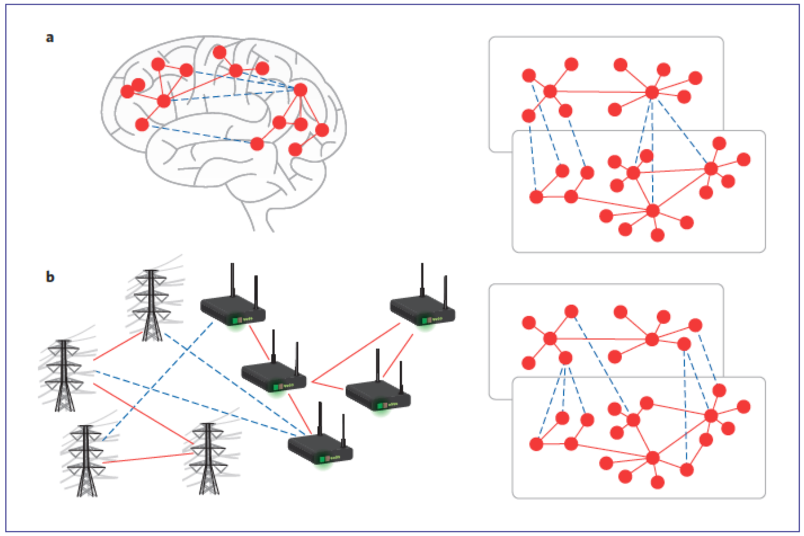

Short Course on Complex Networks and Point Processes with Applications

Oxford, Sep 11-12, 2017

Multilayer Networks

Ginestra Bianconi

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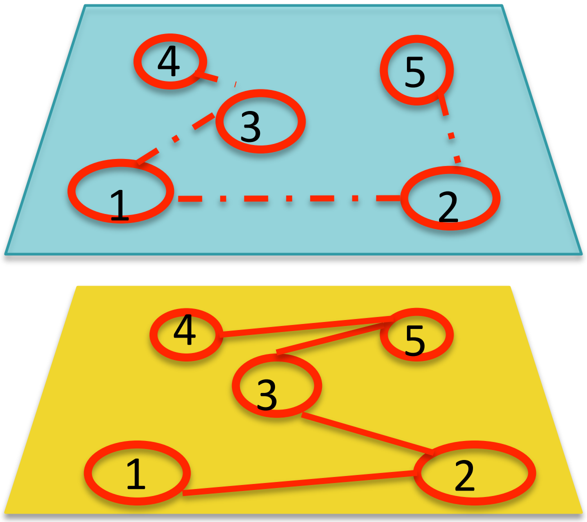
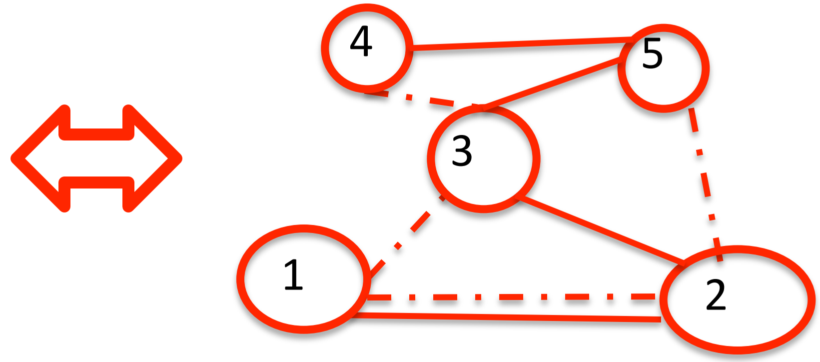
Queen Mary
University of London





Multiplex Networks:

centrality measures

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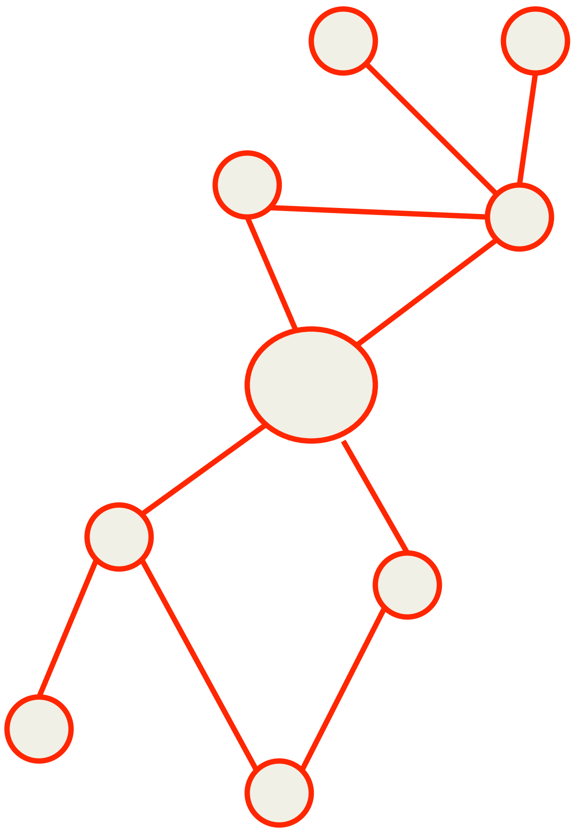
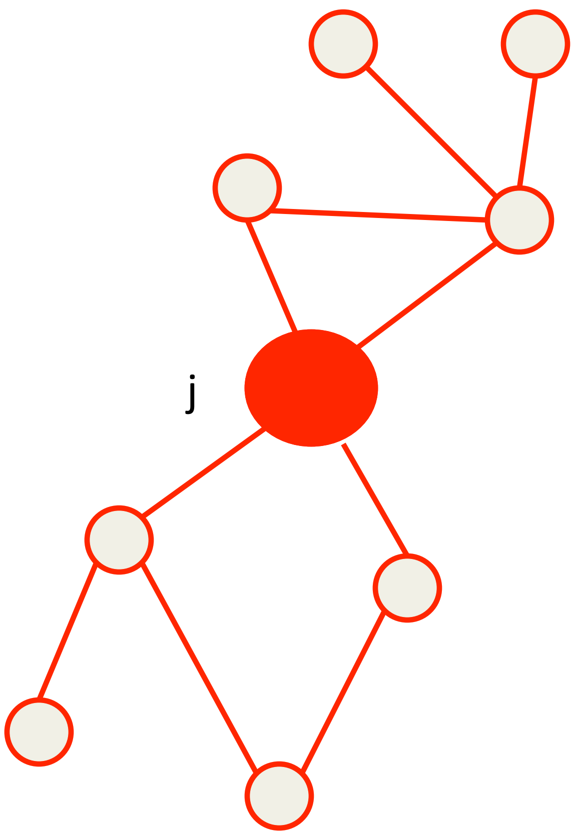
PageRank

The Page Rank is based on a random walk

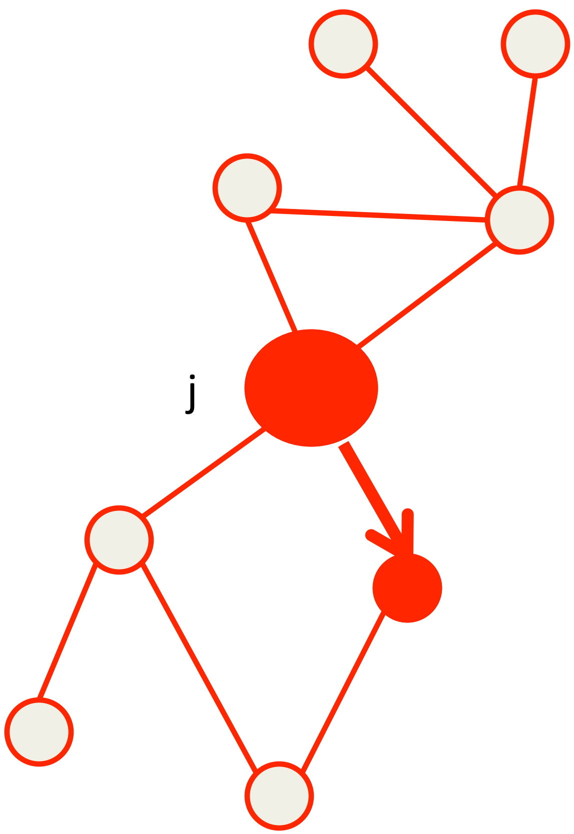


The Page Rank is based on a random walk



- We assume to have a random walker on the node j of the network

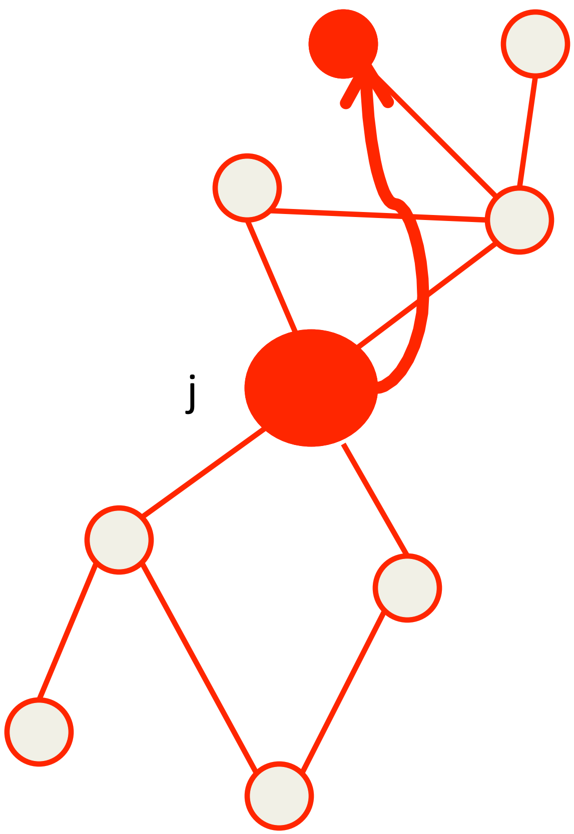
The Page Rank is based on a random walk



- We assume to have a random walker on the node j of the network

- With probability $\tilde{\alpha}$ the random walker hops to a neighbor node

The Page Rank is based on a random walk



- We assume to have a random walker on the node j of the network
- With probability $\tilde{\alpha}$ the random walker hops to a neighbor node
- With a probability $1-\tilde{\alpha}$ it jumps to a random node

PageRank

The PageRank x_i of node i is the probability that in the stationary state we find the random walker on node i

$$x_i = \tilde{\alpha} \sum_j \frac{A_{ij}}{g_j} x_j + \beta$$

with

$g_j = \max(k_j, 1)$, k_i indicating the degree of node i

$$A_{ij} = \begin{cases} 1 \text{ if node } j \text{ links to node } i \\ 0 \quad \text{otherwise} \end{cases}, \quad \tilde{\alpha} = 0.85$$

Quantifying the

centrality of the nodes

with the

Functional Multiplex PageRank

Influences of multilinks

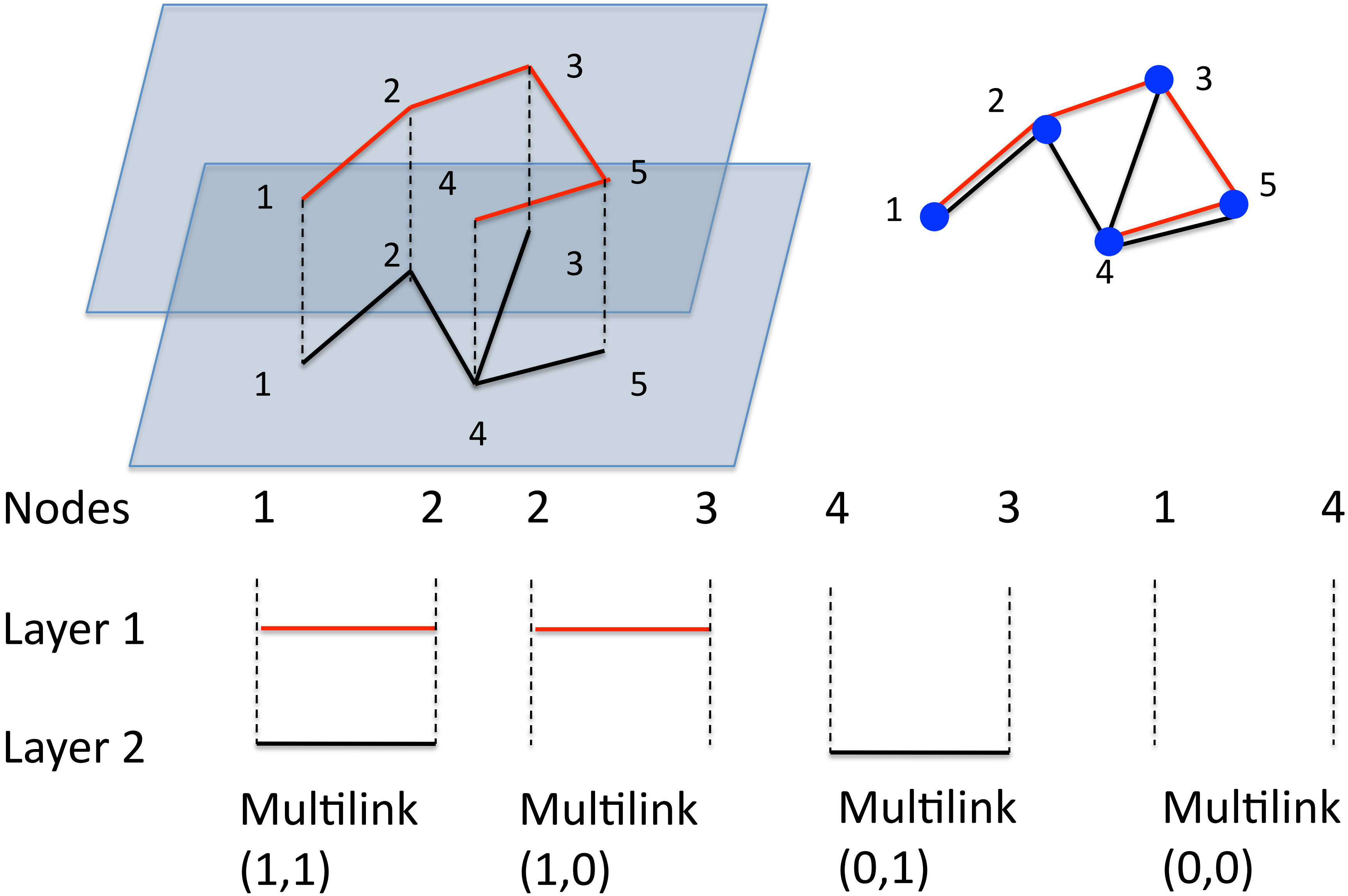
In a multiplex network different pattern of connections might contribute differently to the centrality of a node

The influence of a multilink $\vec{m}$
is indicated by $z^{\vec{m}}$

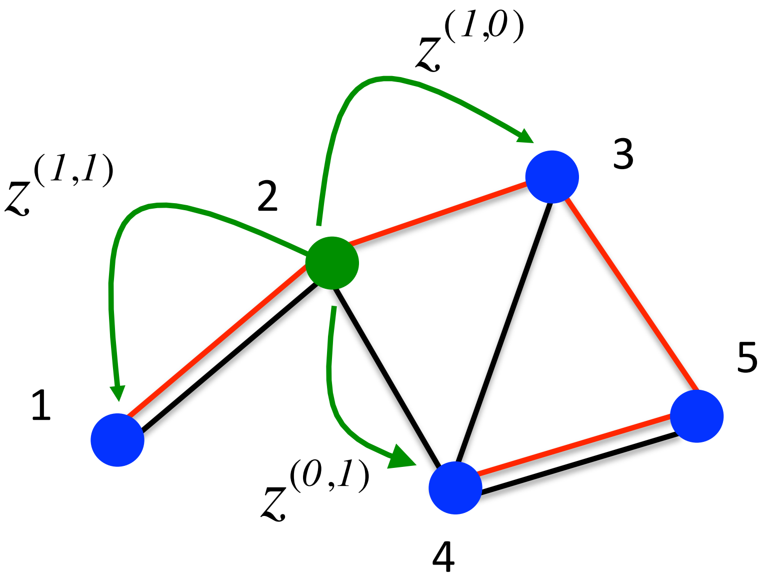
J. Iacovacci et al. 2016

Multilinks

G. Bianconi
PRE (2013)

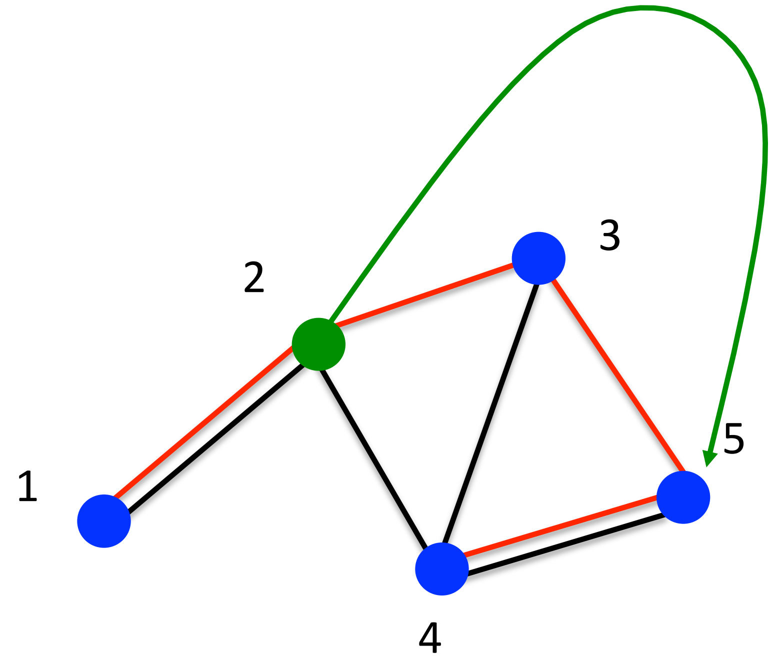


Functional Multiplex PageRank



(a)

The random walker can jump to a neighbor node, with a probability proportional to the influence of the corresponding multilink



(b)

The random walker can jump to a random node (teleportation)

Functional Multiplex PageRank

The centrality of a node i

is a function

$$X_i(\mathbf{z})$$

depending on the values of the influences **z** attributed to multilinks

For a duplex network

$$\mathbf{z} = (z^{(1,0)}, z^{(0,1)}, z^{(1,1)})$$

Non-linear effects due to the overlap of the links

The Functional Multiplex PageRank allows

for the inclusion of strong non-linear effects

due to the overlap of the links.

For example, in a duplex network we can have

$$z^{(1,1)} \neq z^{(0,1)} + z^{(1,0)}$$

and we can weight multilinks (1,1) much more or much less than the sum of the weight of multilinks (0,1) and (1,0).

Absolute Multiplex PageRank

From the
Functional Multiplex PageRank
we can extract the
Absolute Multiplex PageRank
given by

$$X_i^* = \max_{\mathbf{z}} X_i(\mathbf{z})$$

which can provide an overall ranking of the nodes
of the multiplex network

The case of a duplex network (M=2)

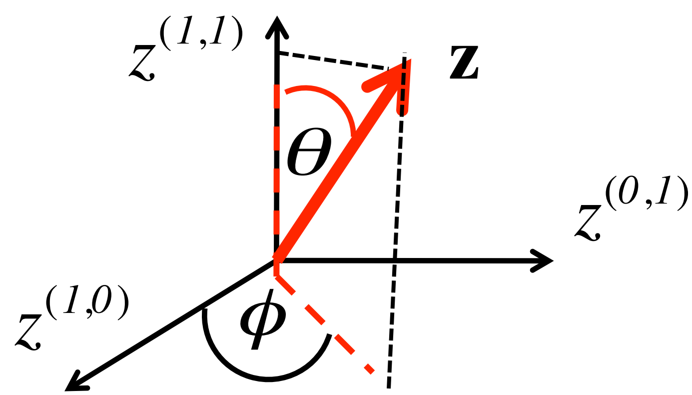
The Functional Multiplex PageRank,

depends only of the direction of the vector of influences **z,** therefore we take

$$z^{(1,0)} = \sin\theta\cos\phi$$
$$z^{(0,1)} = \sin\theta\sin\phi$$
$$z^{(1,1)} = \cos\theta$$

with

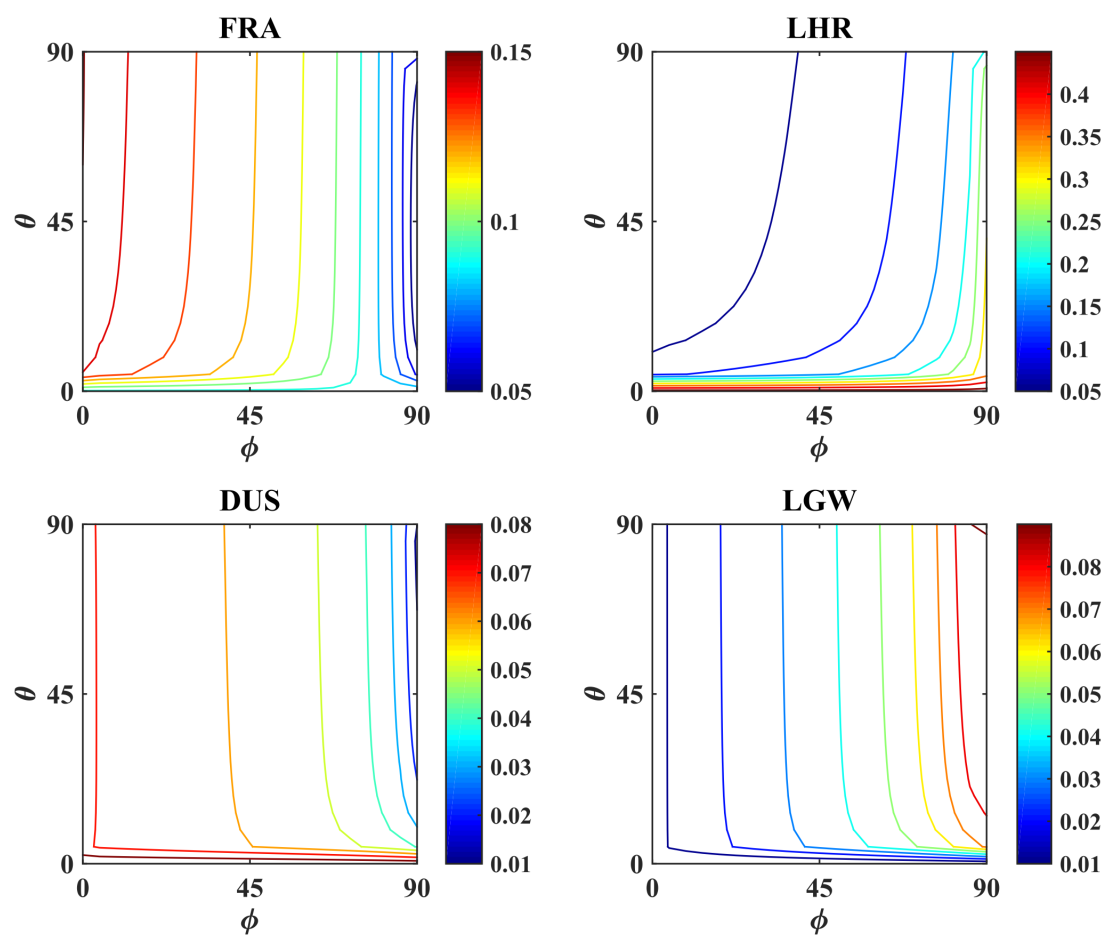
$$\theta,\phi \in [0,\pi/2]$$



Top ranked airports in the duplex Lufthansa/British Airways network according to the Absolute Multiplex PageRank

Rank	Airport
1	Heathrow Airport (LHR)
2	Munich Airport (MUC)
3	Frankfurt Airport (FRA)
4	Gatwick Airport (LGW)

Different pattern to success of major airports



For $\phi=0^o$ $\theta=90^o$ multilinks (1,0) have major influence

For $\phi=90^o$ $\theta=90^o$ multilinks (0,1) have major influence

For $\theta=0^o$ multilinks (1,1) have major influence

Correlations between the *pattern to success*

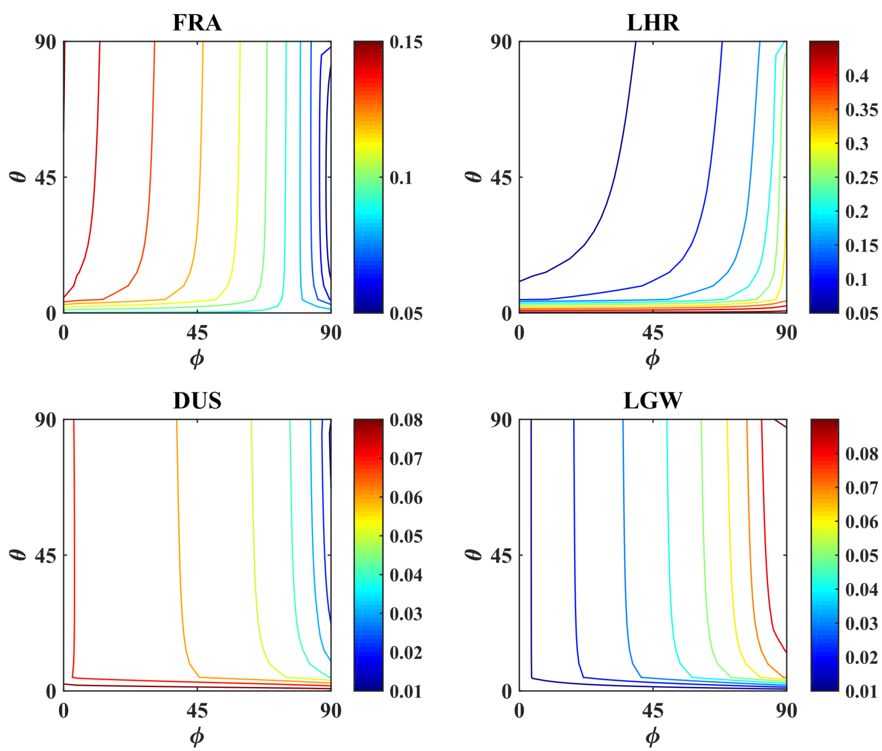
$$\rho = \frac{\langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle}{\sigma_i \sigma_j}$$

where the average and the standard deviation are calculated on a grid (ϕ_r, θ_s) with $r=1,2,..,N_\phi$ and $s=1,2,...,N_\theta$

$$\langle Y \rangle = \frac{1}{N_\phi N_\theta} \sum_{r=1..N_\phi} \sum_{s=1..N_\theta} Y(\phi_r, \theta_s)$$

$$\sigma_i = \sqrt{\langle X_i^2 \rangle - \langle X_i \rangle^2}$$

Correlations between the pattern to success between major airports



ρ	LHR	FRA	LGW	DUS
LHR	1	-0.797	0.484	0.351
FRA	-0.797	1	-0.983	0.275
LGW	0.484	-0.983	1	-0.729
DUS	0.351	0.2758	-0.729	1

Duplex connectome network of C.elegans

Top ranked neurons

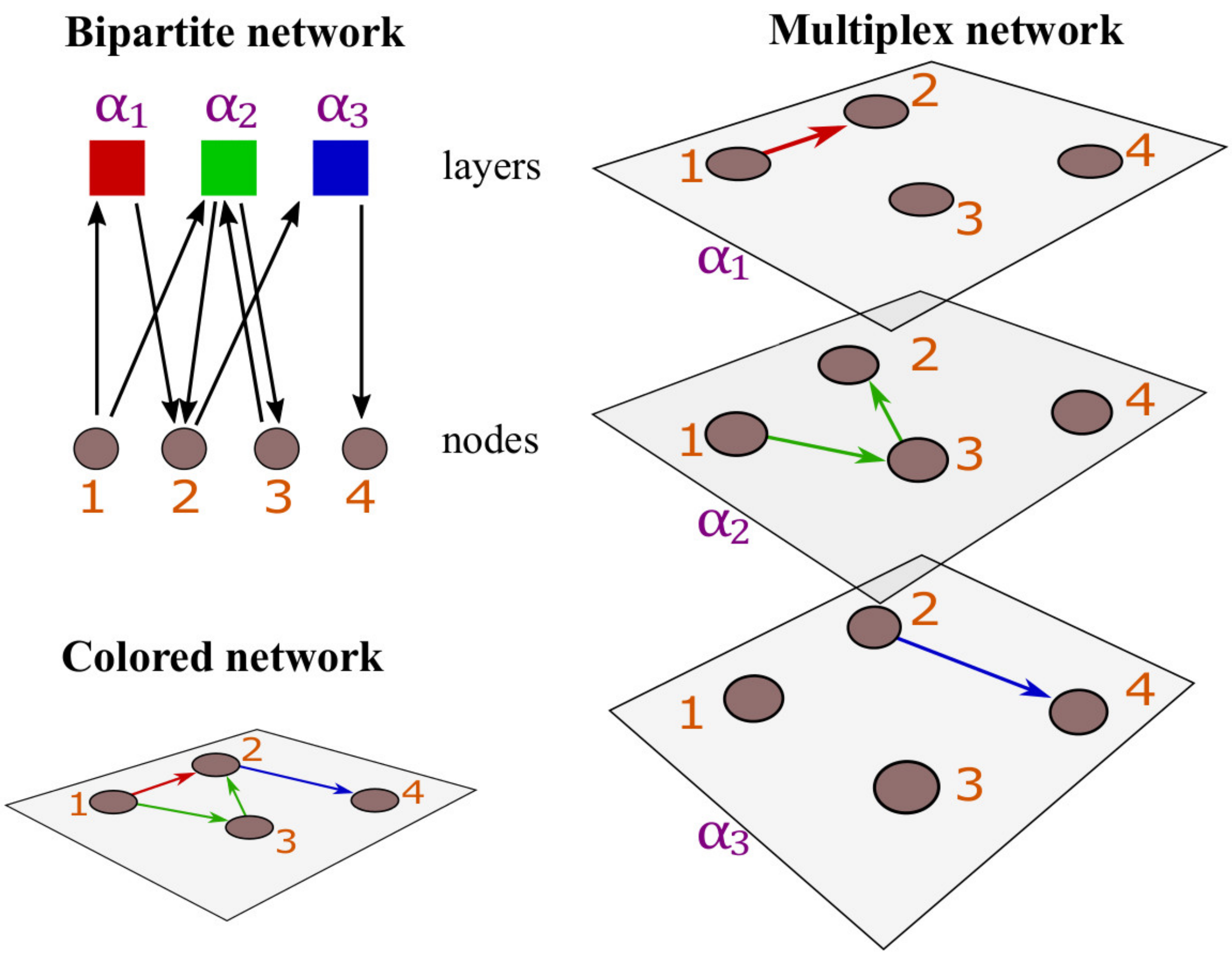
Rank	Neuron	Rank	Neuron
1	AVAR	6	PVCR
2	AVAL	7	AVDR
3	AVBL	8	AVER
4	AVBR	9	AVEL
5	PVCL	10	DVA

Similar neurons types have correlated *pattern to success*

Pearson correlations

MultiRank
Centrality of nodes and
Influence of layers
in large multiplex networks

Multiplex network representation



Rahmede et al. (2017)

MultiRank

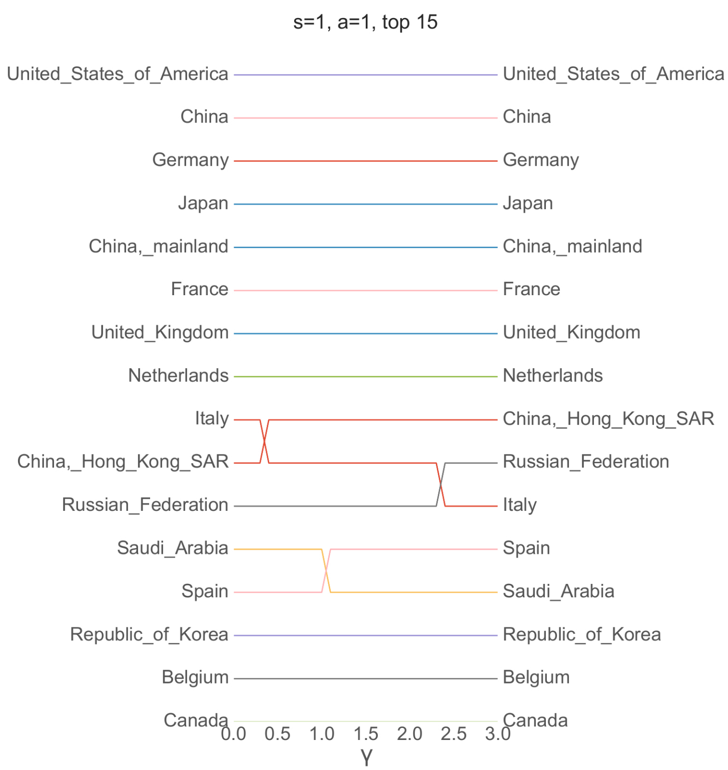
Node are central

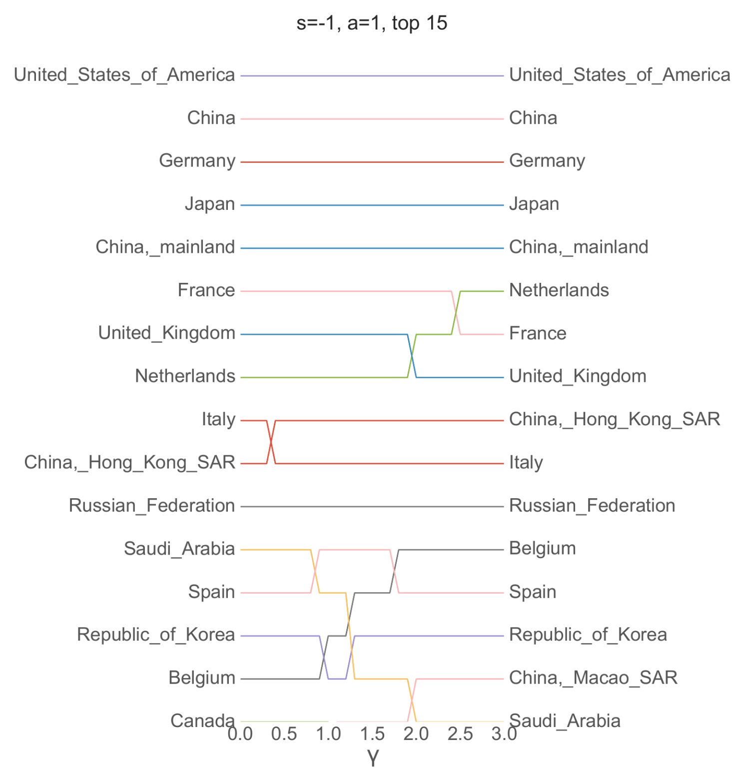
if already central nodes point to them in very influent layers

Layers are more influent

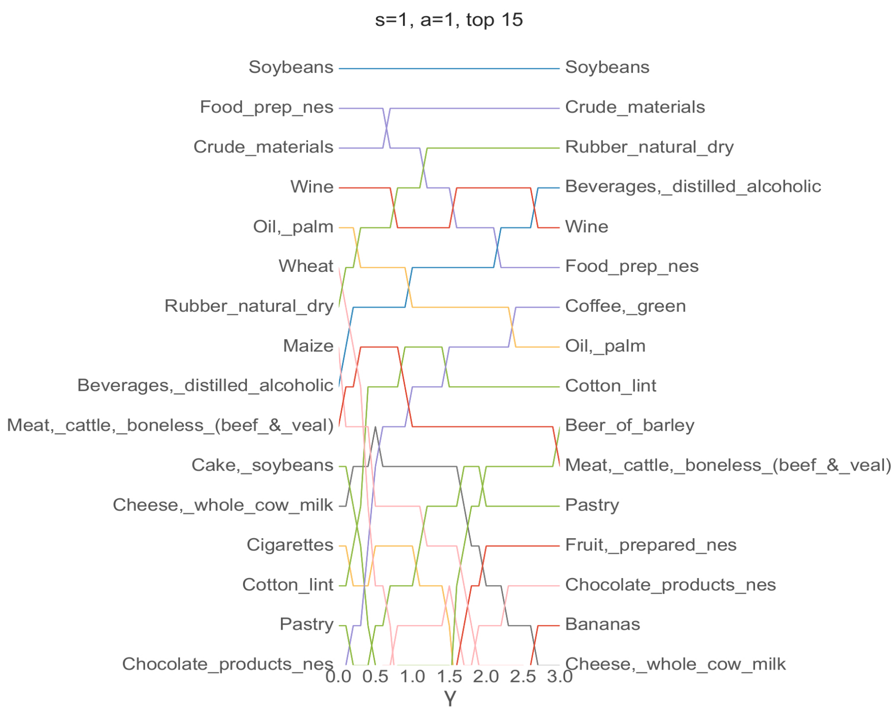
if very central nodes are active in it

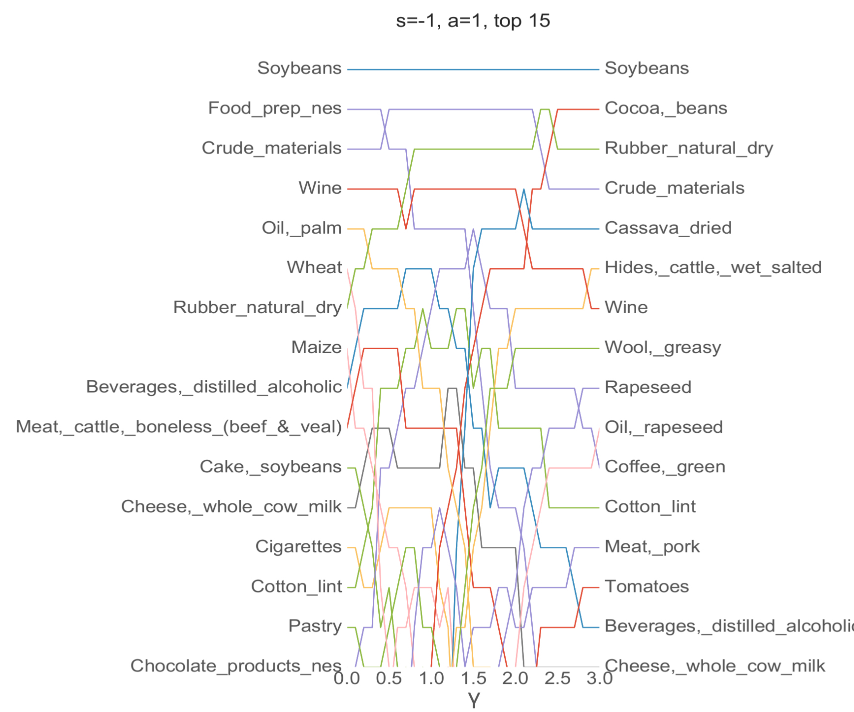
Centrality of countries in the FAO Multiplex Trade Networks



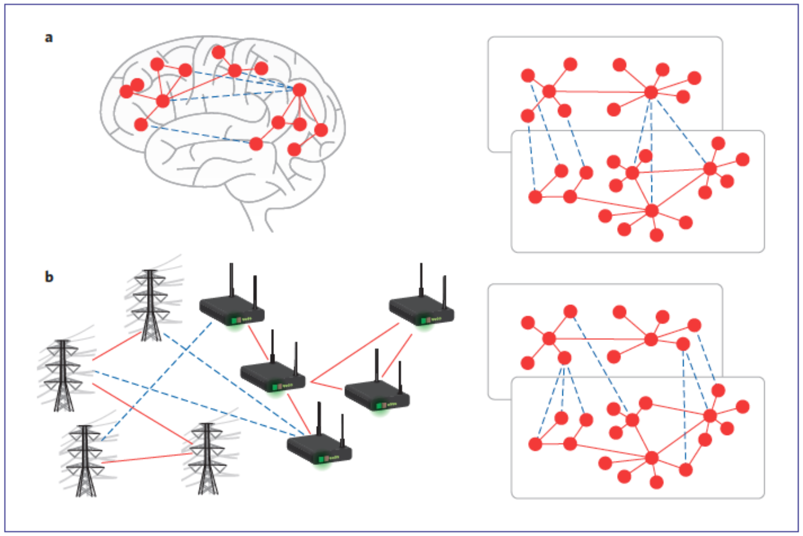


Influences of layers in the FAO Multiplex Trade Networks





Rahmede et al. (2017)



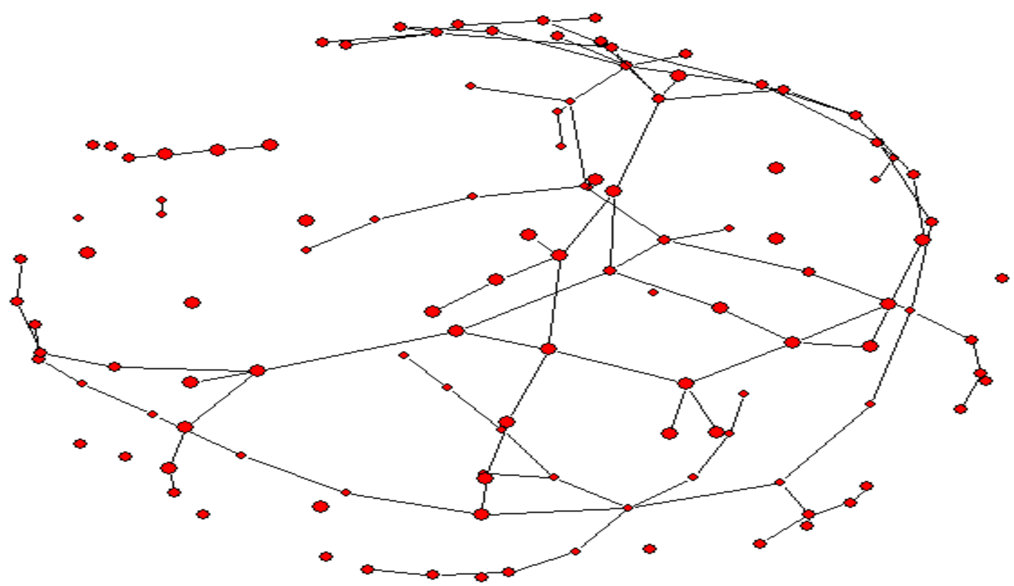
Robustness
of
Multilayer Networks

Percolation in single networks

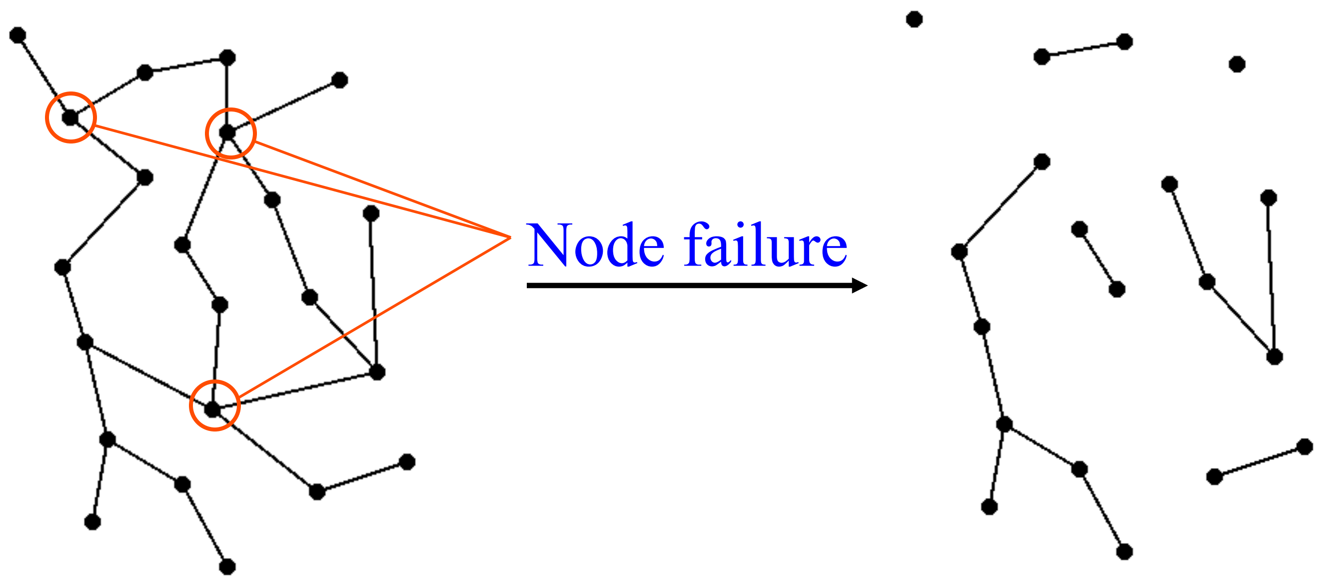
Giant component

➢A connected component of a network is a subgraph induced by any set of nodes such that for each pair of nodes in the subgraph there is at least one path connecting them and such that no other node is connected to them by any path.

➢The giant component is the connected component of the network which contains a number of nodes of the same order of magnitude of the total number of nodes in the network.

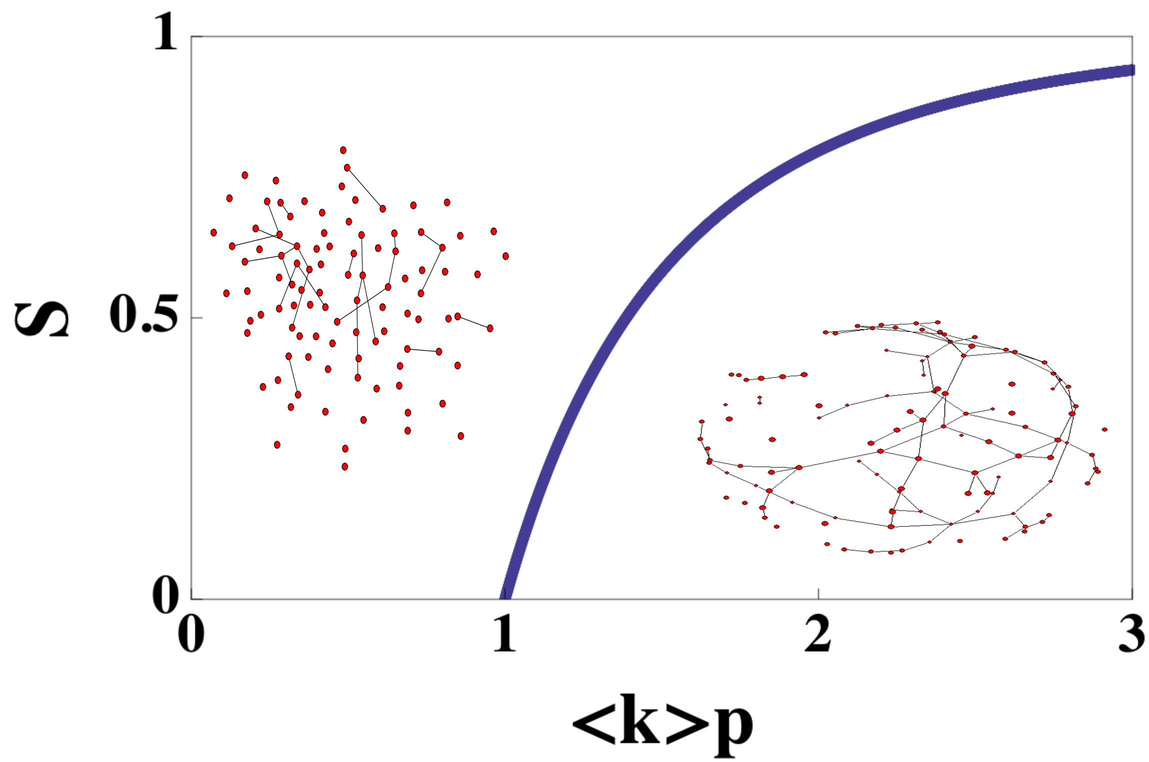


Robustness of complex networks



We assume that a fraction 1-p of nodes is damaged.
We evaluate the robustness of the network by calculating the fraction S of nodes in the giant component after this inflicted damage.

Percolation transition in Poisson networks



S is the fraction of nodes in the giant component

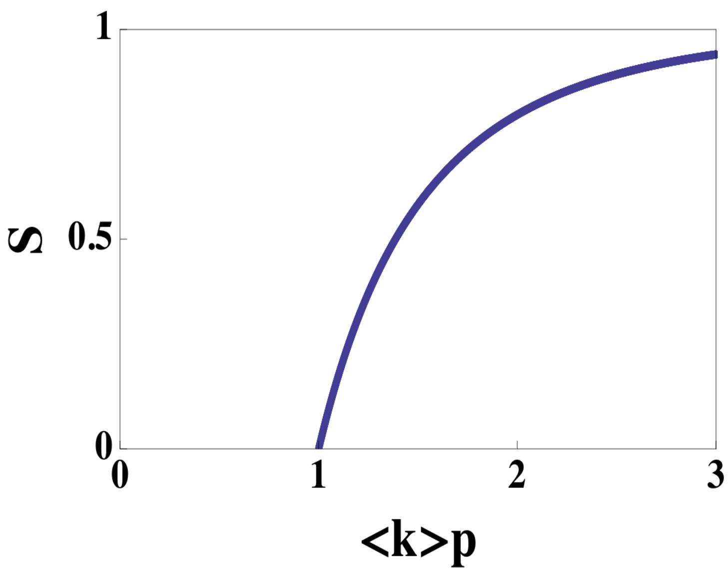
$$S = \begin{cases} (p - p_c)^{\beta} & for\ p \geq p_c \\ 0 & for\ p < p_c \end{cases}$$

$$p_c = 1/\langle k \rangle$$

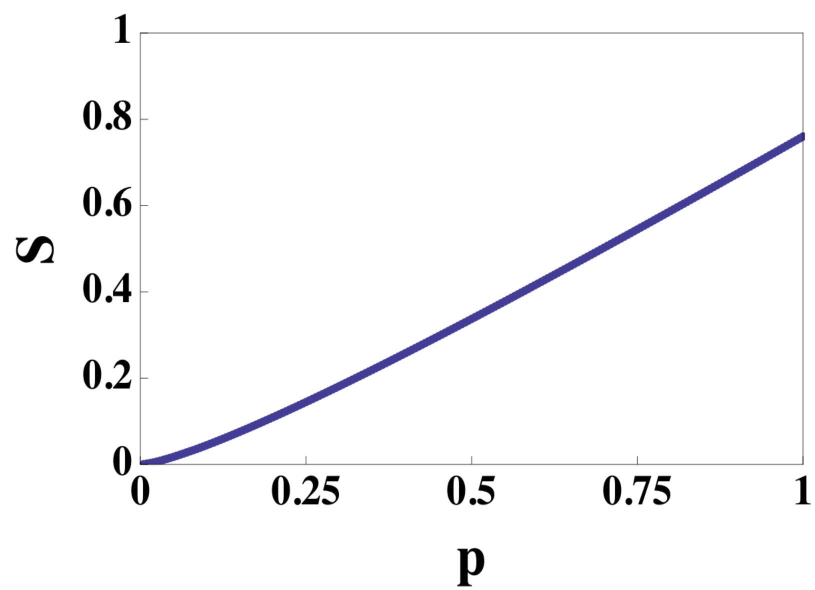
$$\beta = 1$$

Robustness of Scale-free networks under random failure

Poisson network



Scale-free network with $\gamma \in (2,3]$

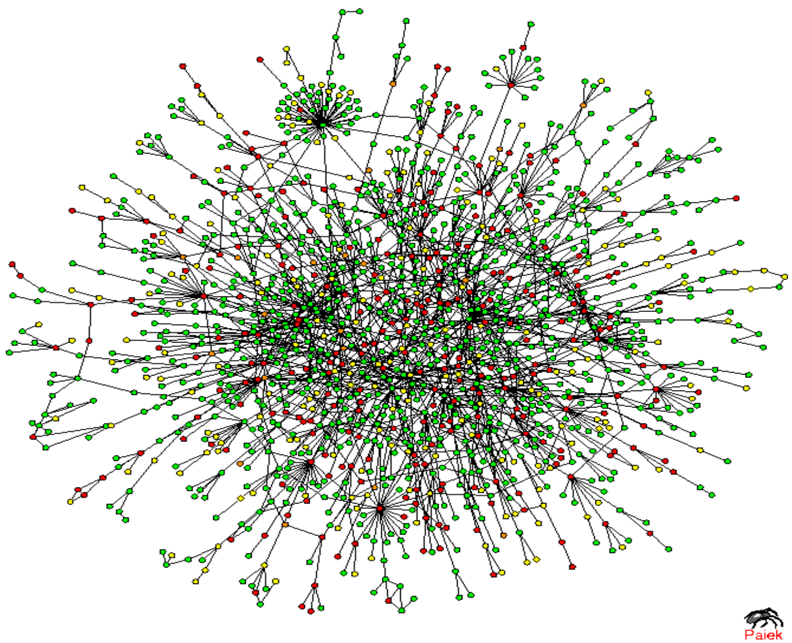


Condition for having a giant component

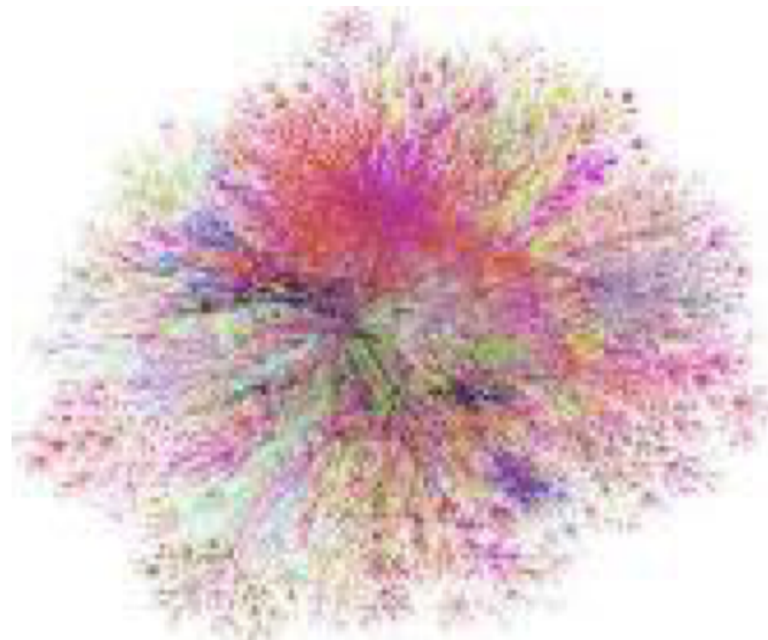
$$p > \frac{1}{\langle k \rangle} \Leftarrow \boxed{p > \frac{\langle k \rangle}{\langle k(k-1) \rangle}} \Rightarrow p > 0 \;\; for \; \gamma \in (2,3]$$

Robustness of Scale-free networks

Protein-protein interaction network



Internet



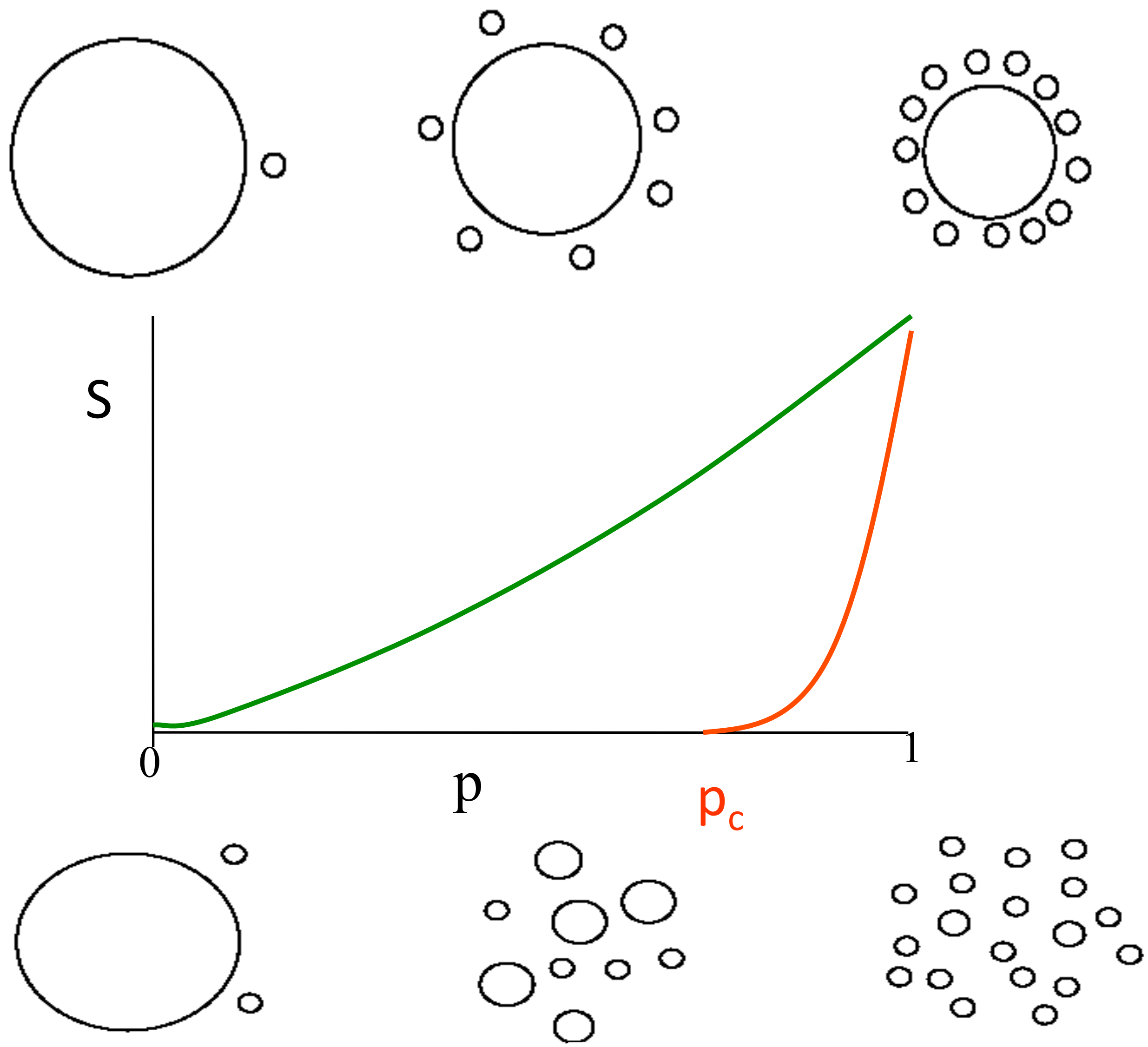
Complex scale free networks are very robust against random damage

Scale-free networks and targeted attack toward nodes of high degree

Failures

Topological error tolerance

R. Albert et al. 2000
Cohen et al. 2000
Cohen et al. 2001
R. Albert et al 2001

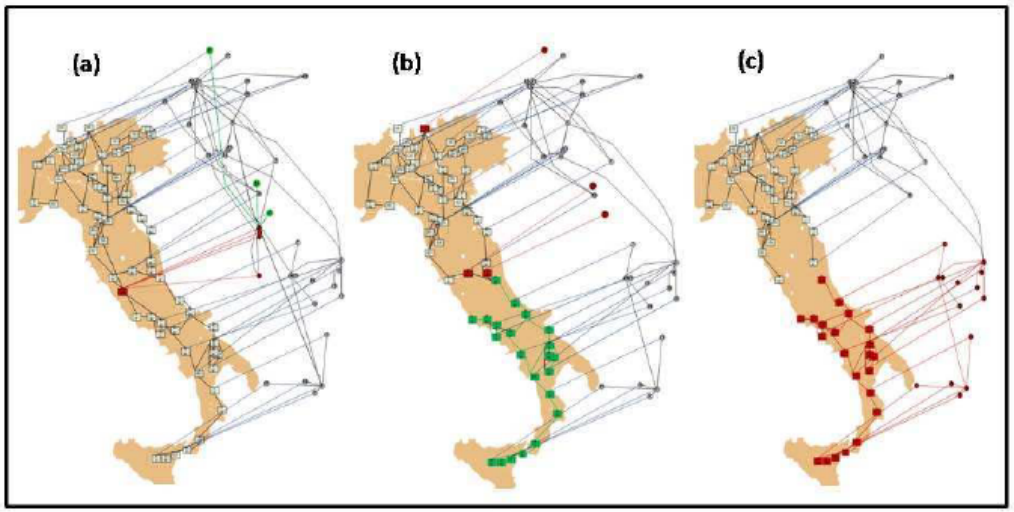


Attacks

Generalized percolation in multiplex networks

Interacting infrastructure networks

Complex infrastructures are interdependent and a failure in one network can generate a cascade of failures in the Interdependent networks



Buldyrev et al. Nature 2010

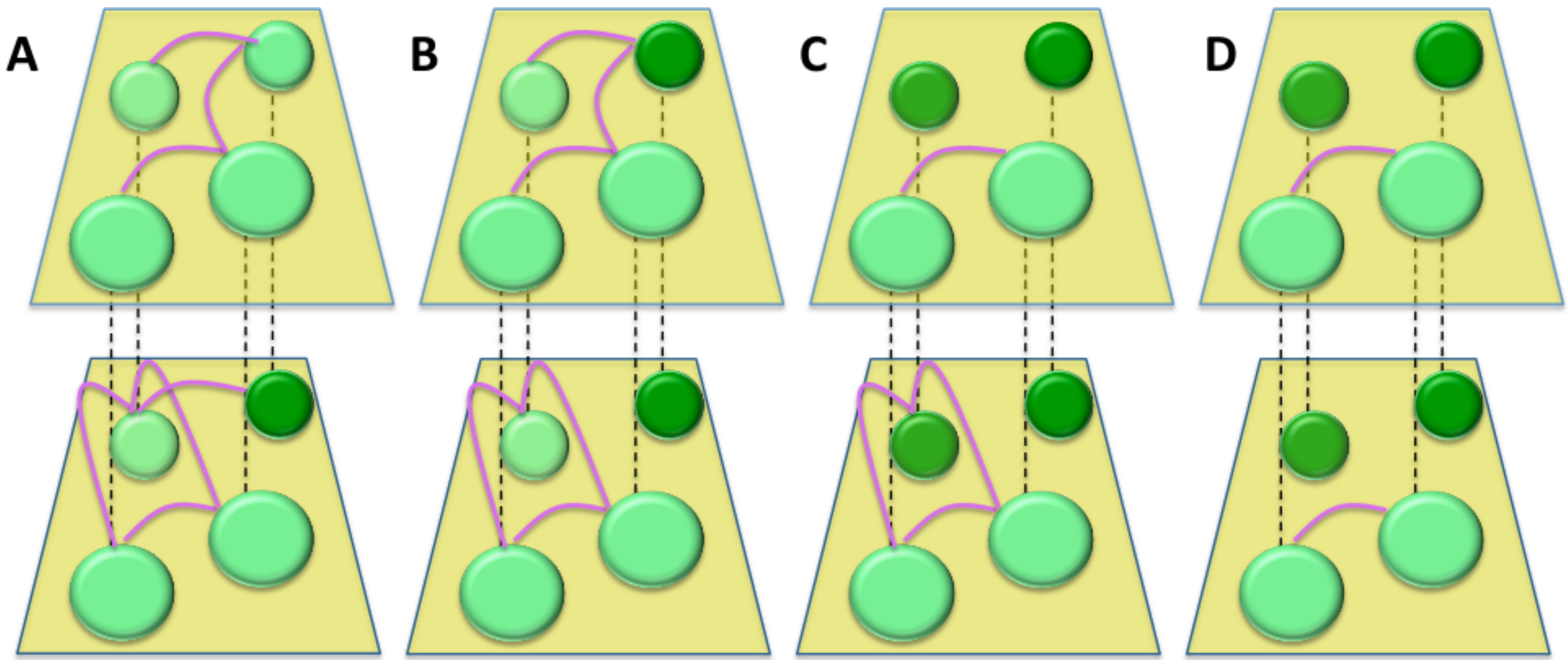
Interdependent multiplex networks

A multiplex network is **interdependent** if all the interlinks imply the interdependence of the connected replica nodes.

Two nodes are interdependent if the damage of one node implies the damage of the other interdependent node, independently on the rest of the network.

Mutually connected giant component

Any two nodes of the mutually connected giant component are connected by at least one path in each layer of the multiplex network



Buldyrev et al. 2010

Case of a Poisson multiplex network with M Layers

Nodes are damaged with probability 1-p

Fraction of nodes in the GC of single Poisson layer with average degree c:

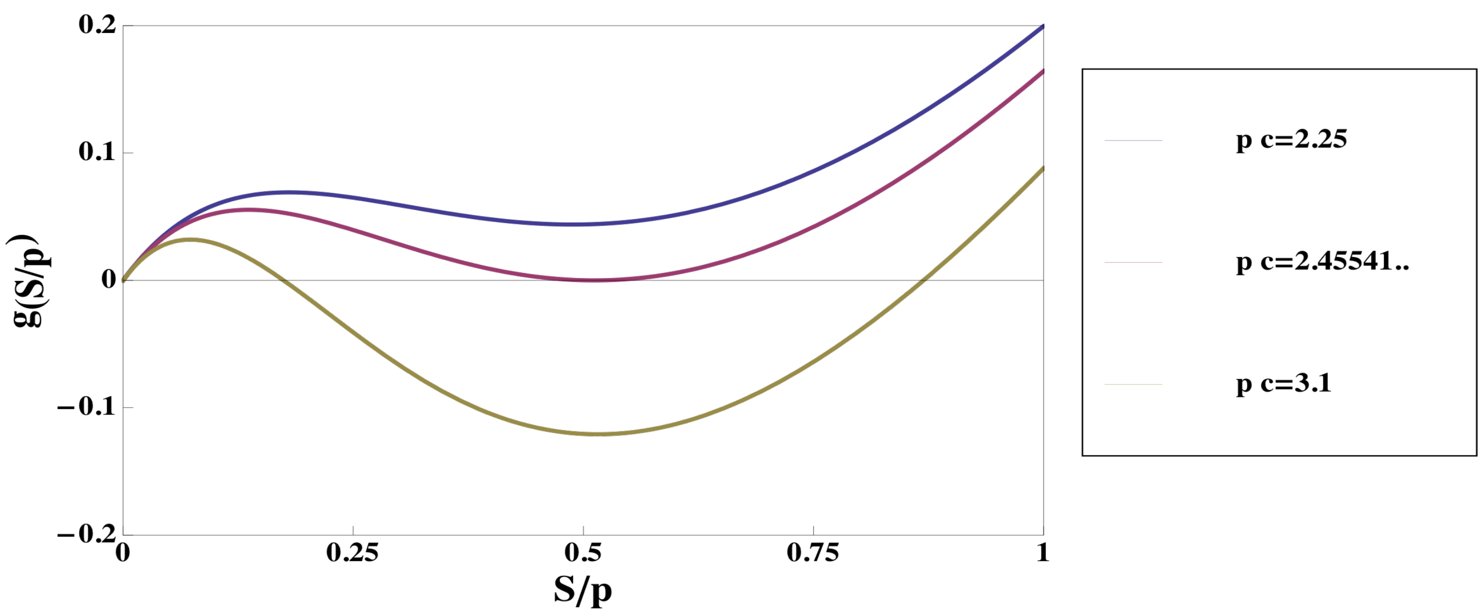
$$S = p\left(1 - e^{-cS}\right)$$

Fraction of nodes in the MCGC of multiplex network with M Poisson layers of average degree c:

$$S = p\left(1 - e^{-cS}\right)^{M}$$

Percolation on two interdependent Poisson networks with average degree c

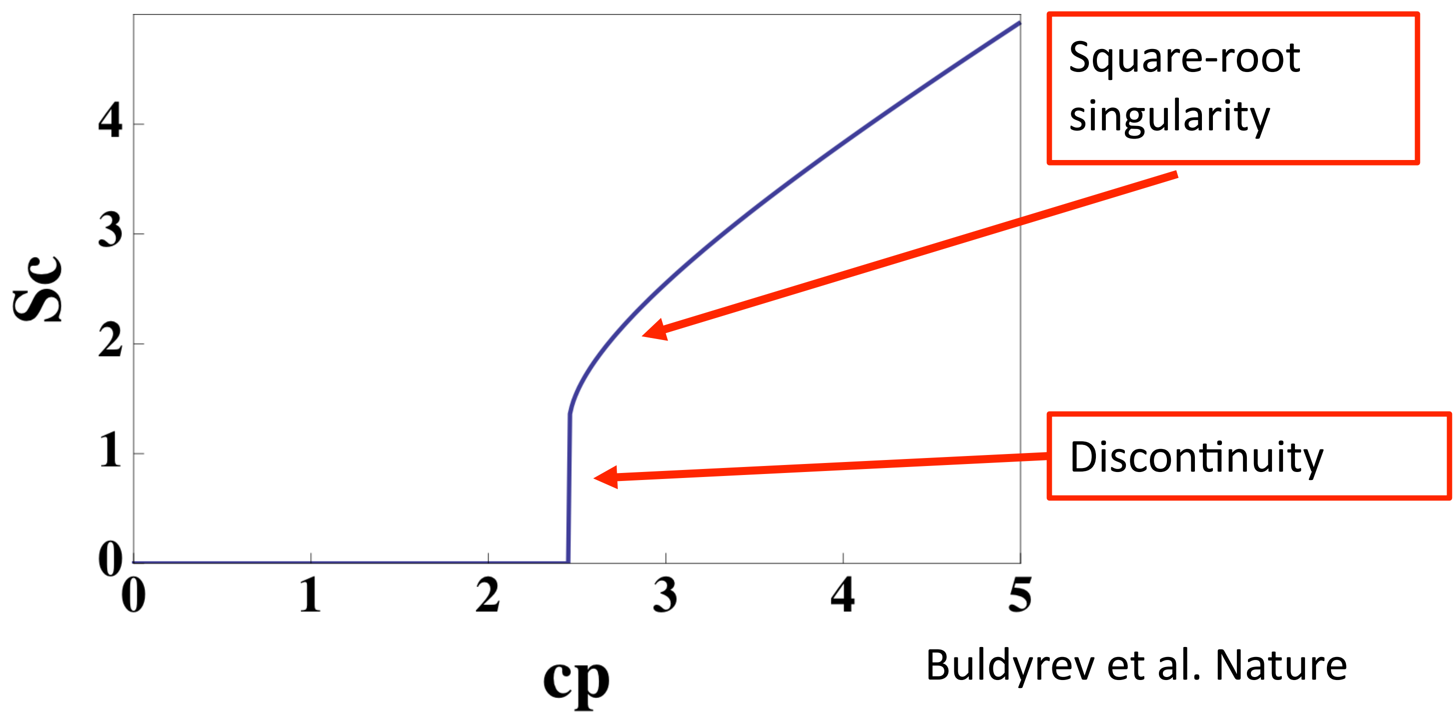
$$g(x = S/p) = x - \left(1 - e^{-cpx}\right)^2 = 0$$



The percolation transition at cp=2.455…
is discontinuous!

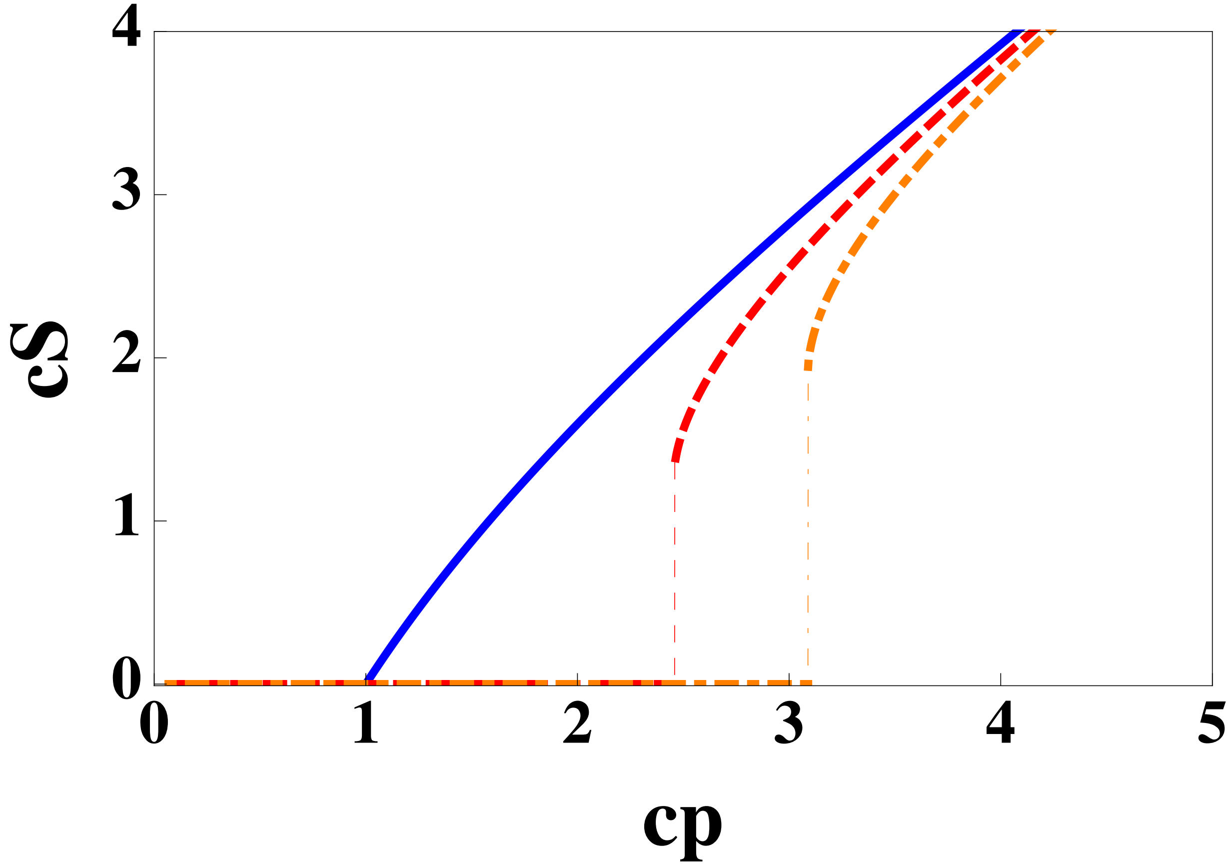
Discontinuous hybrid transition

Mutually connected giant component
in a muplex network with M=2 Poisson layers
of average degree c

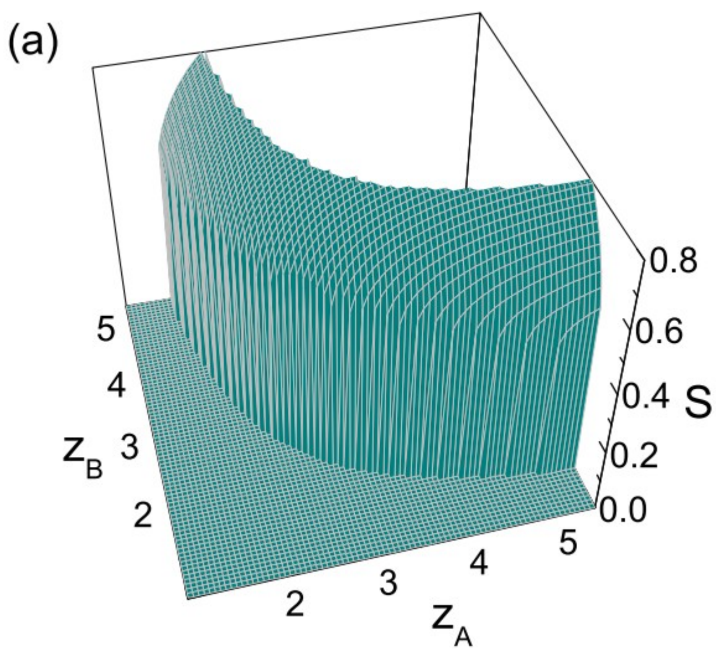


Buldyrev et al. Nature

Mutual connected component of a Poisson multiplex network with no link overlap

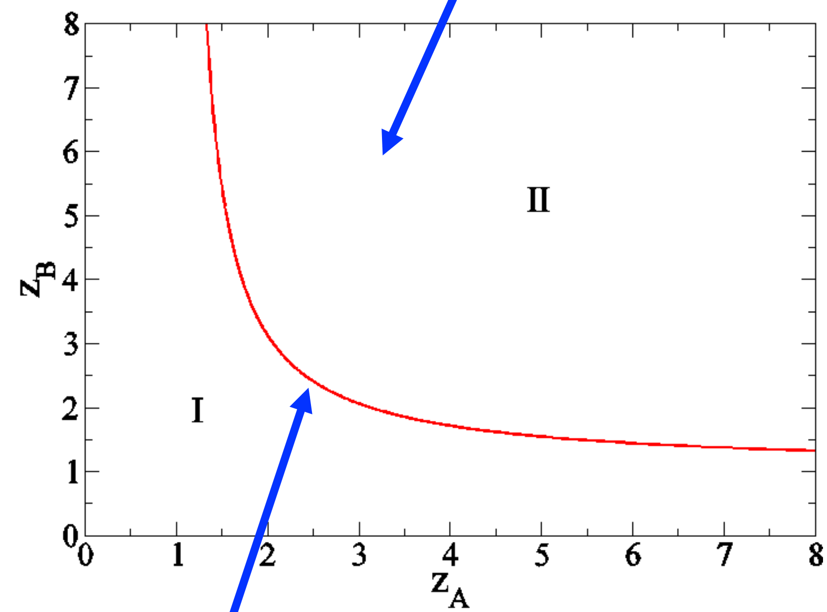


Phase diagram of ER-ER interdepedent networks
With average degree z_A and z_B



Son S.-W., et al. EPL(2012)

Region II: percolating

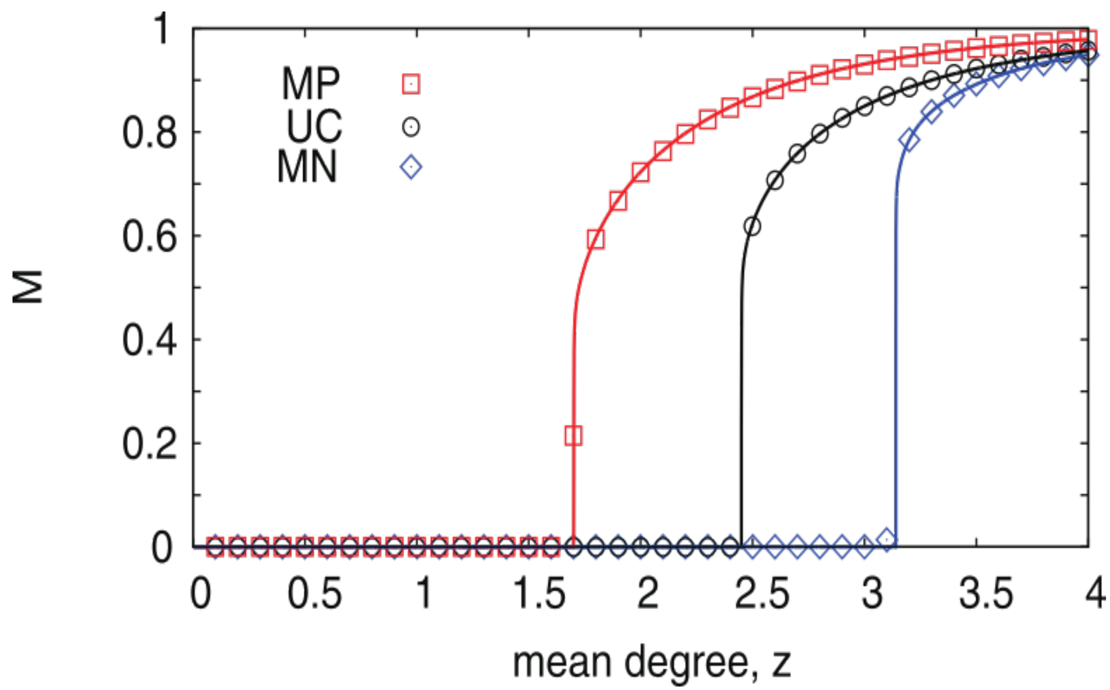


Region I: non percolating

Effects of degree correlations

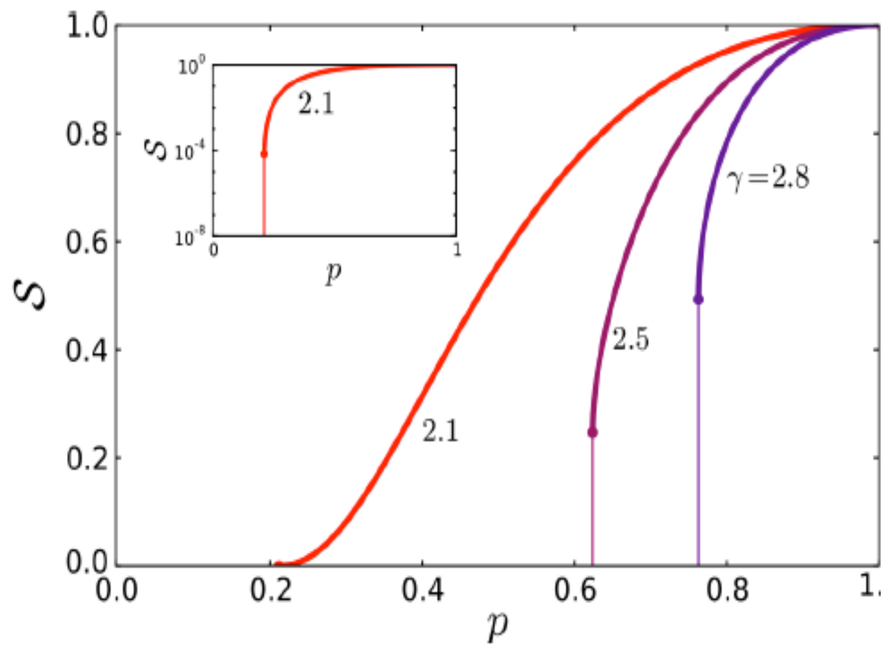
Positive degree correlations improve the robustness of a multiplex network.
MP maximally positive Degree correlations

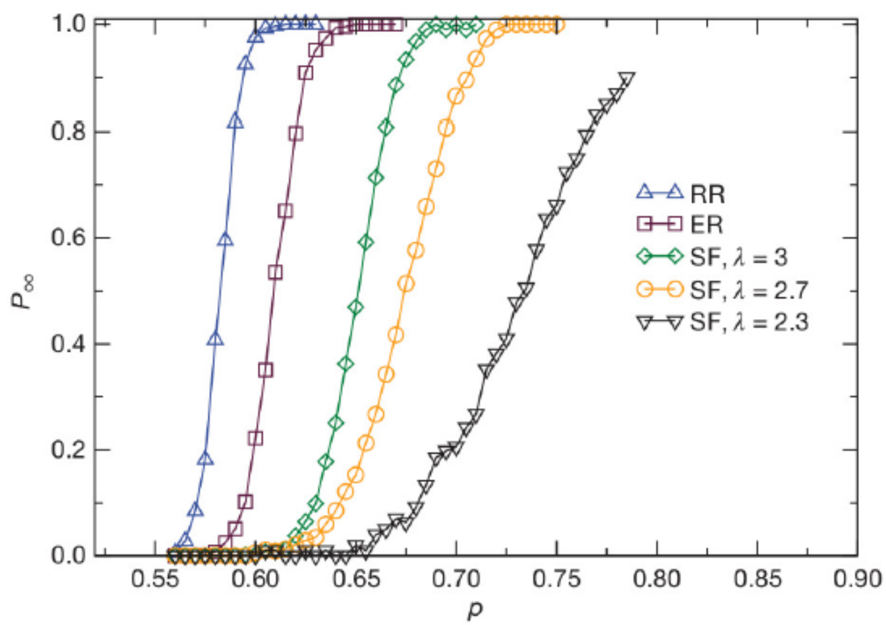
Negative degree correlations reduce the robustness of a multiplex network.
MN maximally negative degree correlations



Mutually connected component in scale free multiplex network

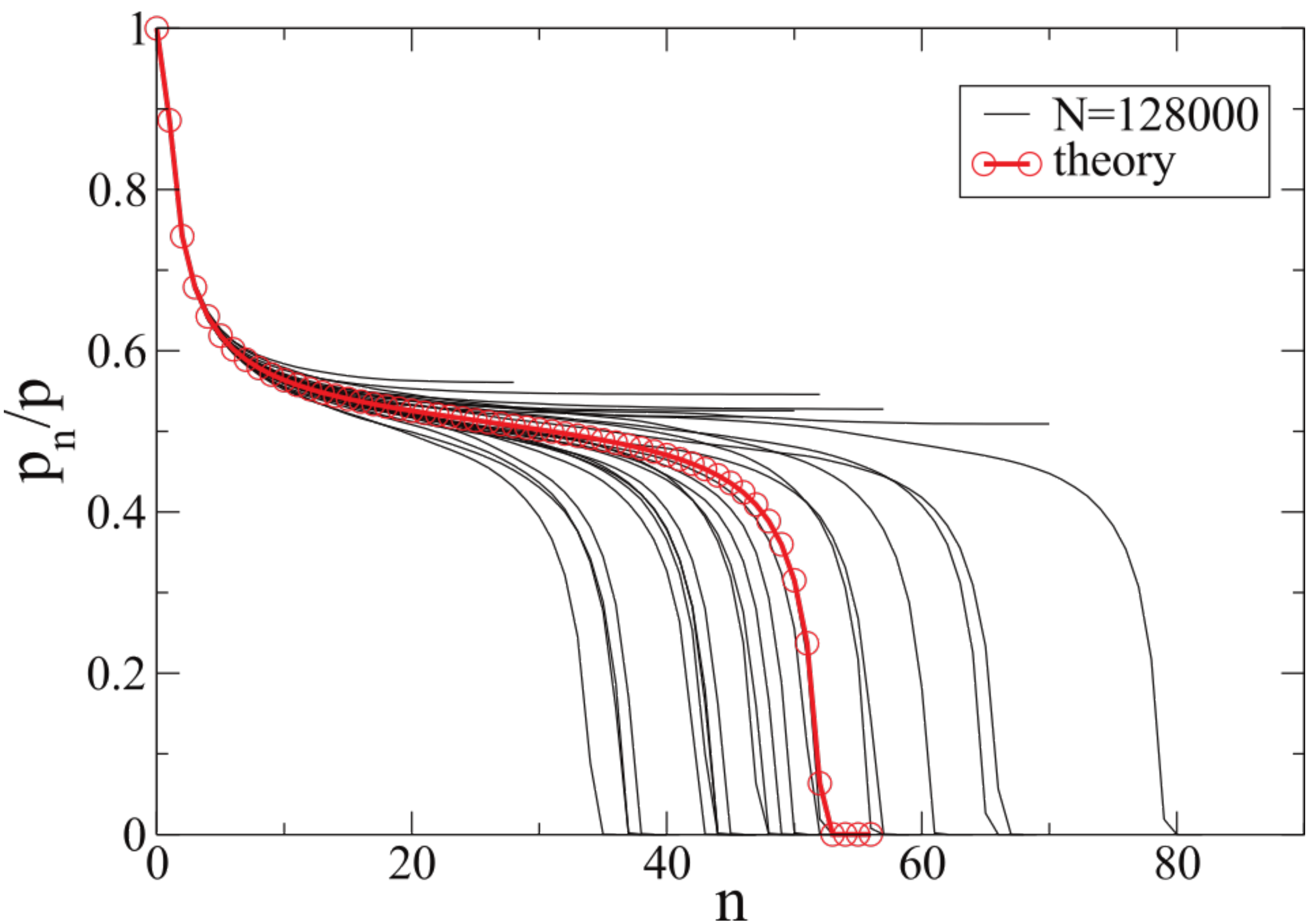
Fixed minimal degree (Baxter et al.)   Fixed average degree (Parshani et al.)





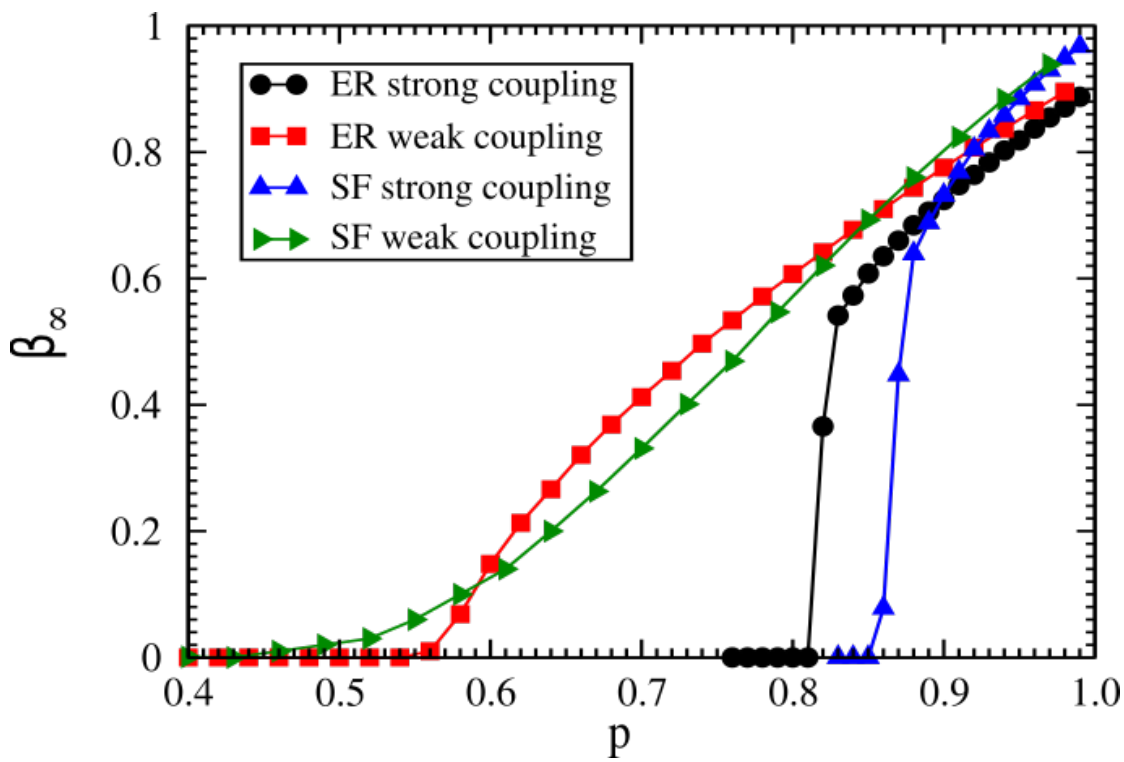
The discontinuity decreases, p_c increases with decreasing γ exponent

Cascade of failure events at the percolation transition



Buldyrev et al. Nature

Partial interdependence changes the nature of the percolation transition



Allowing for partial interdependence can change the nature of the transition from discontinuous to continuous.

Duplex network with Poisson Layers and Link Overlap

Duplex networks with Poisson multidegree distribution with

$$\langle k^{01}\rangle = \langle k^{10}\rangle = c_1$$
$$\langle k^{11}\rangle = c_2$$

MCGC

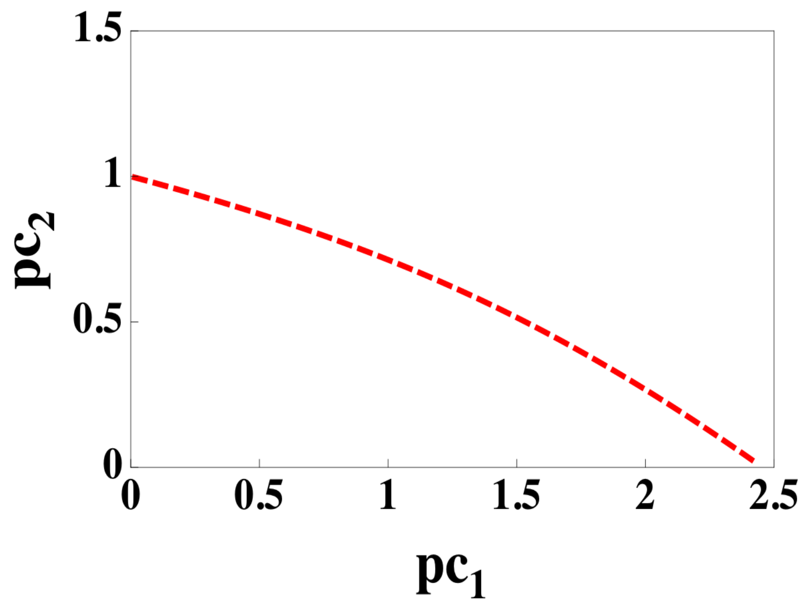
$$S = p\left(1 - 2e^{-c_1 S - c_2(S+S_{2,1})} + e^{-2c_1 S - c_2(S+S_{2,1})}\right)$$
$$S_{(1,1),(1,0)} = S_{2,1} = p\left(e^{-c_1 S - c_2(S+S_{2,1})} - e^{-2c_1 S - c_2(S+2S_{2,1})}\right)$$

Phase diagram for the MCGC in a duplex network

Duplex networks with Poisson multidegree distribution with

$$\langle k^{01} \rangle = \langle k^{10} \rangle = c_1$$

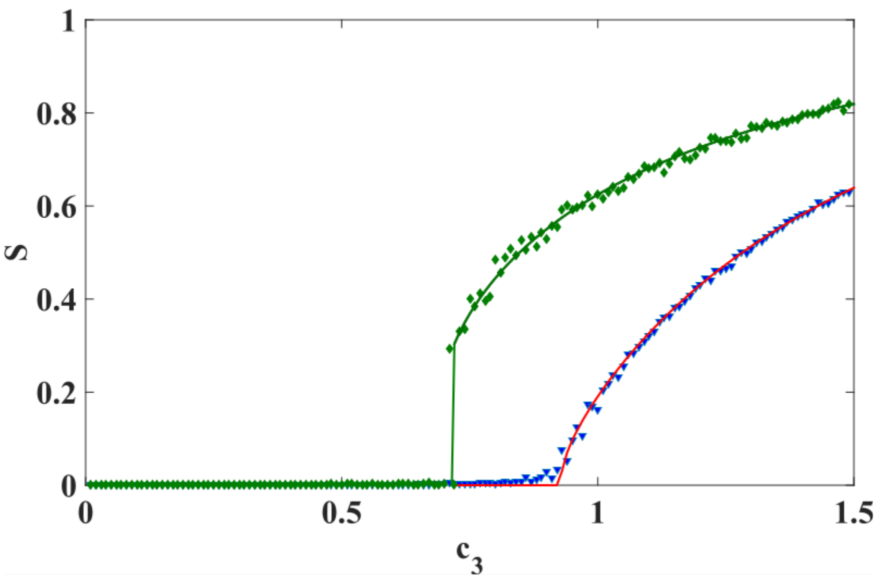
$$\langle k^{11} \rangle = c_2$$



Cellai et al (2016)

Multiplex network with three Poisson layers and link overlap

Multiplex networks with three layers with Poisson multidegree distribution



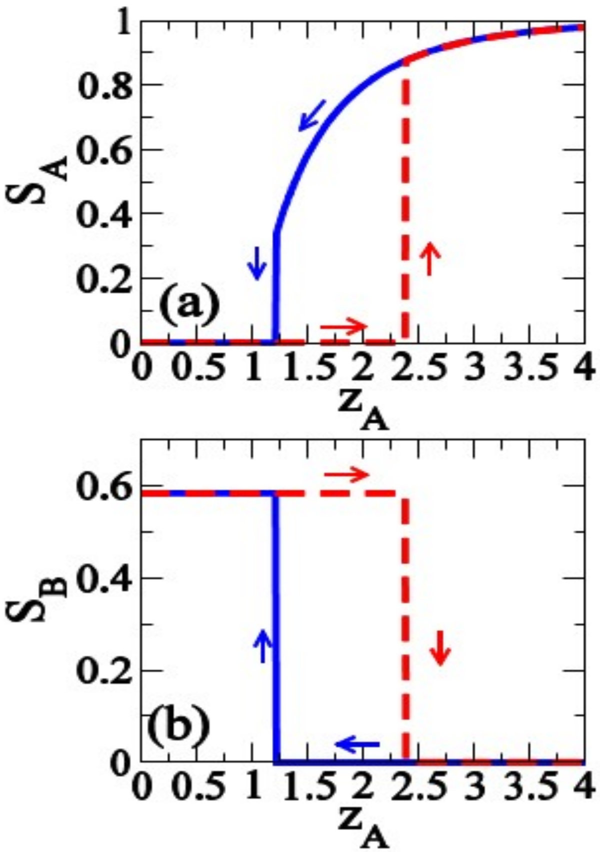
$$\langle k^{001}\rangle = \langle k^{010}\rangle = \langle k^{100}\rangle = c_1$$
$$\langle k^{110}\rangle = \langle k^{101}\rangle = \langle k^{011}\rangle = c_2$$
$$\langle k^{111}\rangle = c_3$$

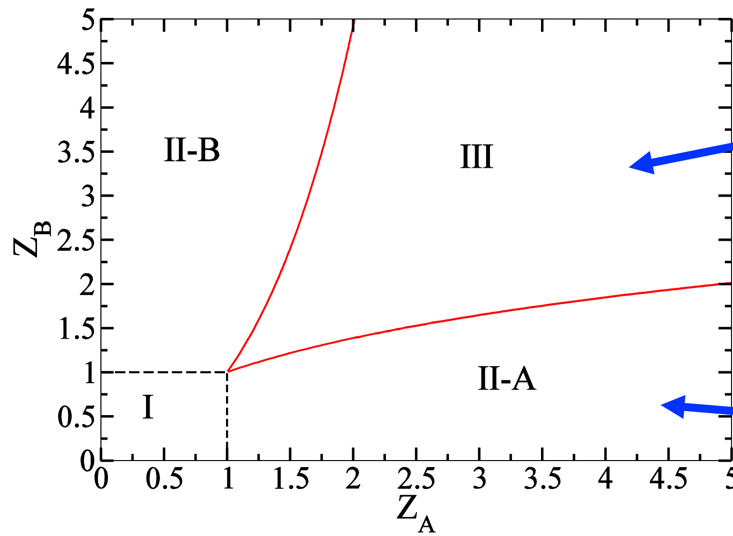
The determination of the MCGC involve solving a non-linear system of three variables

The network has a continuous phase transition only for complete overlap of the links

Competing networks

The function of a node in a network is incompatible with the function of the same node in the other network network





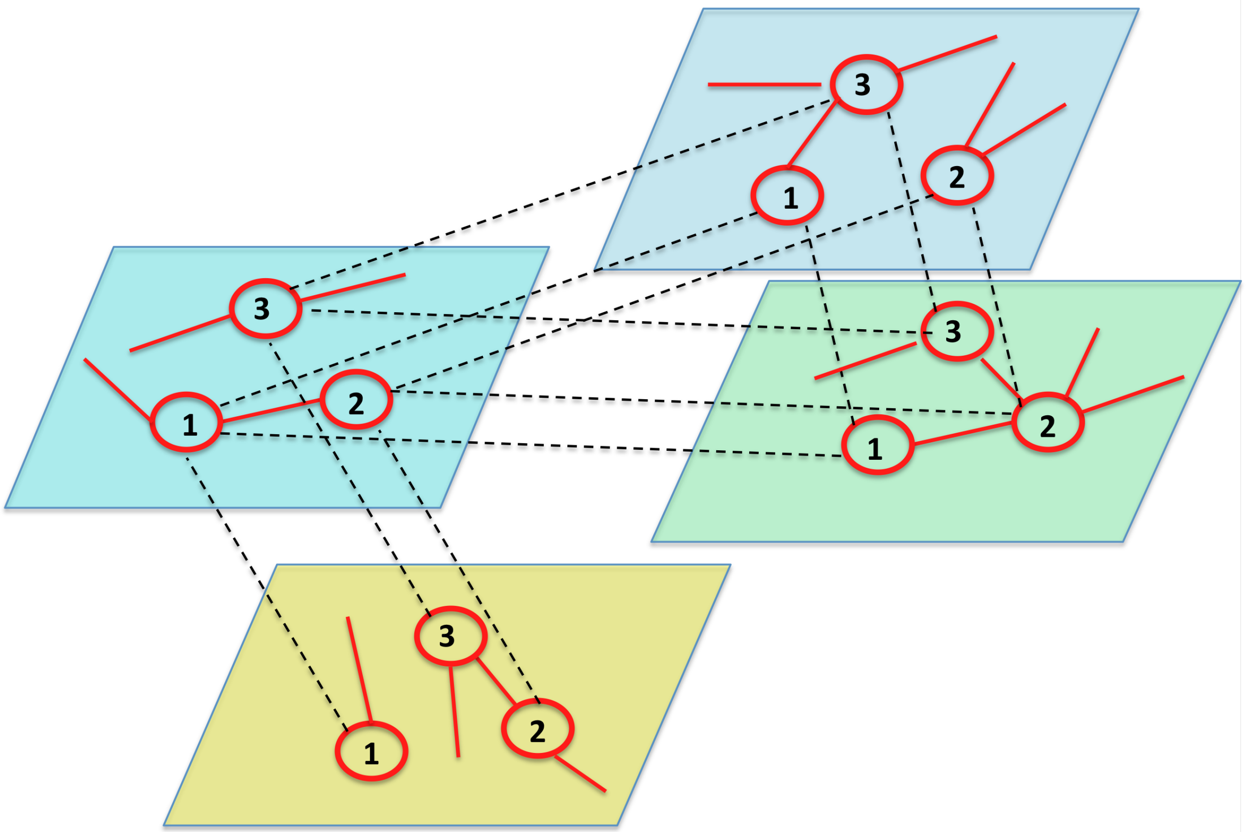
Region III: Bistable region, either one of the networks percolates

Region II: only one network can percolate

K. Zhao et al. JSTAT (2013)

Percolation in network of networks

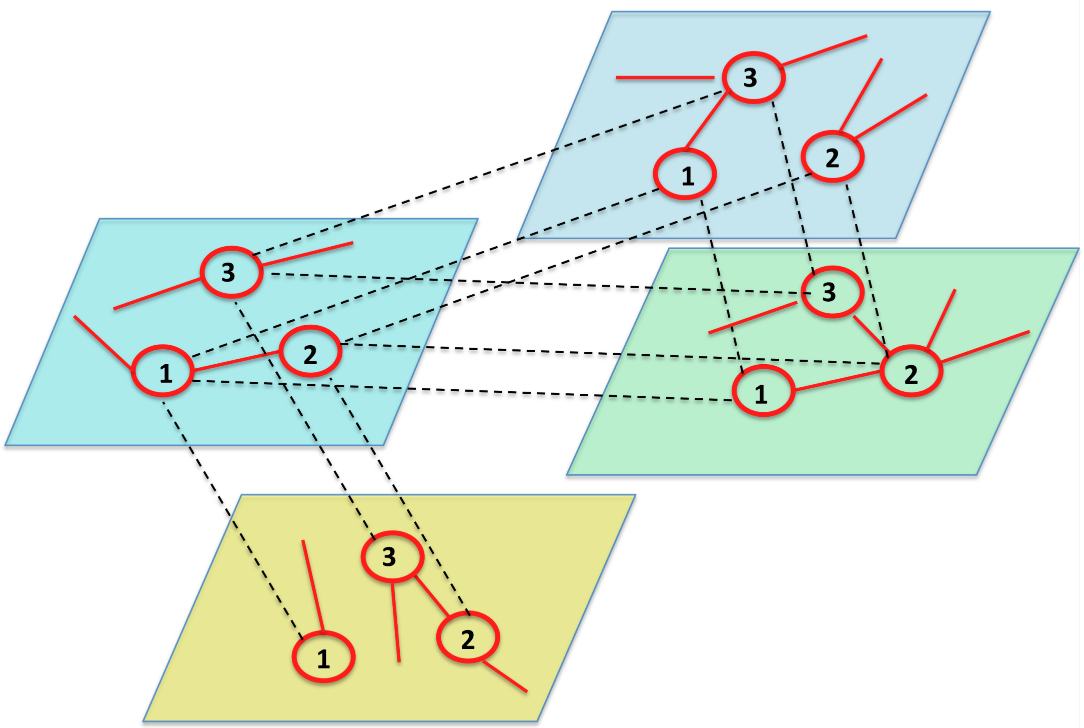
Network of Networks Case I



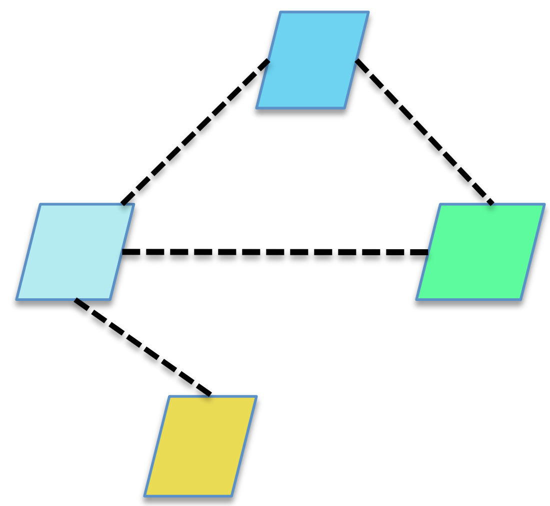
If a network is interacting with another network all the nodes of the network are interdependent with their “replica nodes” on the other network and vice versa.

The network of networks

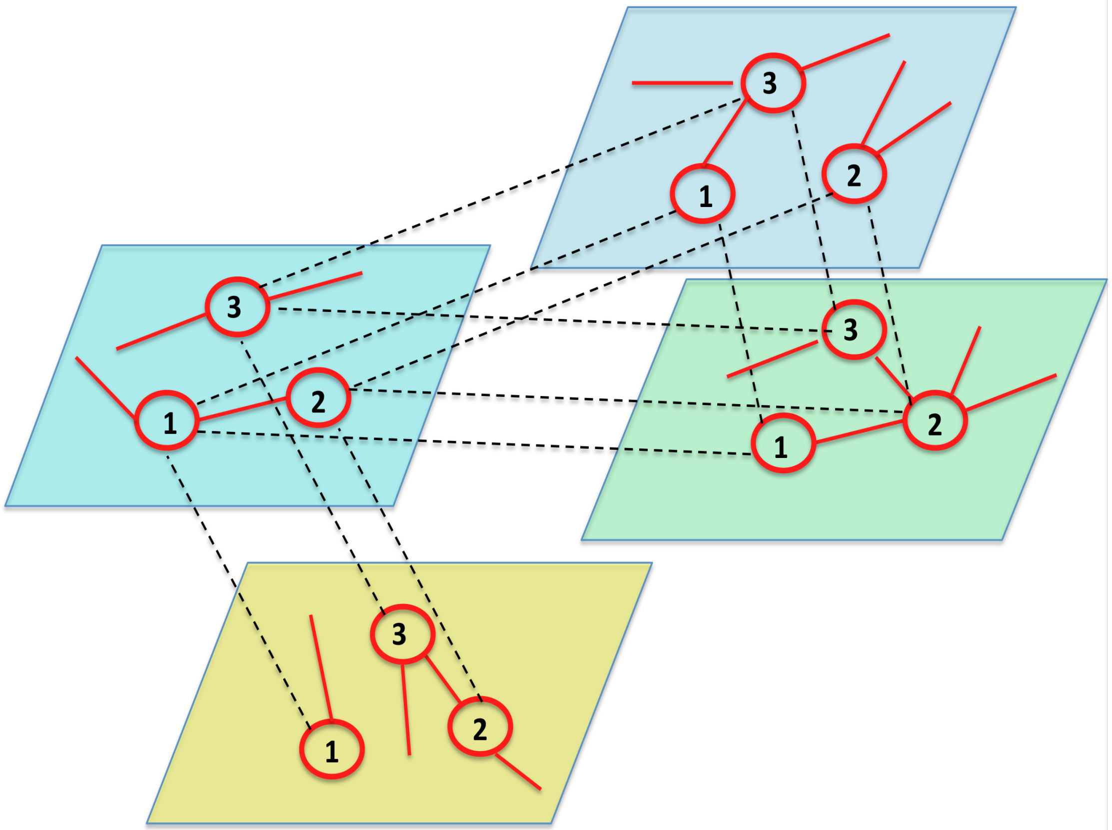
Interacting networks



Supernetwork



Network of Networks Case I



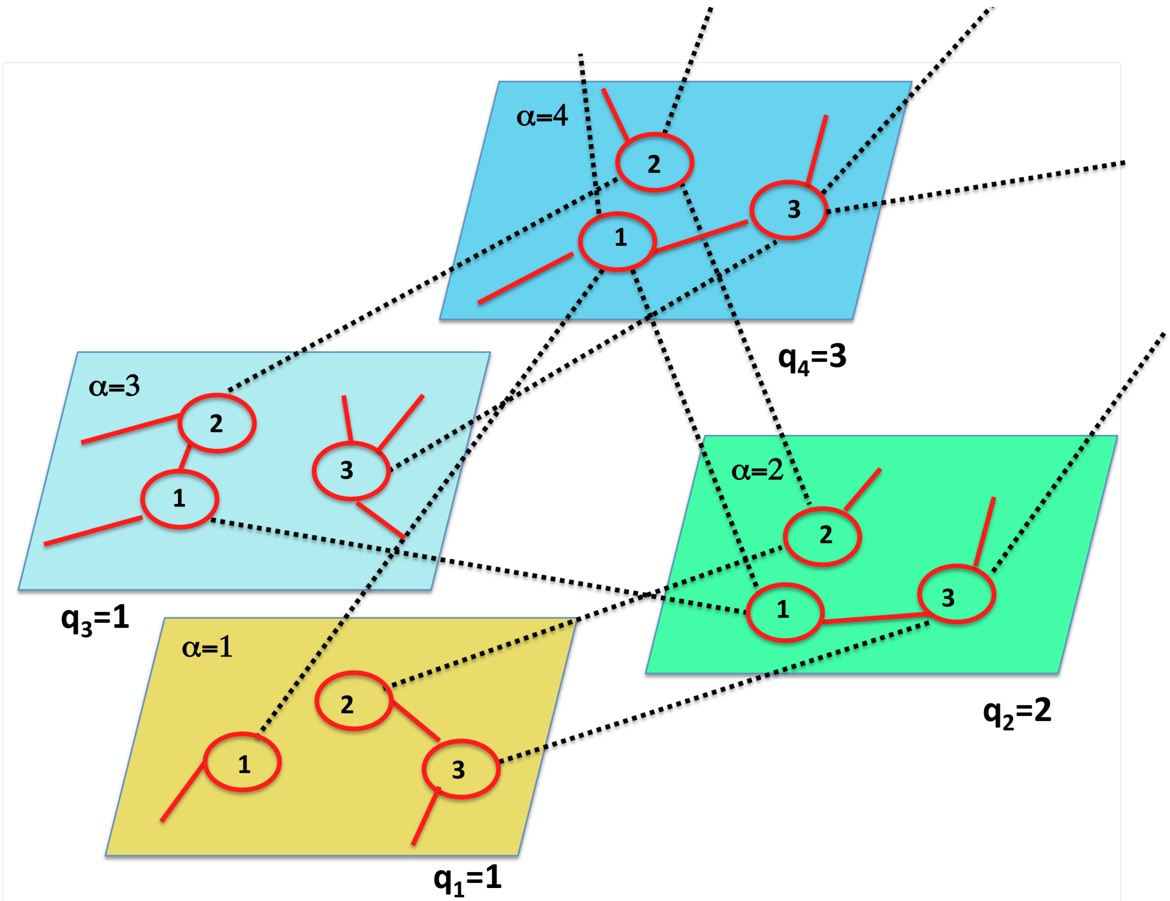
A node is in the mutually connected giant component if all the nodes that can be reached by interlinks have at least one neighbor in their layer that is in the percolation cluster.

Robustness of the network of networks

- **The robustness of a network of networks belonging to the case I is independent on the structure of the network of networks as long as the network of networks is connected.**
- All the layers start to percolate when the fraction of non-damaged nodes $p>p_c$
- The transition is discontinuous as long as M>1 if the layers are not correlated.

G. Bianconi, S.N. Dorogovtsev and J. F. F. Mendes 2014

Network of networks Case II



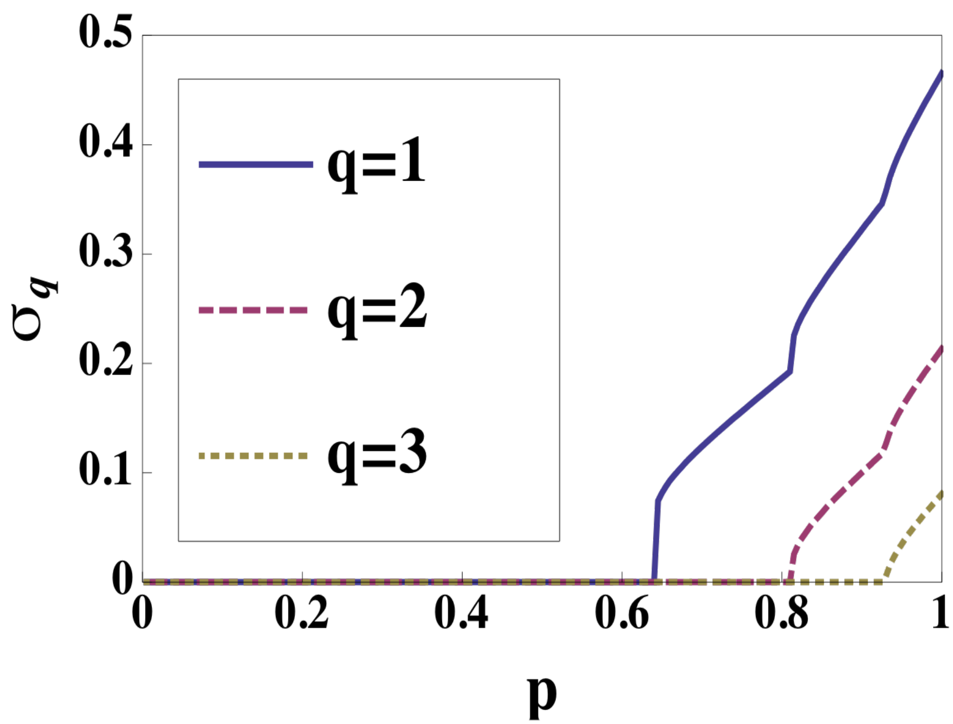
Every layer α has a supradegree $q_{,\alpha}$.

Therefore every node of layer α has q_α links to q_α replica nodes in some other layer chosen randomly

Main results for case II

- **The layers with higher superdegree are more fragile than layers with low superdegree.**
- **In the networks there are multiple percolation transitions corresponding to the activation of layers with increasing value of the superdegree.**
- Each of these transitions is discontinuous is the networks in the different layers are not correlated for r=1
- **If r<1 some of these transitions can become continuous**

Percolation in layers with superdegree q



Case γ=2.8 c=20

Multiple phase transitions!

Layers with larger superdegree are more vulnerable!

G. Bianconi and S.N Dorogovstev (2014)

Nature Physics News & Views

MULTILAYER NETWORKS

Dangerous liaisons?

Many networks interact with one another by forming multilayer networks, but these structures can lead to large cascading failures. The secret that guarantees the robustness of multilayer networks seems to be in their correlations.

Ginestra Bianconi

Natural complex systems evolve according to chance and necessity — trial and error — because they are driven by biological evolution. The expectation is that networks describing natural complex systems, such as the brain and biological networks within the cell, should be robust to random failure. Otherwise, they would have not survived under evolutionary pressure. But many natural networks do not live in isolation; instead they interact with one another to form multilayer networks — and evidence is mounting that random networks of networks are acutely susceptible to failure. Writing in *Nature Physics*, Saulo Reis and colleagues[1] have now identified the key correlations responsible for maintaining robustness within these multilayer networks.

In the past fifteen years, network theory[2,3] has granted solid ground to the expectation that natural networks resist failure. It has also extended the realm of robust systems to man-made self-organized networks that do not obey a centralized design, such as the Internet or the World Wide Web. In fact, it has been shown that many isolated complex biological, technological and social networks are scale free, meaning that their nodes

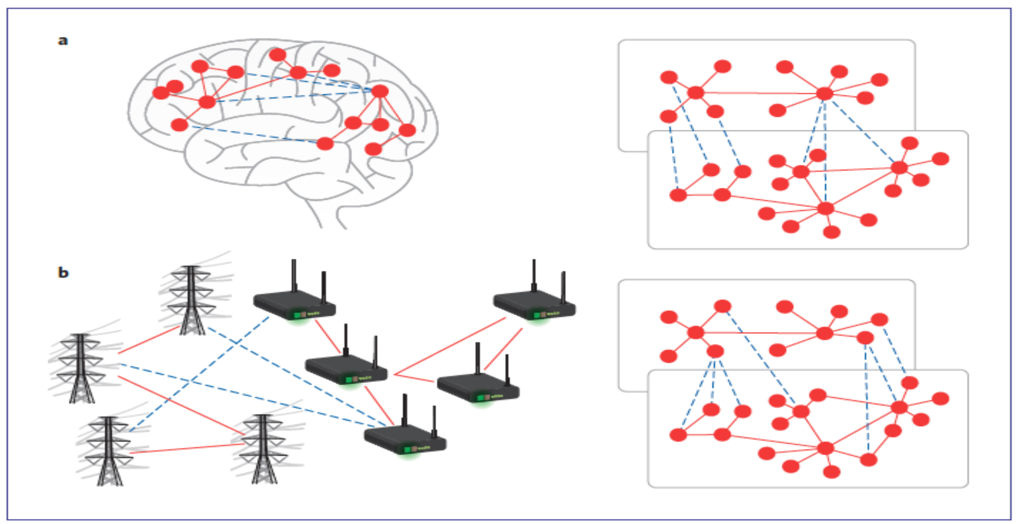
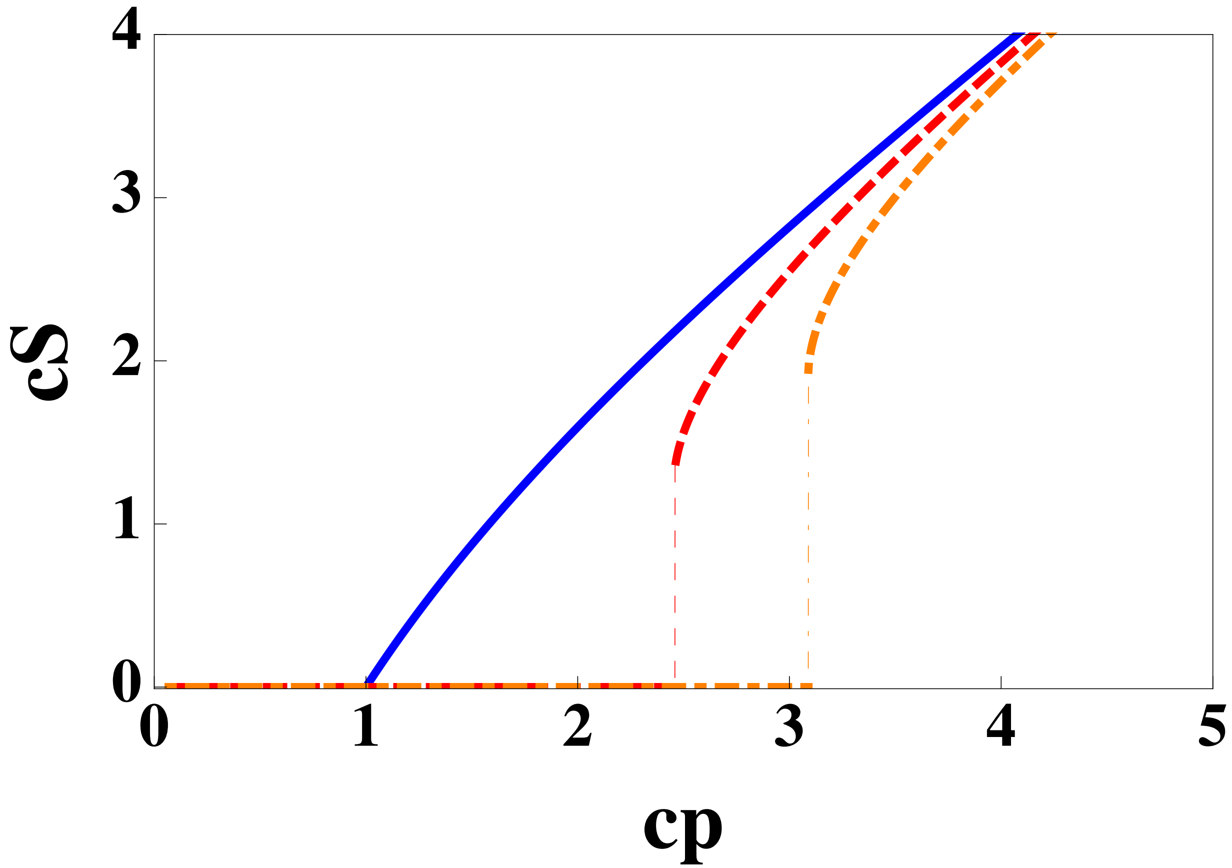


Figure 1 | Reis *et al.*[1] have shown that correlations between intra- (red) and interlayer (blue dotted) interactions influence the robustness of multilayer networks. **a**, In the brain, each network layer has multilayer assortativity and the hubs in each layer are also the nodes with more interlinks, so liaisons between layers are trustworthy. **b**, In complex infrastructures (such as power grids and the Internet), if the interlinks are random, the resulting multilayer network is affected by large cascades of failures[6], and liaisons can be considered dangerous.

Mutual connected component of a Poisson multiplex network with no link overlap

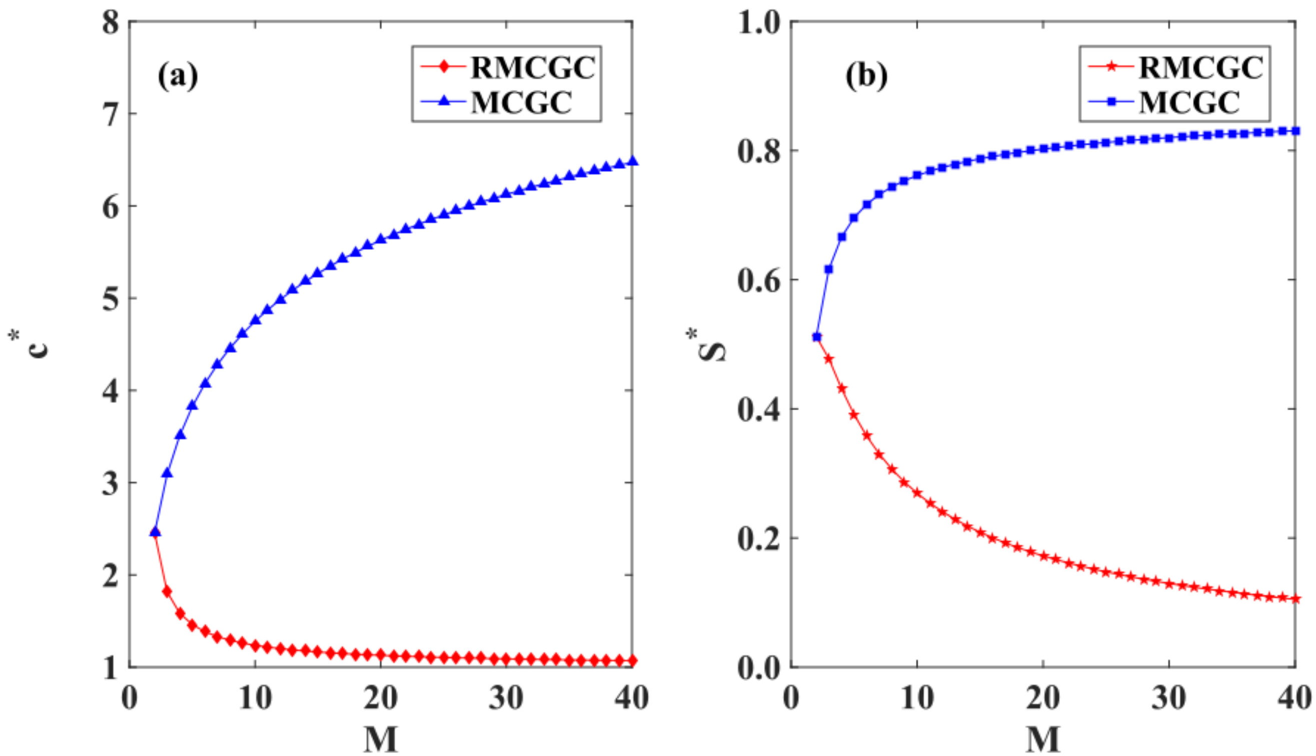


Redundant interdependencies

Is a multiplex network more or less robust if we add new layers?

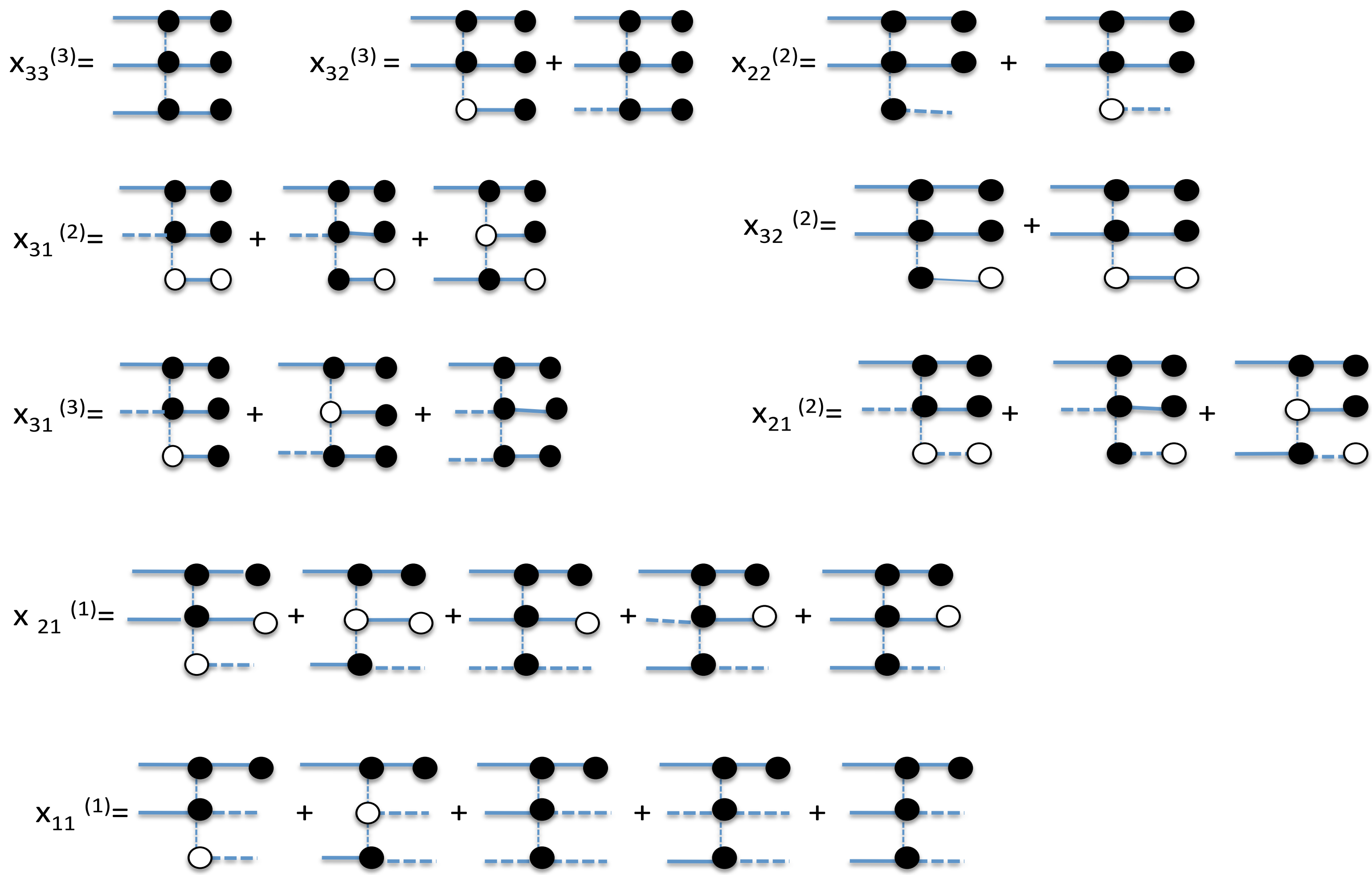
If interdependencies are redundant and a node can be in the Redundant MCGC as long as at least one replica node is active, then the more layers we add to the network the more robust it becomes

Redundant Mutually Connected Giant Component

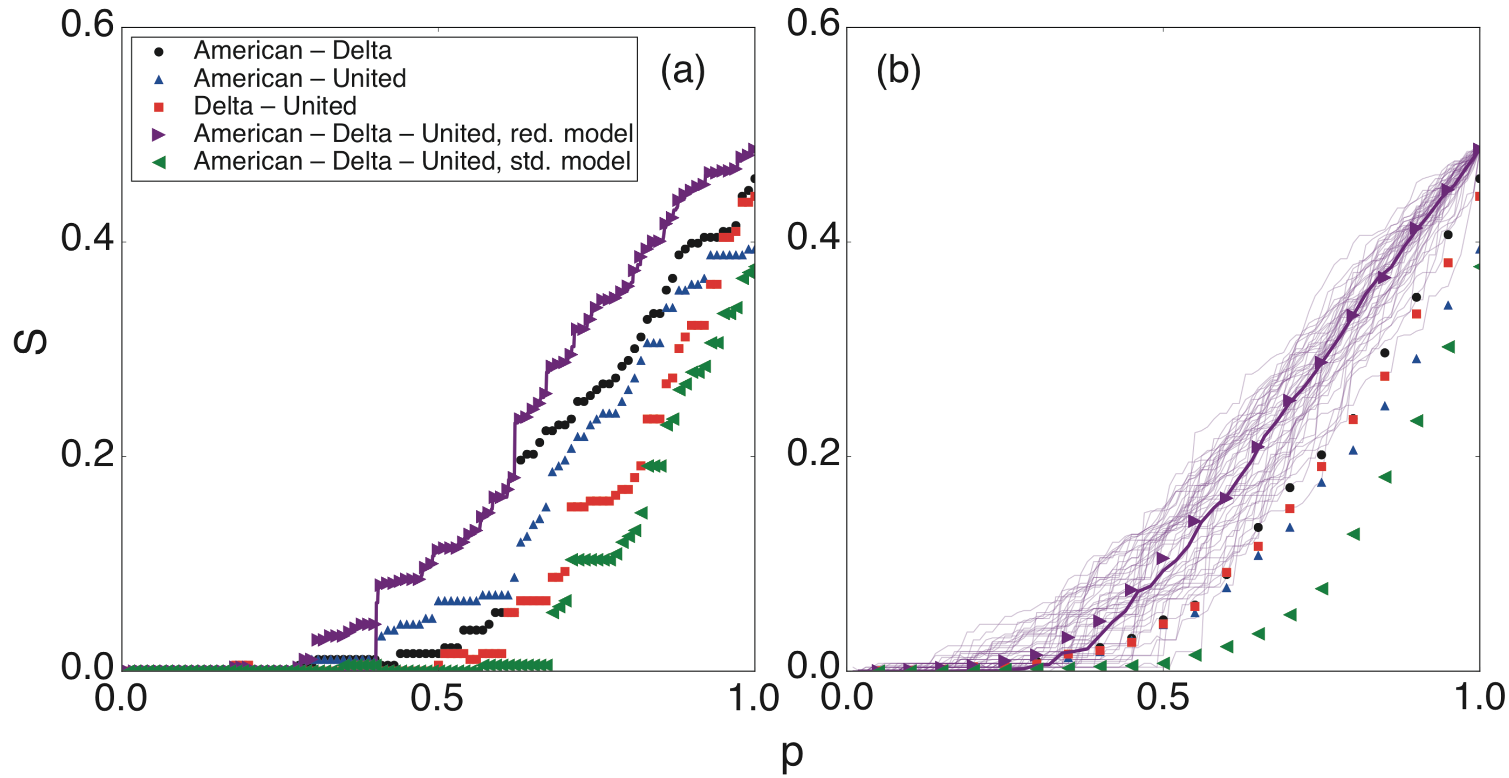


Radicchi Bianconi PRX (2017)

Equations



Redundant interdependencies boost the robustness of multilayer networks



Radicchi Bianconi PRX (2017)

Conclusions

Percolation on interdependent networks
captures the possible mechanisms
yielding fragile multilayer networks

Percolation in multilayer interdependent networks
display surprising novel phenomena

- In presence of interdependencies, the percolation transition becomes discontinuous and hybrid and is characterized by large avalanches of failure events.
- In presence of partial interdependencies it can become continuous.
- In network of networks it is possible to observe multiple phase transition.
- Redundant interdependencies might explain why many natural made networks have many layers as in this framework the robustness increases with the number of layers.

Message passing algorithm for percolation

Message passing algorithms are widely used for characterizing critical phenomena and dynamical systems in complex networks

- **Percolation on single networks**

(Karrer, Newman, Zdeborova PRL 2014)

- **Network control**

(Liu, Slotine & Barabasi Nature 2011)

- **Epidemic spreading in multi-slice networks**

(Valdano et al. PRX 2015)

Message passage algorithm for the Giant Component of a single network

The initial node damage is indicated by the variables s_i associated to the nodes of the network: s_i=0 if node i is damaged and s_i=1 otherwise.

The message going from node i to not j follows

$$\sigma_{i\to j} = s_i\left(1-\prod_{l\in N(i)\setminus j}(1-\sigma_{l\to i})\right)$$

The node i is in the giant component if σ_i=1 otherwise σ_i=0 where

$$\sigma_i = s_i\left(1-\prod_{l\in N(i)}(1-\sigma_{l\to i})\right)$$

Message passage algorithm for the Mutually Connected Giant Component in absence of link overlap

The initial node damage is indicated by putting
$s_i=0$ if node i is damaged and $s_i=1$ otherwise.
The generic message going from node i to node j is updated according to

$$\sigma_{i\to j} = s_i \prod_{\alpha=1,M}\left(1-\prod_{l\in N_\alpha(i)\setminus j}(1-\sigma_{l\to i})\right)$$

A node i is in the MCGC if $\sigma_i=1$ where

$$\sigma_i = s_i \prod_{\alpha=1,M}\left(1-\prod_{l\in N_\alpha(i)}(1-\sigma_{l\to i})\right)$$

Percolation in multiplex networks with overlap of the links: the message passing approach

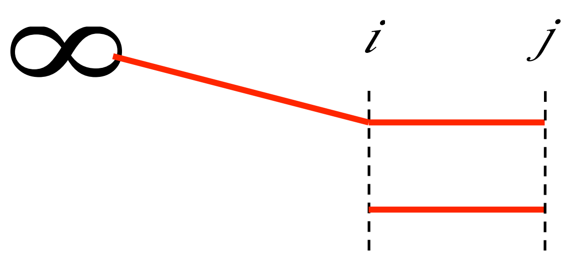
Directed percolation problem

Nodes in the directed mutually connected giant component (DMCGC) can be found by using the same algorithm used in absence of overlap of the links

In absence of overlap of the links

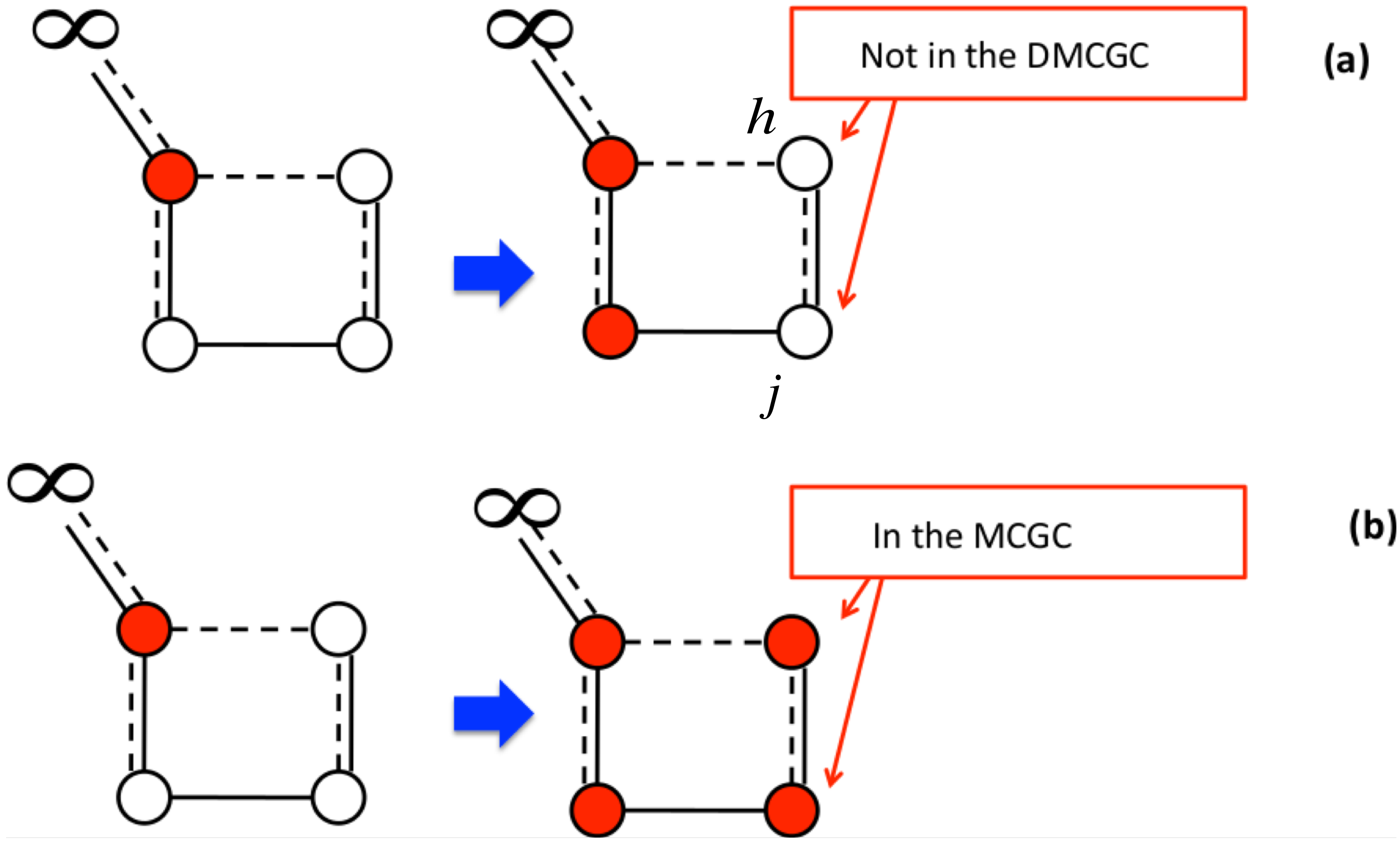
DMCGC=MCGC

Specifically we will have



$$\sigma_{i\to j} = 0$$

Difference between the DMCGC and the MCGC



Min et al. (2015) Cellai et al. (2016)

Required properties of the message passing algorithm for the MCGC

- The MCGC must be of maximum size:
  - the messages are polarized
  - the sender node must assume that the target node is in the MCGC.
- The messages must indicate the set of layers that connect the sender node to the MCGC.

$$\vec{n} = (n_1, n_2 \ldots, n_M)$$

The algorithm

The message

$$\vec{n}_{i\to j} = (n_{i\to j}^{[1]}, n_{i\to j}^{[2]}, \ldots n_{i\to j}^{[M]}), \quad n_{i\to j}^{[\alpha]} = 0,1$$

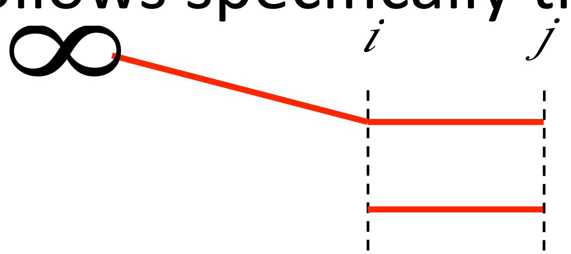
indicates that

assuming that j belongs to the MCGC

- node i must be in the MCGC

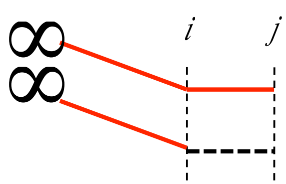
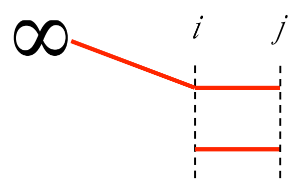
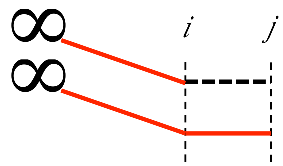
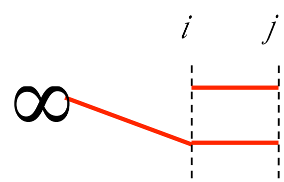
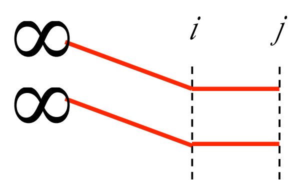
- node i connects node j to the MCGC *exclusively* through the layers α with $n_{i\to j}^{[\alpha]} = 1$

It follows specifically that we have

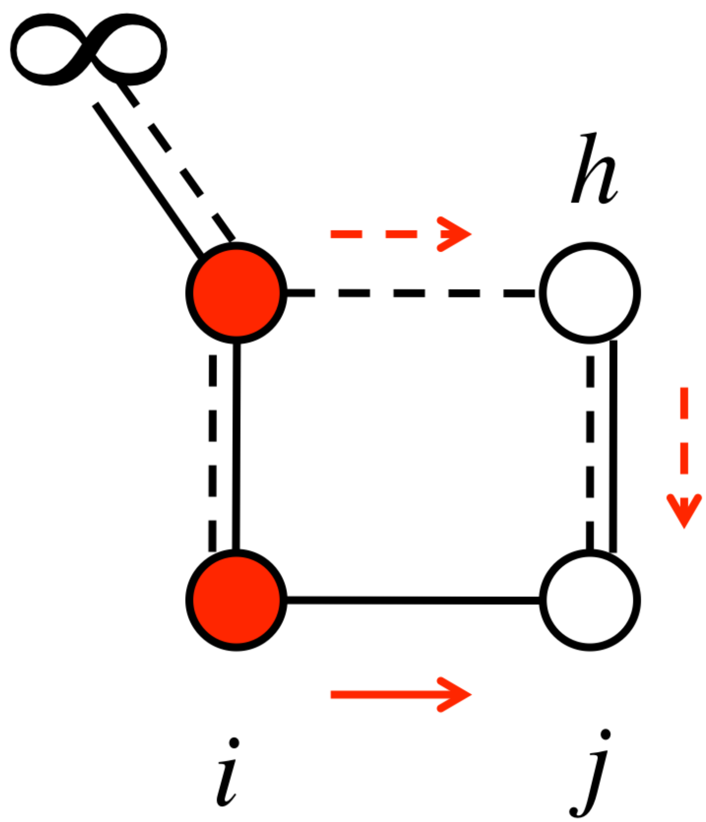


$$\vec{n}_{i\to j} = (1,0)$$

Non-trivial cases for M=2

	$\vec{m}_{ij} = (1,0)$	$\vec{m}_{ij} = (0,1)$	$\vec{m}_{ij} = (1,1)$
$\vec{n}_{i \to j} = (1,0)$			
$\vec{n}_{i \to j} = (0,1)$			
$\vec{n}_{i \to j} = (1,1)$			

How this algorithm can predict that node j and h are in the MCGC



Cellai et al. (2016)

Duplex network with Poisson Layers and Link Overlap

Duplex networks with Poisson multidegree distribution with

$$\langle k^{01}\rangle = \langle k^{10}\rangle = c_1$$
$$\langle k^{11}\rangle = c_2$$

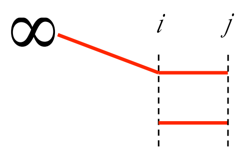
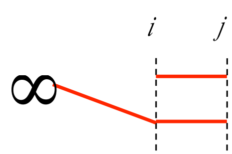
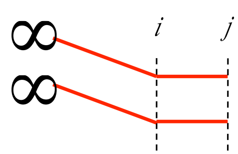
MCGC

$$S = p\left(1 - 2e^{-c_1 S - c_2(S+S_{2,1})} + e^{-2c_1 S - c_2(S+S_{2,1})}\right)$$
$$S_{(1,1),(1,0)} = S_{2,1} = p\left(e^{-c_1 S - c_2(S+S_{2,1})} - e^{-2c_1 S - c_2(S+2S_{2,1})}\right)$$

DMGC

$$S = p\left(1 - 2e^{-(c_1+c_2)S} + e^{-(2c_1+c_2)S}\right)$$

DMGC and MCGC messages

	DMCGC	MCGC
	$\sigma_{i\to j} = 0$	$\vec{n}_{i\to j} = (1,0)$
	$\sigma_{i\to j} = 0$	$\vec{n}_{i\to j} = (0,1)$
	$\sigma_{i\to j} = 1$	$\vec{n}_{i\to j} = (1,1)$

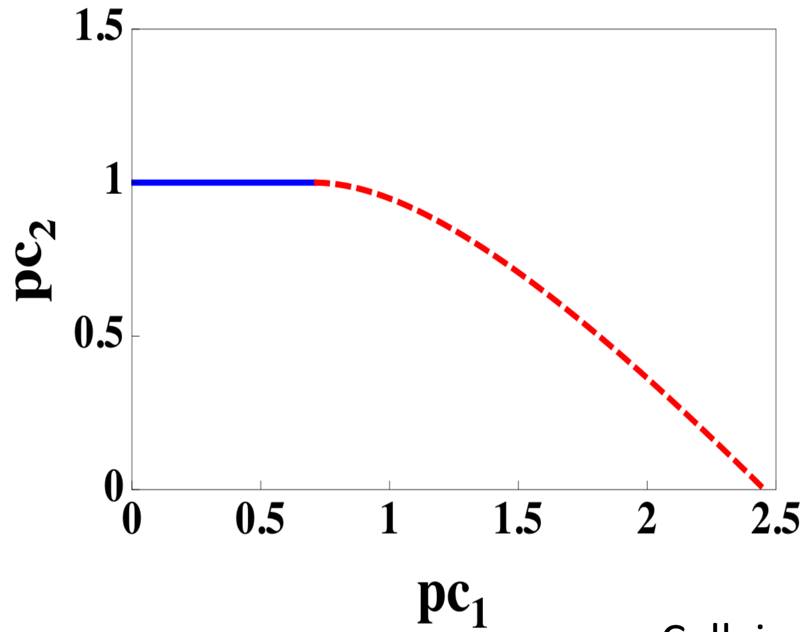
Phase diagram for DMCGC and MCGC

Duplex networks with Poisson multidegree distribution with

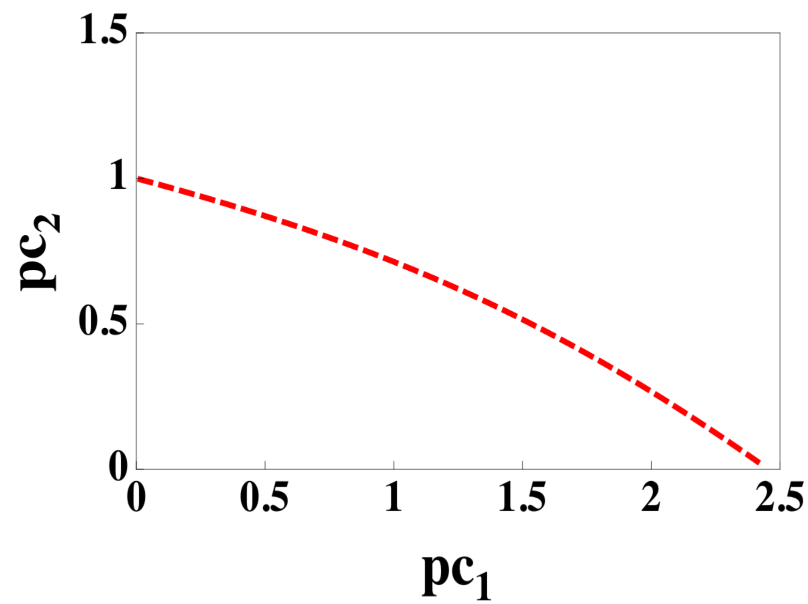
$$\langle k^{01}\rangle = \langle k^{10}\rangle = c_1$$

$$\langle k^{11}\rangle = c_2$$

DMCGC



MCGC



Cellai et al. PRE (2013); Cellai et al (2016)

Conclusions

We have formulated a message passing theory for percolation and directed percolation in multiplex network with link overlap.

- *Both algorithms reduce to percolation in multiplex network in absence of overlap and to percolation on single network in presence of complete overlap.*
- *The algorithm for directed percolation has an epidemic spreading interpretation. The algorithm for percolation does not have a feed-forward character.*
- *The two critical phenomena have different phase diagrams.*

The algorithm for the MCGC can be used to study

1. *the robustness of real multiplex networks and*
2. *to study the percolation phase diagram of multiplex networks with link overlap and arbitrary number of layers.*