\right) \\
p_{\mathrm{off}}^{(c)} &= \frac{3.5^{3.5}\Gamma(5.5+N_{\mathrm{RB}})}{\Gamma(3.5)N_{\mathrm{RB}}\Gamma(3+N_{\mathrm{RB}})} \frac{(\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}})^{2+N_{\mathrm{RB}}}}{(3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}})^{5.5+N_{\mathrm{RB}}}} {}_2F_1\left(2, 5.5 + N_{\mathrm{RB}}, 3 + N_{\mathrm{RB}}, \frac{\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}{3.5+\lambda_{\mathrm{MT}}/\lambda_{\mathrm{BS}}}\right)
\end{aligned}$$

… Practical Radiation Patterns



	$G_q\left(\theta_q\right)$
Omni-directional	1
3GPP [19]	$\gamma_q^{(\mathrm{3GPP})} 10^{-(6/5)\left(\theta_q/\phi_q^{(\mathrm{3dB})}\right)^2} \mathbb{1}_{\left[0,\phi_q^{(\mathrm{3GPP})}\right]}\left(\lvert\theta_q\rvert\right)+\gamma_q^{(\mathrm{3GPP})} 10^{-A_q/10} \mathbb{1}_{\left[\phi_q^{(\mathrm{3GPP})},\pi\right]}\left(\lvert\theta_q\rvert\right)$
UWLA [20]	$\gamma_q^{(\mathrm{UWLA})}\left\lvert N_q^{-1}\sin\left(N_q\pi\nu^{-1}\cos\left(\theta_q\right)d_q\right)\sin^{-1}\left(\pi\nu^{-1}\cos\left(\theta_q\right)d_q\right)\right\rvert^2$
Three-Sector [21]	$\gamma_q^{(1,\mathrm{sec})}\mathbb{1}_{\left[0,\phi_q^{(1,\mathrm{sec})}\right]}\left(\lvert\theta_q\rvert\right)+\gamma_q^{(2,\mathrm{sec})}\mathbb{1}_{\left[\phi_q^{(3,\mathrm{sec})},\pi\right]}\left(\lvert\theta_q\rvert\right)$ $+\gamma_q^{(1,\mathrm{sec})}\left(1-\left(\lvert\theta_q\rvert-\phi_q^{(1,\mathrm{sec})}\right)\Big/\epsilon_q\right)\mathbb{1}_{\left[\phi_q^{(2,\mathrm{sec})},\phi_q^{(2,\mathrm{sec})}\right]}\left(\lvert\theta_q\rvert\right)$ $+\left(2g_q^{(2,\mathrm{sec})}\left(\lvert\theta_q\rvert-\phi_q^{(2,\mathrm{sec})}\right)\Big/\epsilon_q\right)\mathbb{1}_{\left[\phi_q^{(2,\mathrm{sec})},\phi_q^{(3,\mathrm{sec})}\right]}\left(\lvert\theta_q\rvert\right)$
Two-lobe [10]	see (12) with $\mathrm{K}_q=2$

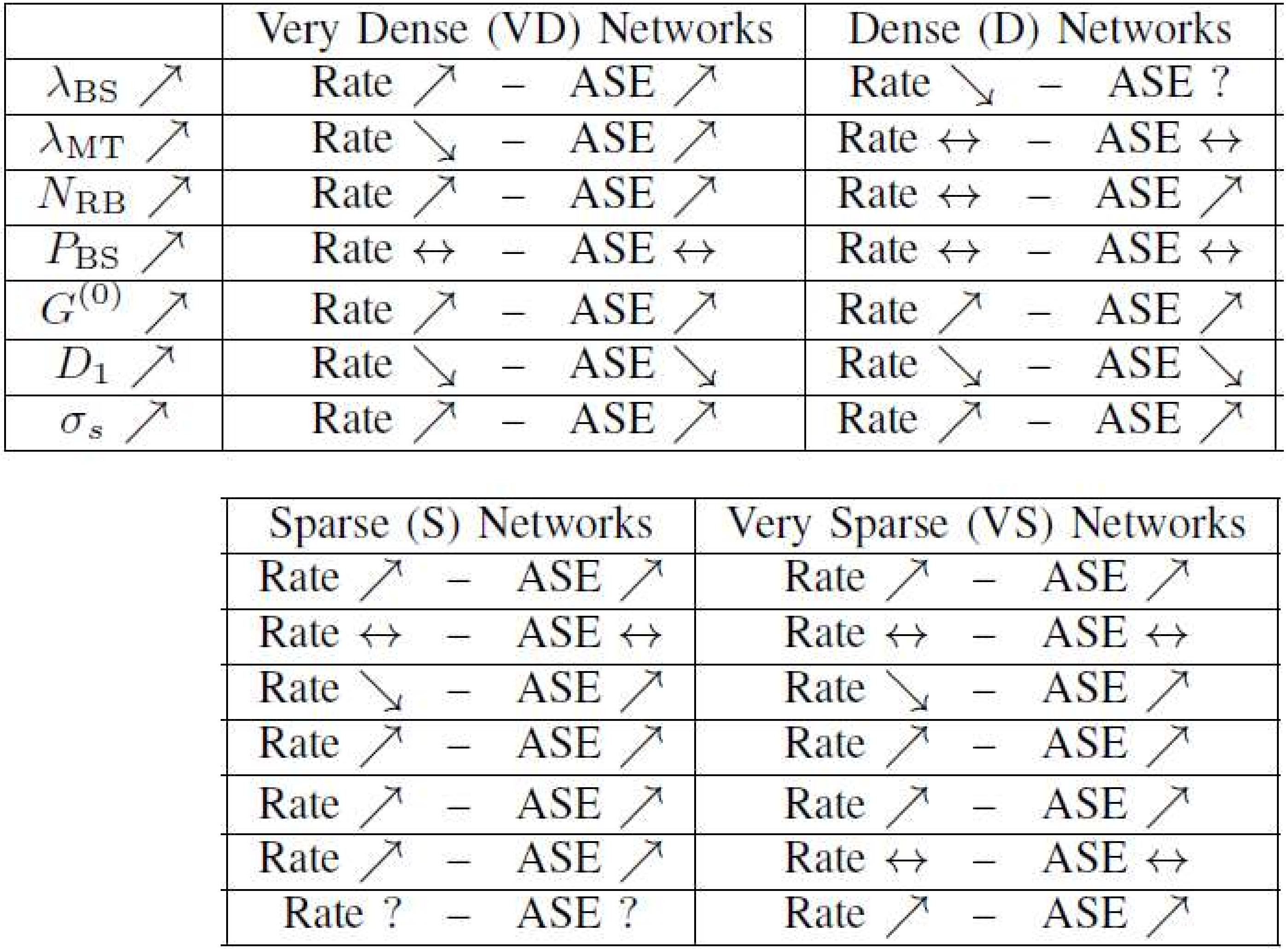
… Relevant Key Performance Indicators

$$\mathrm{ASE} = \left(\lambda_{\mathrm{MT}} p_{\mathrm{sel}} / \ln\left(2\right)\right) \ln\left(1 + \mathrm{SINR}\right); \quad \mathrm{PT} = \lambda_{\mathrm{MT}} p_{\mathrm{sel}} \log_2\left(1 + \mathrm{T}\right) \Pr\left\{\mathrm{SINR} \geq \mathrm{T}\right\}$$

$$\boxed{\begin{aligned}
&\mathcal{R} = \textstyle\sum_{s \in \mathsf{S}} \mathbb{E}_{L_s^{(0)}} \left\{ \mathbb{E}\left\{ \ln\left(1 + \mathrm{SINR}\left(L_s^{(0)}\right)\right) \middle| L_s^{(0)} \right\} \Pr\left\{ L^{(0)} = L_s^{(0)} \right\} \right\} \\
&= \textstyle\sum_{s \in \mathsf{S}} \mathbb{E}_{L_s^{(0)}} \left\{ \left(\int_0^\infty \exp\left(-\sigma_N^2 z\right) \bar{\mathcal{M}}_{g_s^{(0)}} \left(\frac{P_{\mathrm{RB}} G^{(0)} z}{L_s^{(0)}} \middle| L_s^{(0)} \right) \mathcal{M}_{I_{\mathrm{agg}}\left(L_s^{(0)}\right)} \left(z \middle| L_s^{(0)} \right) \frac{dz}{z} \right) \Upsilon_s\left(L_s^{(0)}\right) \right\} \\
&= \textstyle\sum_{s \in \mathsf{S}} \int_0^\infty \int_0^\infty \exp\left(-\sigma_N^2 z\right) \bar{\mathcal{M}}_{g_s^{(0)}} \left(\frac{P_{\mathrm{RB}} G^{(0)} z}{x} \middle| x \right) \mathcal{M}_{I_{\mathrm{agg}}(x)} \left(z \middle| x \right) \Upsilon_s\left(x\right) f_{L_s^{(0)}}\left(x\right) \frac{dz dx}{z}
\end{aligned}}$$

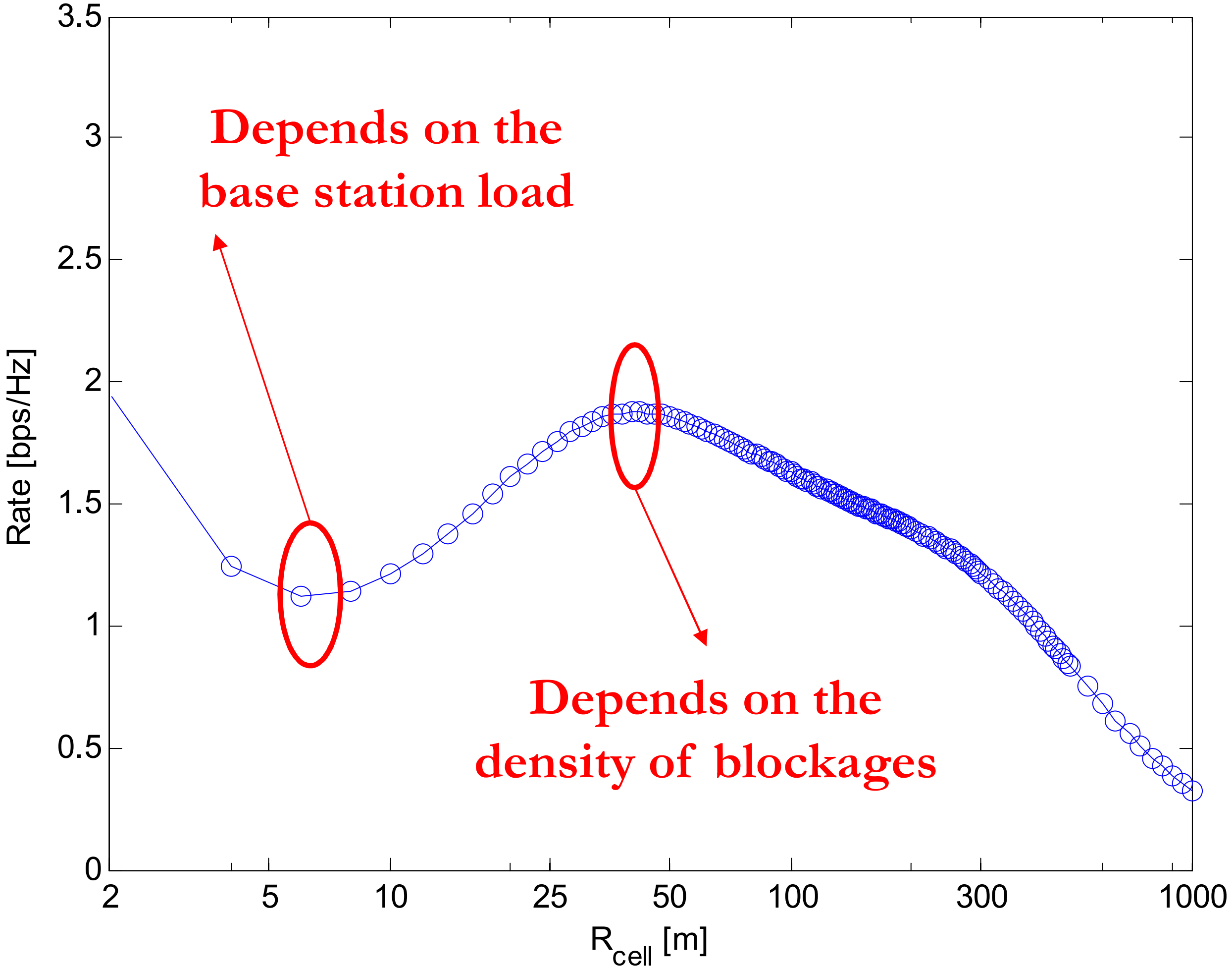
$$\boxed{\begin{aligned}
&\mathcal{C}\left(\mathrm{T}\right) = \textstyle\sum_{s \in \mathsf{S}} \mathbb{E}_{L_s^{(0)}} \left\{ \Pr\left\{ \mathrm{SINR}\left(L_s^{(0)}\right) \geq \mathrm{T} \middle| L_s^{(0)} \right\} \Pr\left\{ L^{(0)} = L_s^{(0)} \right\} \right\} \\
&= \textstyle\sum_{s \in \mathsf{S}} \mathbb{E}_{L_s^{(0)}} \left\{ \Pr\left\{ I_{\mathrm{agg}}\left(L_s^{(0)}\right) \leq \frac{1}{\mathrm{T}} \frac{P_{\mathrm{RB}} G^{(0)} g_s^{(0)}}{L_s^{(0)}} - \sigma_N^2 \middle| L_s^{(0)} \right\} \Upsilon_s\left(L_s^{(0)}\right) \right\} \\
&= \sum_{s \in \mathsf{S}} \int\limits_0^\infty \left(\frac{1}{2} - \frac{1}{\pi} \int\limits_0^\infty \mathrm{Im}\left\{ \exp\left(\jmath z \sigma_N^2\right) \mathcal{M}_{g_s^{(0)}} \left(\jmath z \frac{P_{\mathrm{RB}} G^{(0)}}{\mathrm{T} x} \middle| x \right) \mathcal{M}_{I_{\mathrm{agg}}(x)} \left(\jmath z \middle| x \right) \right\} \frac{dz}{z} \right) \Upsilon_s\left(x\right) f_{L_s^{(0)}}\left(x\right) dx
\end{aligned}}$$

… Proof of Several Performance Trends



	Very Dense (VD) Networks	Dense (D) Networks	Sparse (S) Networks	Very Sparse (VS) Networks
$\lambda_{\mathrm{BS}} \nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\searrow$ – ASE ?	Rate $\nearrow$ – ASE $\nearrow$	Rate $\nearrow$ – ASE $\nearrow$
$\lambda_{\mathrm{MT}} \nearrow$	Rate $\searrow$ – ASE $\nearrow$	Rate $\leftrightarrow$ – ASE $\leftrightarrow$	Rate $\leftrightarrow$ – ASE $\leftrightarrow$	Rate $\leftrightarrow$ – ASE $\leftrightarrow$
$N_{\mathrm{RB}} \nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\leftrightarrow$ – ASE $\nearrow$	Rate $\searrow$ – ASE $\nearrow$	Rate $\searrow$ – ASE $\nearrow$
$P_{\mathrm{BS}} \nearrow$	Rate $\leftrightarrow$ – ASE $\leftrightarrow$	Rate $\leftrightarrow$ – ASE $\leftrightarrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\nearrow$ – ASE $\nearrow$
$G^{(0)} \nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\nearrow$ – ASE $\nearrow$
$D_1 \nearrow$	Rate $\searrow$ – ASE $\searrow$	Rate $\searrow$ – ASE $\searrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\leftrightarrow$ – ASE $\leftrightarrow$
$\sigma_s \nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate $\nearrow$ – ASE $\nearrow$	Rate ? – ASE ?	Rate $\nearrow$ – ASE $\nearrow$

… Proof of Several Performance Trends

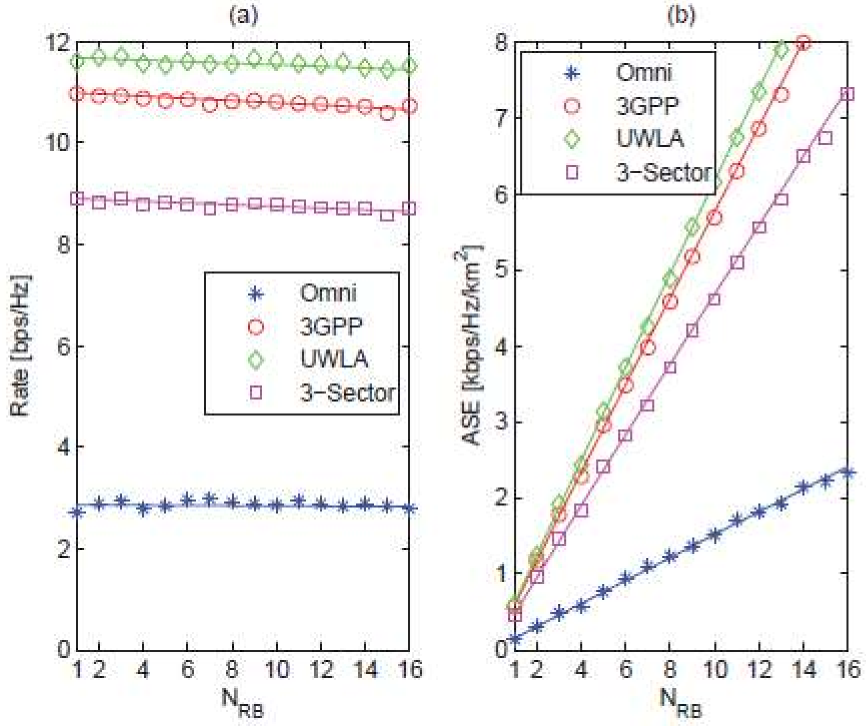


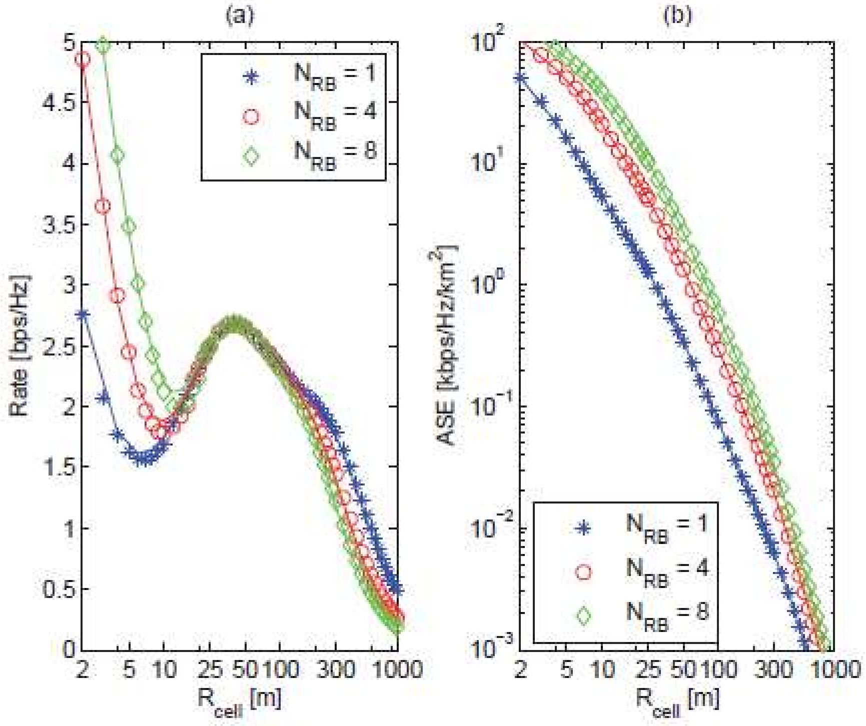
Depends on the base station load

Depends on the density of blockages

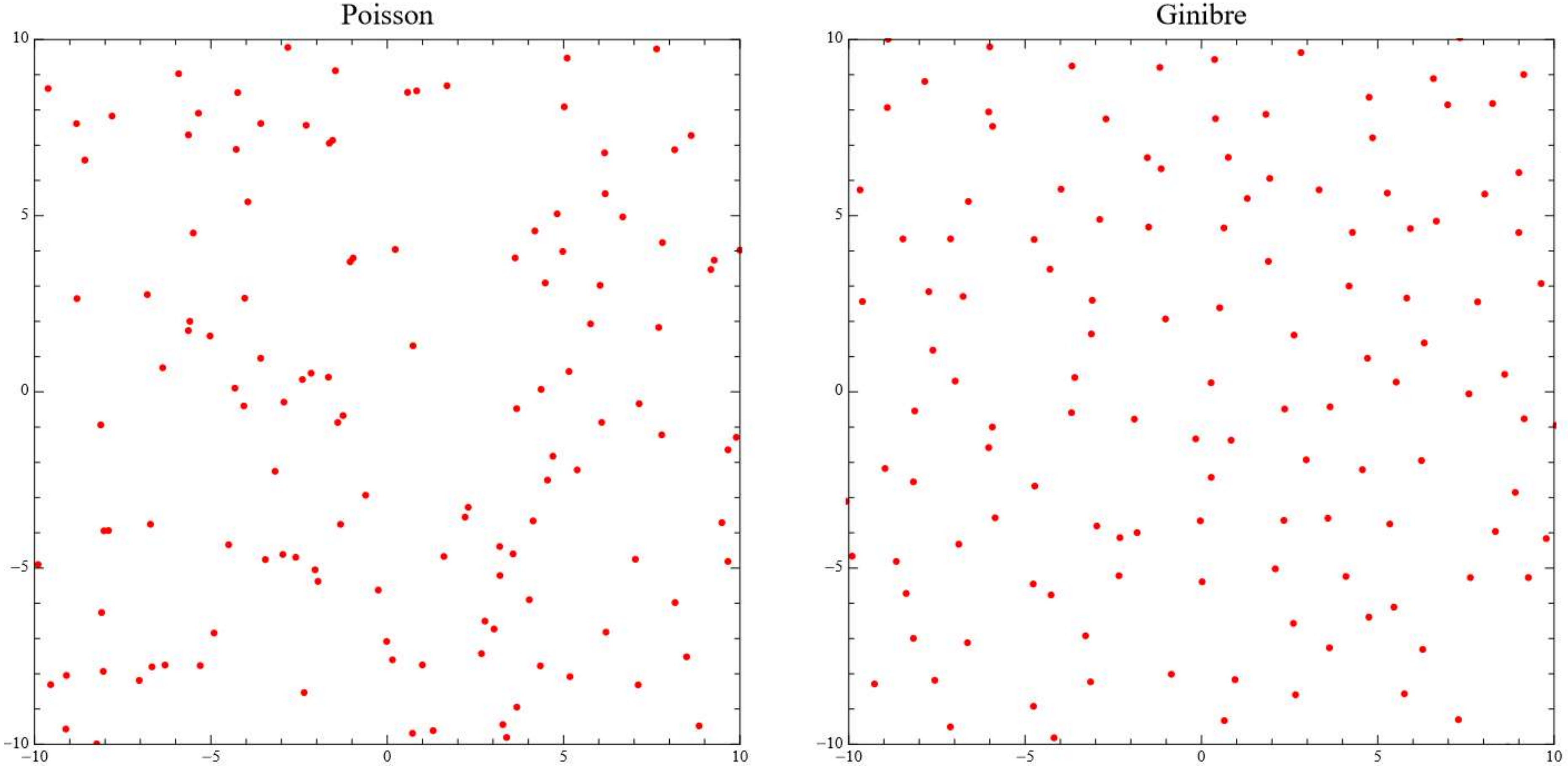
… Experimental Validation

Path-Loss	$\alpha_{\rm LOS} = 2.6$, $\alpha_{\rm NLOS} = 3.8$, $\kappa_{\rm LOS} = \kappa_{\rm NLOS} = (4\pi f_0/c_0)^2$ with $f_0 = 2.1$ GHz, $c_0 \approx 3 \cdot 10^8$ m/s
Shadowing, fading	$\sigma_{\rm LOS} = 4$ dB, $\sigma_{\rm NLOS} = 10$ dB, $\Omega_{\rm LOS} = \Omega_{\rm NLOS} = 1$, $m_{\rm LOS} = 2.8$, $m_{\rm NLOS} = 1$
BS power, noise	$P_{\rm BS} = 20$ dBm, $\sigma_N^2 = -174 + 10\log_{10}({\rm B_W}) + \mathcal{F}$ dBm with ${\rm B_W} = 180$ kHz, $\mathcal{F} = 10$ dB
Link-state	3GPP [8]: $a_{\rm 3G} = 18$, $b_{\rm 3G} = 36$, $c_{\rm 3G} = 1$; RS [23]: $a_{\rm RS} = 1$, $b_{\rm RS} = 0.046$ m^{-1}
Empirical	BSs: O2 in [7, Table 1], [7, Fig. 1], $R_{\rm cell} \approx 83.4$ m; buildings: London [7, Fig. 1], [7, Sec. 2.3.1]
$\lambda_{\rm MT} = \left(\pi R_{\rm MT}^2\right)^{-1}$	$R_{\rm MT} \approx \{3.9, 7.6, 11.9, 50, 100\}$ m is the population density of Paris, London, Rome, Pennsylvania, Texas

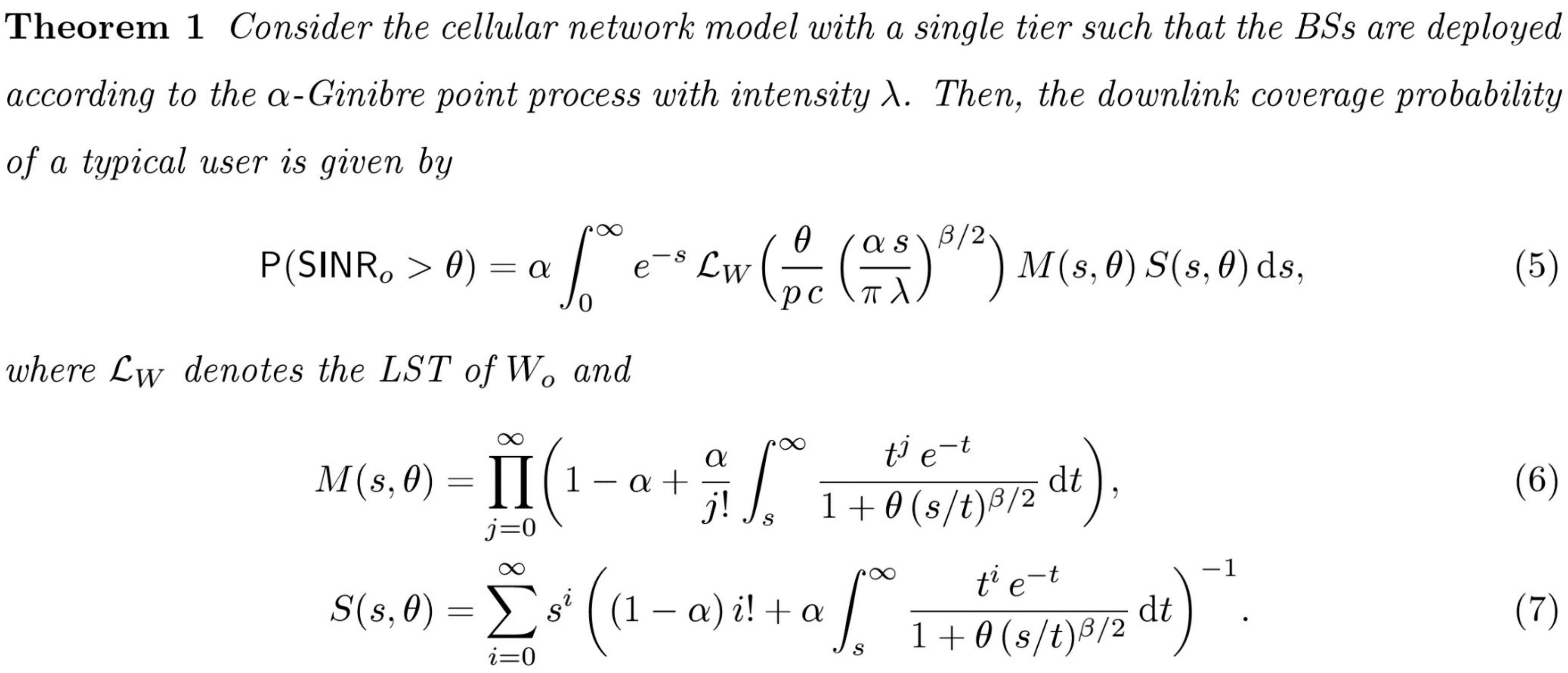




Beyond the PPP-Based Modeling: The α-Ginibre PP



Beyond the PPP-Based Modeling: The α-Ginibre PP



Theorem 1 *Consider the cellular network model with a single tier such that the BSs are deployed according to the α-Ginibre point process with intensity λ. Then, the downlink coverage probability of a typical user is given by*

$$\mathsf{P}(\mathsf{SINR}_o > \theta) = \alpha \int_0^\infty e^{-s}\, \mathcal{L}_W\Bigl(\frac{\theta}{p\,c}\Bigl(\frac{\alpha\, s}{\pi\,\lambda}\Bigr)^{\beta/2}\Bigr)\, M(s,\theta)\, S(s,\theta)\, \mathrm{d}s, \tag{5}$$

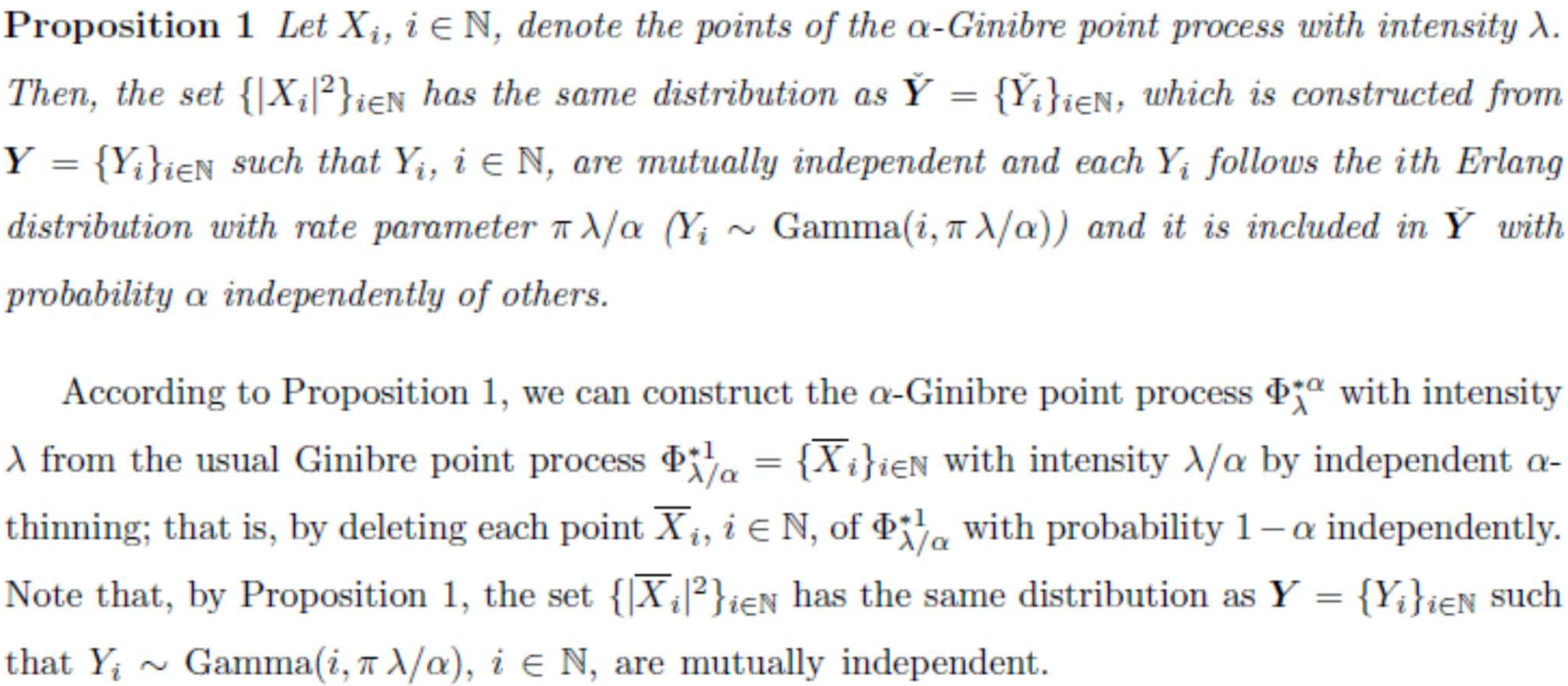
where $\mathcal{L}_W$ denotes the LST of W_o and

$$M(s,\theta) = \prod_{j=0}^{\infty}\Bigl(1 - \alpha + \frac{\alpha}{j!}\int_s^\infty \frac{t^j\, e^{-t}}{1 + \theta\,(s/t)^{\beta/2}}\, \mathrm{d}t\Bigr), \tag{6}$$

$$S(s,\theta) = \sum_{i=0}^{\infty} s^i \Bigl((1-\alpha)\, i! + \alpha \int_s^\infty \frac{t^i\, e^{-t}}{1 + \theta\,(s/t)^{\beta/2}}\, \mathrm{d}t\Bigr)^{-1}. \tag{7}$$

I. Nakata and N. Miyoshi, "Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations", Research Reports on Mathematical and Computing Sciences (ISSN 1342-2804), Oct. 2013, B-473.

Beyond the PPP-Based Modeling: The α-Ginibre PP

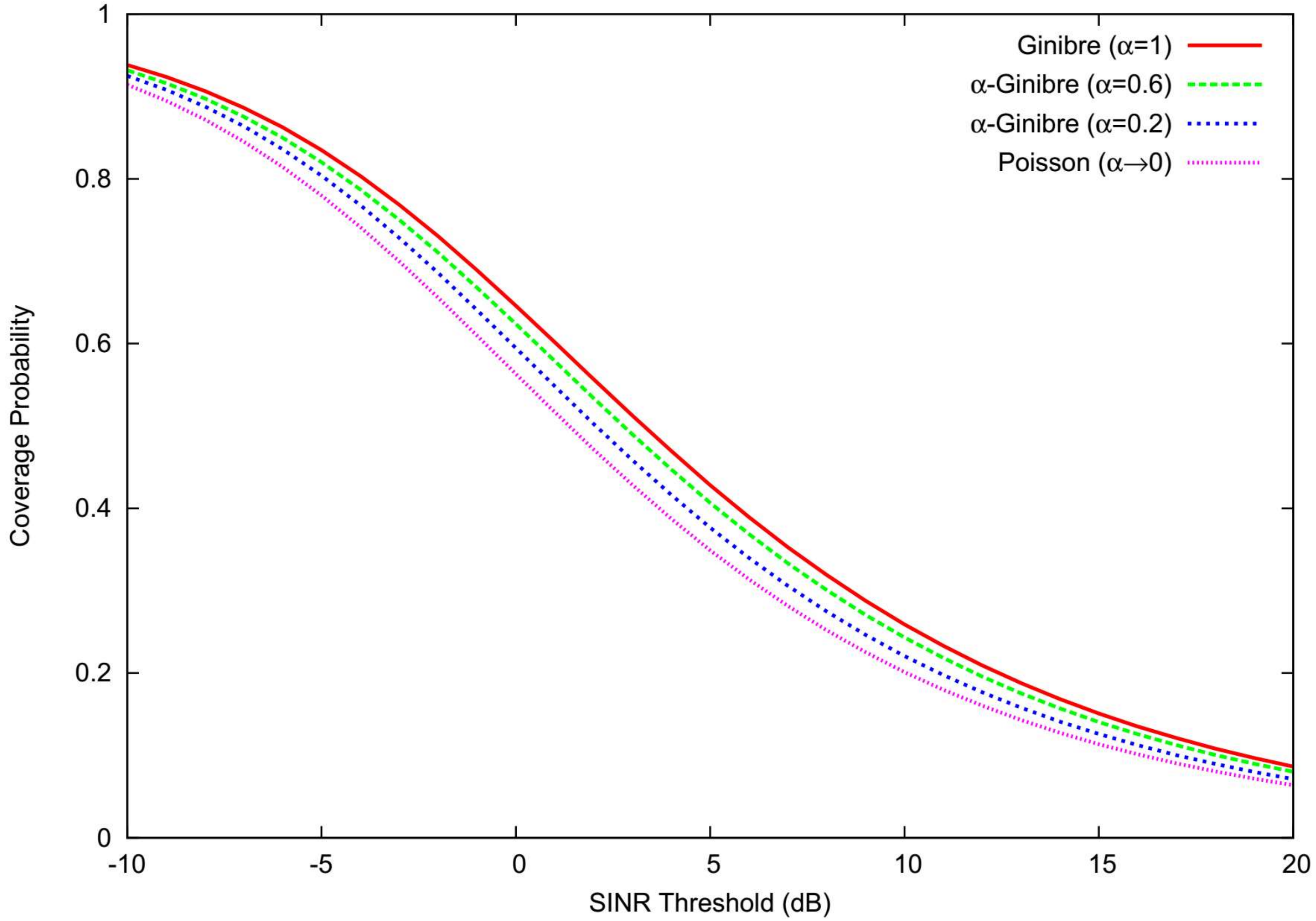


Proposition 1 *Let X_i, $i \in \mathbb{N}$, denote the points of the α-Ginibre point process with intensity λ. Then, the set $\{|X_i|^2\}_{i\in\mathbb{N}}$ has the same distribution as $\check{\boldsymbol{Y}} = \{\check{Y}_i\}_{i\in\mathbb{N}}$, which is constructed from $\boldsymbol{Y} = \{Y_i\}_{i\in\mathbb{N}}$ such that Y_i, $i \in \mathbb{N}$, are mutually independent and each Y_i follows the ith Erlang distribution with rate parameter $\pi\,\lambda/\alpha$ ($Y_i \sim \mathrm{Gamma}(i, \pi\,\lambda/\alpha)$) and it is included in $\check{\boldsymbol{Y}}$ with probability α independently of others.*

According to Proposition 1, we can construct the α-Ginibre point process $\Phi_\lambda^{*\alpha}$ with intensity λ from the usual Ginibre point process $\Phi_{\lambda/\alpha}^{*1} = \{\overline{X}_i\}_{i\in\mathbb{N}}$ with intensity λ/α by independent α-thinning; that is, by deleting each point $\overline{X}_i$, $i \in \mathbb{N}$, of $\Phi_{\lambda/\alpha}^{*1}$ with probability $1-\alpha$ independently. Note that, by Proposition 1, the set $\{|\overline{X}_i|^2\}_{i\in\mathbb{N}}$ has the same distribution as $\boldsymbol{Y} = \{Y_i\}_{i\in\mathbb{N}}$ such that $Y_i \sim \mathrm{Gamma}(i, \pi\,\lambda/\alpha)$, $i \in \mathbb{N}$, are mutually independent.

I. Nakata and N. Miyoshi, "Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations", Research Reports on Mathematical and Computing Sciences (ISSN 1342-2804), Oct. 2013, B-473.

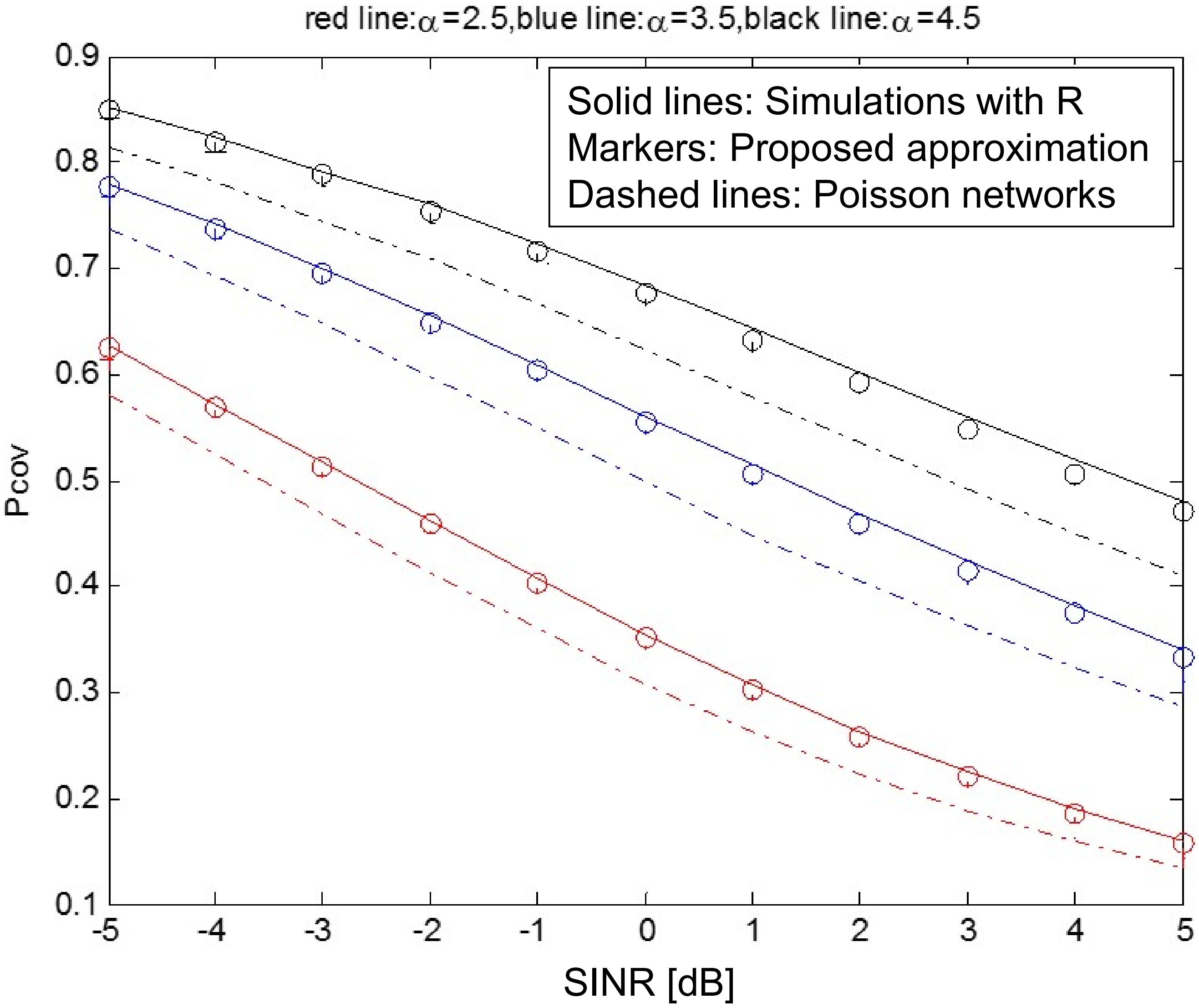
Beyond the PPP-Based Modeling: The α-Ginibre PP



I. Nakata and N. Miyoshi, "Spatial stochastic models for analysis of heterogeneous cellular networks with repulsively deployed base stations", Research Reports on Mathematical and Computing Sciences (ISSN 1342-2804), Oct. 2013, B-473.

The Road Ahead: IM Approach for non-Poisson Nets

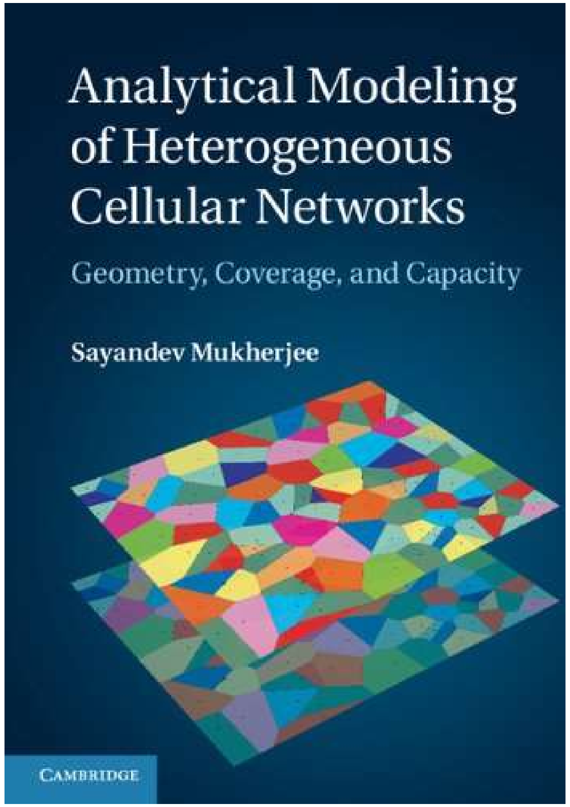
… Cauchy determinantal point process (spatially-repulsive) …



… final thoughts …

The Role of Stochastic Geometry in Communications

1.4 The role of analytic modeling



The analytic-modeling-based investigation of deployment scenarios has two phases. In the first phase, we use probabilistic models for the locations of the BSs to determine analytic expressions for the CCDF of the SINR in the deployment region. In other words, the use of a stochastic model (Poisson point process, or PPP) for the locations of the BSs allows us to write an analytic expression for the expectation of (1.4) with respect to either the joint distribution of $(R_0, R_1, \ldots, R_M)$ or the conditional joint distribution of $(R_1, \ldots, R_M)$ given $R_0 = r_0$. Further, these results can be extended to arbitrary fading distributions and arbitrary numbers of tiers of BSs.

As we shall see, this has the benefit of providing insights into the *combinations* of deployment parameters that affect the CCDF of the SINR, and therefore the different sets of deployment parameters that are *equivalent* in that they yield the same CCDF of the SINR. This analytic phase allows us to sift through the large space of combinations of deployment parameters to settle quickly on certain equivalence classes of deployment parameters, each class corresponding to some desired CCDF of the SINR. The service provider may then choose a set of deployment parameters from one of these equivalence classes based on its economic utility function.

In the next phase of the network design, the shortlist of deployment scenarios (as defined by the deployment parameters) chosen in the first phase may be investigated in depth via simulation. This effectively uses the power of detailed simulation, incorporating all relevant aspects whose behavior and impact on performance is to be investigated, for a few selected deployment scenarios.

S. Mukherjee: "Analytical Modeling of Heterogeneous Cellular Networks", Cambridge University Press, January 2014.

The Renaissance of (Network) Communication Theory

… IEEE TCOM Nov. 2011 – now …

THE IMPACT OF COMMUNICATION THEORY ON TECHNOLOGY DEVELOPMENT:

IS THE BEST BEHIND US, OR AHEAD?”

aka “Is Communication Theory Dead”?

PLENARY PANEL: “The Impact of Communication Theory on Technology Development: Is the Best Behind us or Ahead?”, IEEE Communications Theory Workshop, May 2010.

Bottom Line

- ❑ Stochastic geometry provides suitable mathematical models and appropriate statistical methods to study and analyze heterogeneous (future deployments) cellular networks

- ❑ It is instrumental for identifying subsets of candidate (feasible, relevant) solutions based on which finer-grained simulations can be conducted, thus significantly reducing the time and cost of optimizing complex communication networks

- ❑ Its application to cellular network designs, however, necessitates to abandon conventional and comfortable assumptions
  - ➢ Poisson (complete spatially random) modeling
  - ➢ Simplistic path-loss models
  - ➢ Simplistic transmission schemes
  - ➢ …

- ❑ Relying upon adequate approximations to avoid oversimplifying the system model is not an option. **CAUTION is, however, mandatory.**

The System-Level Side of 5G – YouTube Video

On System-Level Analysis & Design of Cellular Networks:
The Magic of Stochastic Geometry

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IEEE International Conference on Communications (ICC)
London, United Kingdom, June 8-12, 2015

https://youtu.be/MB8IvOYYvB0

Thank You for Your Attention

- **ETN-5Gwireless (H2020-MCSA, grant 641985)**

  An European Training Network on 5G Wireless Networks

  http://cordis.europa.eu/project/rcn/193871_en.html (Jan. 2015, 4 years)

5Gwireless

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