COMPLEX NETWORKS AND POINT PROCESSES WITH APPLICATIONS

Connection functions and connectivity

September 11, 2017

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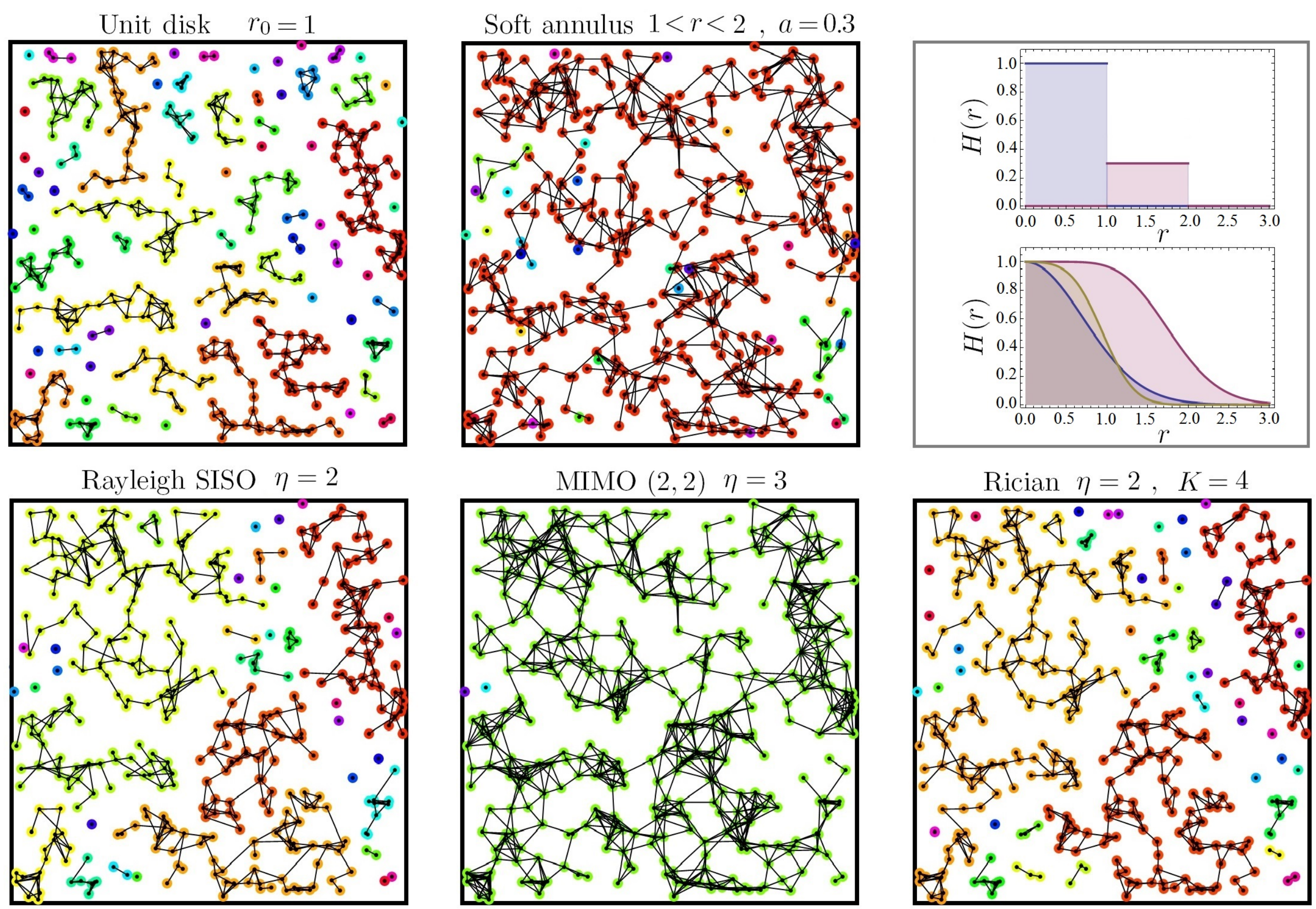
Summary

1. The unit disk model and generalisations
2. Pair connections in wireless communications
3. Mean degree
4. Connectivity and boundaries



The unit disk model

Introduced by E. N. Gilbert [Gilbert61]:

> *Recently random graphs have been studied as models of communications networks. Points (vertices) of a graph represent stations; lines of a graph represent two-way channels. In the literature [Austin et al 1959, Erdős-Rényi 1960 and Gilbert 1959] each pair of stations has some probability (the same for all pairs of stations regardless of their separation) of being joined by a channel. Such a model cannot represent accurately a network of shortrange stations spread over a wide area. The random plane networks of this paper provide a simple model in which the range of the stations is a parameter.*
>
> *To construct a random plane network [now called* **Random Geometric Graph (RGG)***], first pick points from the infinite plane by a Poisson process with density D [here, ρ] points per unit area. Next join each pair of points by a line if the pair is separated by distance less than R [here, r_0].*

This model remains very popular due to its simplicity [Walters11]; also called the **unit disk** or **hard connection** model. For now, we are looking at just the connection model; we will see later that its properties are often qualitatively different from random connection models.

Generalisations of the unit disk model

Deterministic:

Hard annulus Nodes connect with mutual distance $r \in [r_{min}, r_{max}]$; helpful to minimise hop count, improve secrecy.

Anisotropic Nodes have an orientation as well as a position, and connect under some condition on their separation and orientations; helpful to take account of beam forming; more on this later.

Partly deterministic:

Quasi unit disk Nodes connect if $r < r_{min}$, do not connect if $r > r_{max}$ and may connect if $r_{min} < r < r_{max}$ depending on other features of the model.

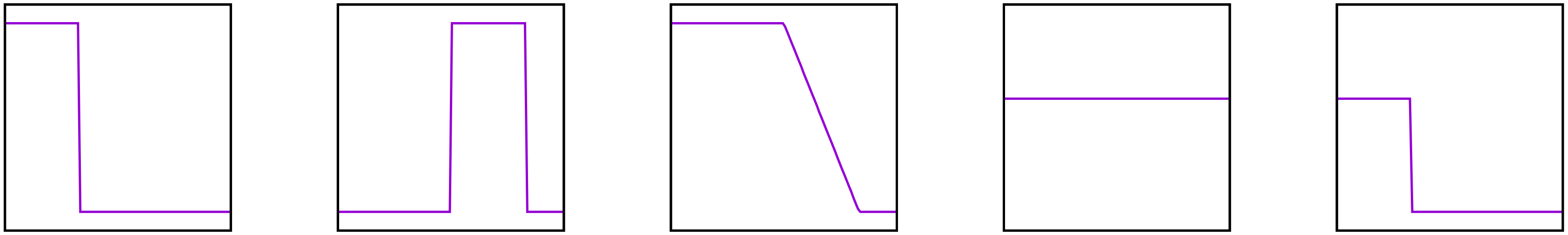
Random: **Soft random geometric graphs** Now two sources of randomness:

1. Node locations (as before).

2. Links are also random.

The most common approach is to take links with a probability $H(r)$, a function of the mutual distance r, chosen independently for each pair of nodes.

Examples so far:



Unit disk $H(r) = \mathbb{1}_{[0,r_0]}(r)$

Hard annulus $H(r) = \mathbb{1}_{[r_{min},r_{max}]}(r)$

Quasi unit disk (one example)

$$H(r) = \begin{cases} 1 & r < r_{min} \\ \frac{r_{max}-r}{r_{max}-r_{min}} & r_{min} < r < r_{max} \\ 0 & r > r_{max} \end{cases}$$

Erdős-Rényi $H(r) = p$, $p \in [0,1]$. No longer a spatial network.

Soft disk $H(r) = p\mathbb{1}_{[0,r_0]}(r)$ Intersection of unit disk and Erdős-Rényi.

Wireless connection models

Connection functions may occur naturally in the analysis of wireless networks. We assume that a connection is made from node i to node j if the signal to interference plus noise ratio (SINR) reaches a specified threshold:

$$H_{ij} = \mathbb{P}(\mathsf{SINR}_{ij} > q)$$

We have

$$\mathsf{SINR}_{ij} = \frac{G_i G_j |h_{ij}|^2 g(r_{ij})}{\mathcal{N} + \mathcal{I}}$$

where G_i and G_j are the relevant antenna gains, h_{ij} is the channel gain, a random variable, $g(r)$ is the path loss function, $\mathcal{N}$ is the noise power and $\mathcal{I}$ the interference power, a sum of contributions from nodes other than i or j.

Since we are considering pair connection functions, we will mostly neglect interference in this lecture; in practice this is appropriate if the MAC protocol allows only one transmission at a time in a given frequency channel. Thus

$$H_{ij} = \mathbb{P}\left(|h_{ij}|^2 > \frac{q\mathcal{N}}{G_i G_j g(r_{ij})}\right) = \bar{F}_{|h|^2}\left(\frac{q\mathcal{N}}{G_i G_j g(r_{ij})}\right)$$

where $\bar{F}$ is the complementary cumulative distribution function (ccdf): $\bar{F}(0) = 1$ decreasing to $\bar{F}(\infty) = 0$.

Path loss

The path loss function is typically

$$g(r) = \frac{1}{(r/r_0)^\eta + \epsilon}$$

where the path loss exponent $\eta = 2$ corresponds to free space propagation (inverse square law), and is often assumed to be in the range $[2, 6]$ for more cluttered environments. The limit $\eta \to \infty$ gives a very sharp transition near $r = r_0$, reducing to the unit or soft disk models.

Interference considerations: If $\epsilon = 0$ then two close nodes transmitting simultaneously cannot receive from any other nodes. If $\eta \leq d$ (where d is the spatial dimension) the total power from distant nodes diverges ("Olbers' paradox.")

In a medium with absorption, an exponential law

$$g(r) = \exp(-r/r_0)$$

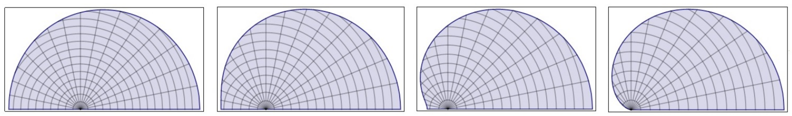
or similar may be appropriate.

Antenna gains

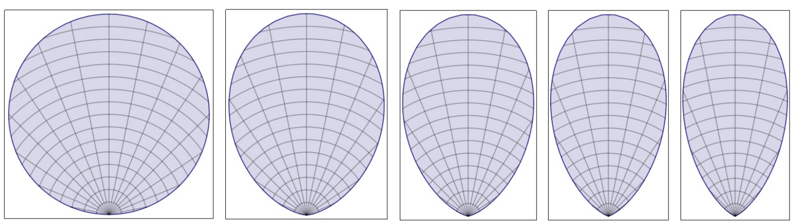
For 2D networks where the radiation patterns have an axis of symmetry perpendicular to the plane, the gains are uniform in the plane, and the antenna gains G_i and G_j may be absorbed into the other constants.

Other models, with θ the angle between the axis of the antenna and the line of sight direction; approximate (and non-normalised) expressions:

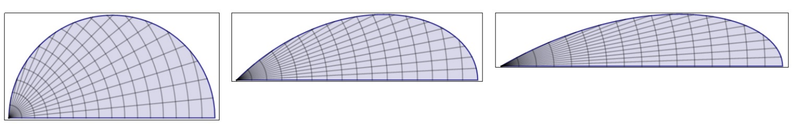
Patch antennas: $G = 1 + \epsilon \cos\theta$, $\epsilon = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$



Dipole antennas: $G = \sin^m \theta$, $\theta \in [0, \pi]$, $m = 1, 2, 3, 4, 5$



Directional (eg horn, array) antennas: $G = \cos \lambda\theta$, $\theta \in [0, \frac{\pi}{2\lambda}]$, $\lambda = 1, 2, 3$



Fading models

The channel gain $|h_{ij}|^2$ is a random variable, assumed iid for different ij and with distribution $F_{|h|^2}$ controlled by the propagation conditions, according to a specific fading model.

Possibilities are:

No fading $h_{ij} = 1$, gives unit disk model or anisotropic equivalent.

Slow fading Fading timescales large compared with communication channel, eg large obstacles. Often modelled by the log-normal distribution

$$\bar{F}_{|h|^2}(x) = \frac{1}{2}\text{erfc}\left[\frac{10\log_{10}(x) - \mu}{\sigma\sqrt{2}}\right]$$

Fast fading Effects due to rapidly varying multipath effects. . .

Fast fading

The distribution of the channel gain depends on the number of specular (eg line of sight (LOS)) and diffuse components. We have

Rayleigh fading (diffuse scattering)

$$\bar{F}_{|h|^2}(x) = e^{-x}$$

Rician fading (line of sight (LOS) plus diffuse)

$$\bar{F}_{|h|^2}(x) = Q_1(\sqrt{2K}, \sqrt{2(K+1)x})$$

where K is the ratio of LOS to diffuse power; $K \to 0$ gives the Rayleigh fading case.

Two or more specular components, with or without diffuse Can be expressed approximately as sums of Q_1 functions; see Ref. [DRW02] for details.

Here, Q_1 is the Marcum Q function

$$Q_M(a,b) = \int_b^\infty x \left(\frac{x}{a}\right)^{M-1} \exp\left(-\frac{x^2+a^2}{2}\right) I_{M-1}(ax)\, dx$$

and I_{M-1} is the modified Bessel function.

Fast fading models are alike

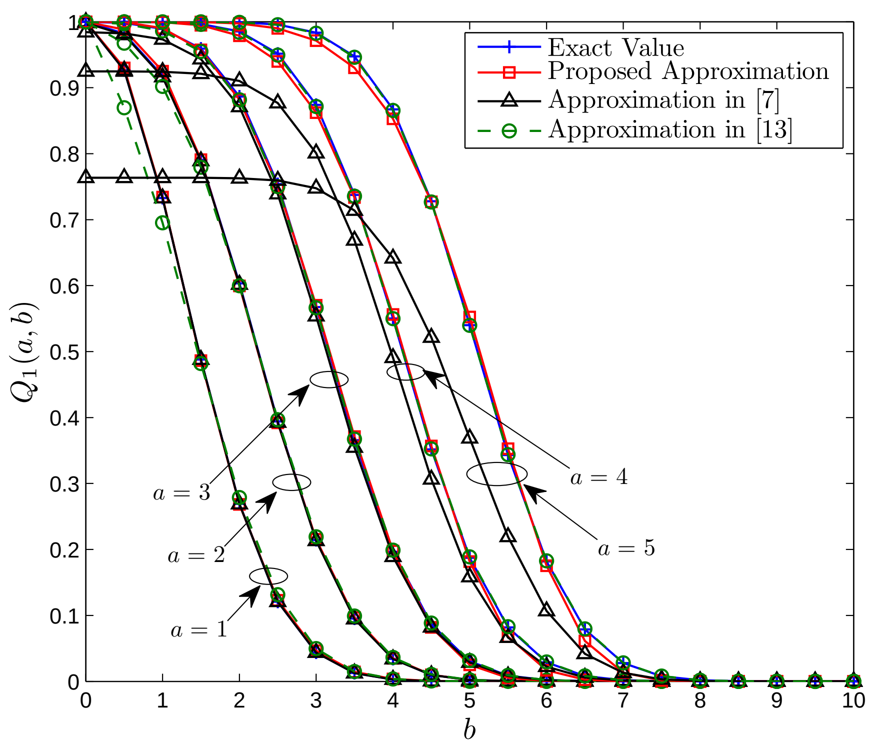
Rayleigh: We find $H(r) = \exp(-(r/r_0)^\eta)$ for some constant r_0.

Rician: We find $H(r) = Q_1(\sqrt{2K}, \sqrt{2(K+1)(r/r_0)^\eta})$.

It turns out [BDC13] that for $a \leq 5$, Q_1 can be approximated by an exponential,

$$Q_1(a,b) \approx \exp(-e^{\nu(a)} b^{\mu(a)})$$

with integrated squared error of less than 10^{-3}. This simplifies the analysis, particularly as we often need to integrate over $H(r)$.



Multi-antenna systems I: Diversity coding using STBC

Each device has m transmit antennas and n receive antennas. Using a space time block coding (STBC) scheme we can replace $|h|^2$ by

$$X = \frac{\zeta_m}{m} \sum_{i=1}^{n} \sum_{j=1}^{m} |h_{i,j}|^2$$

with

$$\zeta_m = \begin{cases} 1 & m \leq 2 \\ 2 & m \geq 3 \end{cases}$$

Note that the double sum is the square of the Frobenius norm of the matrix $\mathbf{H}$ with elements h_{ij}.

Using the same assumptions as for Rayleigh fading, namely complex Gaussian distributed h_{ij} we obtain a χ^2 distribution

$$\bar{F}_X(x) = \frac{\Gamma(mn, x\zeta_m/m)}{\Gamma(mn)}$$

where Γ is the (upper incomplete) gamma function.

Multi-antenna systems II: MIMO-MRC

Each device has m transmit antennas and n receive antennas as before. Now, using Multiple input multiple output (MIMO) with maximum ratio combining (MRC), the overall gain is given by the maximum eigenvalue of $\mathbf{H}^T\mathbf{H}$, where the ^{T} denotes transpose. It is more difficult than STBC to get analytic expressions; some examples, assuming Rayleigh fading:

MISO and SIMO If $m = 1$,

$$\bar{F}_{\lambda_{max}(\mathbf{H}^T\mathbf{H})}(x) = \frac{\Gamma(n, x)}{\Gamma(n)}$$

Similarly, if $n = 1$, replace n by m above.

MIMO (2,2) If $m = n = 2$,

$$\bar{F}_{\lambda_{max}(\mathbf{H}^T\mathbf{H})}(x) = e^{-x}(x^2 + 2 - e^{-x})$$

MIMO, many antennas If $m, n \to \infty$ so that $m/n \to y$ with $0 \leq y < \infty$, then

$$\frac{1}{n}\lambda_{max}(\mathbf{H}^T\mathbf{H}) \xrightarrow{p} (1 + \sqrt{y})^2$$

with convergence in probability. See Ref. [CGD15].

Connection functions in general spatial networks

Many complex networks naturally have a spatial structure; examples include transport, neuronal, climate and nanowire networks [Bart11]. If there is data on both node locations and links, it is possible to construct phomenological connection functions $H(r)$, which may be longer ranged than those considered for wireless networks.

For many other kinds of complex networks, eg social networks, a "latent" space can be constructed in which nearby nodes are more likely to be linked. Here, we have the flexibility to assign locations to nodes.

In each case, the main question is the extent to which the link independence assumption holds.

SRGG: General mathematical setting

Space X Originally $\mathbb{R}^2$, often $[0,1]^2$ or some other subset of $\mathbb{R}^2$ or $\mathbb{R}^3$. In the anisotropic case, we model orientations, so a subset of $\mathbb{R}^d \times SO(d)$.

Measure Λ on X to define the **Poisson Point Process** (PPP), that is, the number of points in a set $S \subset X$ is Poisson distributed with mean $\Lambda(S)$ and independent of the points not in S. Typically $\Lambda = \rho\lambda$ where ρ is a constant density and λ is Lebesgue (more generally Haar) measure. Technicalities: Need associated σ-algebra on X; Λ should be σ-finite and nonatomic.

Metric $d : X \times X \to [0,\infty]$ to define mutual distances between nodes. Normally Euclidean distance, but often "toroidal" distance on $[0,1]^2$, obtained by identifying opposite edges, to have a finite system avoiding boundary effects. A metric satisfies

- **Symmetry** $d(x,y) = d(y,x)$. But we can define a nonsymmetric directed graph from transmitters to receivers.
- **Identity of indiscernables** $d(x,y) = 0 \Rightarrow x = y$. But we can define $d = 0$ for nodes at the same location but different orientations.
- **Triangle inequality** $d(x,z) \leq d(x,y) + d(y,z)$. But we can violate this to model obstacles.

Poisson and binomial point processes

The expected number of nodes in the PPP is

$$\bar{N} = \int \Lambda(dx)$$

If this is finite, we can define the normalised (ie, probability) measure

$$\widehat{\Lambda} = \bar{N}^{-1}\Lambda$$

The **binomial point process** (BPP) is obtained by fixing the number of nodes N and distributing them independently with respect to $\widehat{\Lambda}$, which is equivalent to the PPP conditioned on exactly N nodes. Thus the PPP can be realised by choosing N according to the discrete Poisson distribution

$$\mathbb{P}(N = n) = \frac{\bar{N}^n e^{-\bar{N}}}{n!}$$

and realising the corresponding BPP.

For $\bar{N}$ large, the Poisson distribution is sharply peaked - many results apply to both Poisson and binomial models.

For more on stochastic geometry, see Refs. [BB09], [Haenggi12].

Connectivity mass

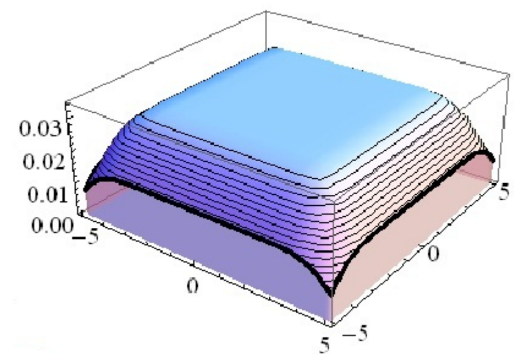
If we condition on a node at some point $x \in X$ in the PPP, the surrounding nodes are still Poisson distributed with intensity Λ. Thus the expected degree of this node is given by the **connectivity mass**

$$\mathbf{M}(x) = \int H(d(x,y))\Lambda(dy)$$

It is natural to assume this is bounded. For example, we want roughly $H(r) < r^{-d}$ as $r \to \infty$ if Λ is uniform on $\mathbb{R}^d$. If in addition $\bar{N} < \infty$, we have

$$\mathbf{M}(x) = \bar{N} \int H(d(x,y))\hat{\Lambda}(dy)$$

In the typical case of a uniform density ρ on a finite domain, $\mathbf{M}(x)$ is almost constant in the bulk (interior), decreasing close to the boundaries. This is responsible for many boundary effects, considered later.



Note that in the literature, "connectivity mass" often refers to the quantity $M = \mathbf{M}/\rho$, which depends only on the connection function $H(r)$ and geometry but not on the density.

Mean degree

The **mean degree** of a graph $\mathcal{K}$ is defined by summing the degrees of nodes and dividing by the number of nodes. For the BPP we distribute the first point at x according to $\hat{\Lambda}$, then the remaining $N-1$ nodes to obtain

$$\mathbb{E}_{BPP}(\mathcal{K}) = \frac{N-1}{\bar{N}^2}\int \mathbf{M}(x)\Lambda(dx)$$

Averaging over this (defining $\mathcal{K} = -1$ for the empty graph) we find

$$\bar{\mathcal{K}} \equiv \mathbb{E}_{PPP}(\mathcal{K}) = \frac{\bar{N}-1}{\bar{N}^2}\int \mathbf{M}(x)\Lambda(dx)$$

In the limit $\bar{N} \to \infty$ we have simply

$$\bar{\mathcal{K}} = \int \mathbf{M}(x)\hat{\Lambda}(dx)$$

In the case of uniform density ($\Lambda = \rho\lambda$) and negligible boundaries, $\mathbf{M}(x)$ is effectively constant:

$$\mathbf{M}(x) = \bar{\mathcal{K}} = \rho\int_{\mathbb{R}^d} H(x)dx = \rho S_{d-1}H_{d-1}$$

where S_{d-1} is the surface of the unit d-ball and H_m the m-th moment of $H(r)$

$$H_m = \int_0^\infty H(r)r^m dr$$

Examples

1. The original random geometric graph has connection function

$$H(r) = \begin{cases} 1 & r < r_0 \\ 0 & r > r_0 \end{cases}$$

and mean degree when nodes are of density ρ in $\mathbb{R}^2$

$$\bar{\mathcal{K}} = \pi r_0^2 \rho$$

2. Isotropic antenna gains, Rayleigh fading:

$$H(r) = e^{-(r/r_0)^\eta}$$

where the typical connection range r_0 is obtained by combining the various constants appearing earlier. If the network is of density ρ in a square domain of side length L, the connectivity mass is

$$\mathbf{M}(x) \approx \begin{cases} \pi r_0^2 \rho \frac{2}{\eta} \Gamma\left(\frac{2}{\eta}\right) & \text{Bulk, far from edge} \\ \pi r_0^2 \rho \frac{1}{\eta} \Gamma\left(\frac{2}{\eta}\right) & \text{Edge, far from corner} \\ \pi r_0^2 \rho \frac{1}{2\eta} \Gamma\left(\frac{2}{\eta}\right) & \text{Exactly at corner} \end{cases}$$

If the boundaries can be ignored ($r_0 \ll L$) the mean degree is

$$\bar{\mathcal{K}} = \pi r_0^2 \rho \frac{2}{\eta} \Gamma\left(\frac{2}{\eta}\right)$$

Anisotropy and mean degree

Again, assume Rayleigh fading and uniform density, neglecting boundary effects. If the separation of two nodes in 2D is at angle ϕ and their orientations are at angles θ_T and θ_R, we find

$$H(r, \phi, \theta_T, \theta_R) = \exp\left[-\frac{(r/r_0)^\eta}{G_T(\phi - \theta_T)G_R(\phi + \pi - \theta_R)}\right]$$

and mean degree

$$\begin{aligned}\bar{\mathcal{K}} &= \frac{\rho}{2\pi}\int H(r, \phi, \theta_T, \theta_R)\; r\; dr\; d\phi\; d\theta_R \\ &= \frac{\rho r_0^2}{2\pi\eta}\Gamma\left(\frac{2}{\eta}\right) S_\eta[G_T] S_\eta[G_R]\end{aligned}$$

where

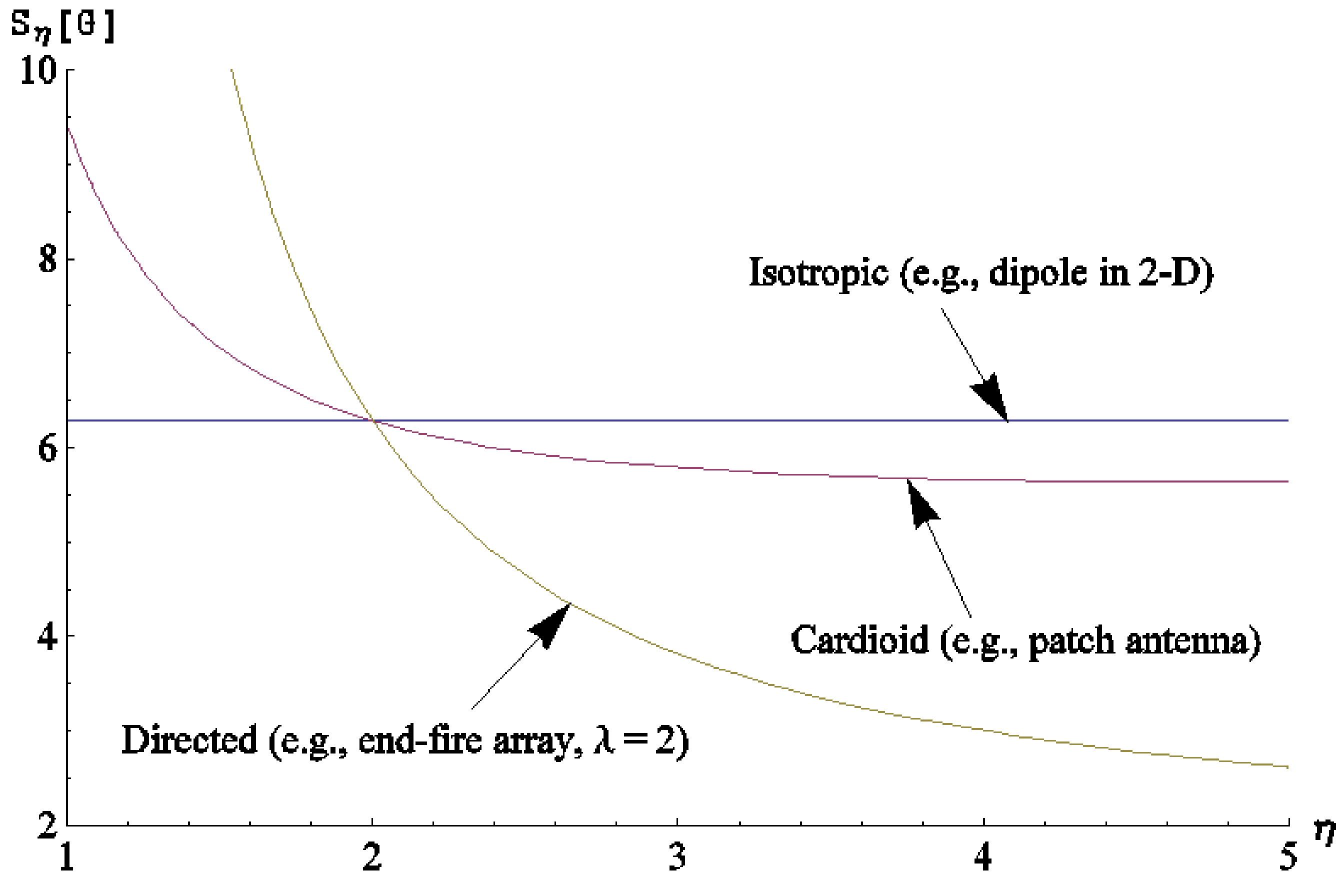
$$S_\eta[G] = \int_0^{2\pi} G(\phi)^{2/\eta} d\phi$$

Optimising radiation patterns

If we normalise the gains by fixing

$$\int_0^{2\pi} G_T(\phi) d\phi = \int_0^{2\pi} G_R(\phi) d\phi = 2\pi$$

the maximal mean degree is given by the isotropic case for $\eta > 2$ and by extremely directional (delta spike) patterns for $\eta < 2$ (see Ref. [CD13]).



Connectivity exponent

Many of the above 2D calculations involve the expression $2/\eta$; the corresponding 3D expressions contain $3/\eta$. Denoting $\mathcal{C} = d/\eta$ as the "connectivity exponent," we find (see Ref. [CGD15])

- Isotropic radiation patterns are preferable to random directional patterns if $\mathcal{C} < 1$.

- Interference from distant nodes is finite if $\mathcal{C} < 1$.

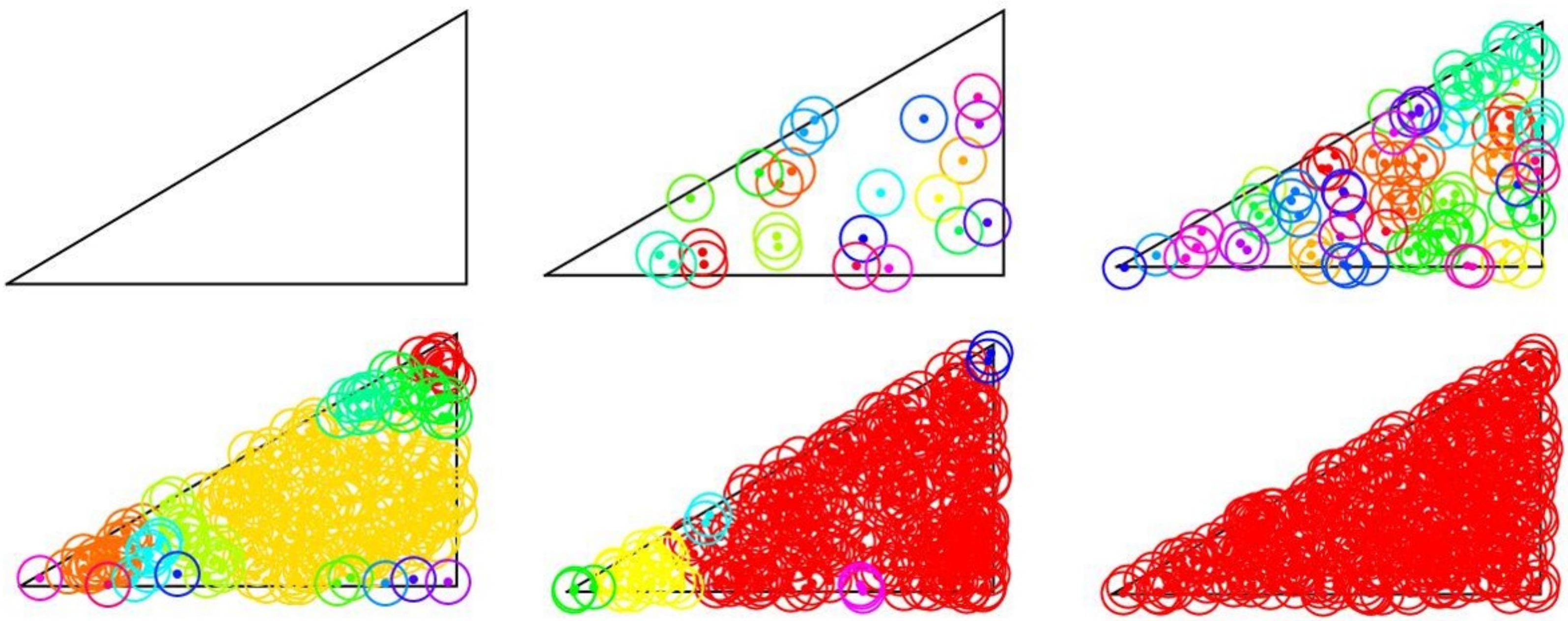
- $\bar{\mathcal{K}} \propto P^{\mathcal{C}}$ in terms of the transmit power P.

- $\bar{\mathcal{K}} \propto n^{\mathcal{C}}$ for STBC (n receive antennas)

- $\bar{\mathcal{K}} \propto (\sqrt{m} + \sqrt{n})^{\mathcal{C}}$ for MIMO-MRC (m transmit and n receive antennas).

(Full) connectivity and boundaries

Now we ask whether the network is connected in a multi-hop fashion, for example in a triangular domain. . .



Isolated nodes occur mostly near the corners.

Dependence on density and geometry

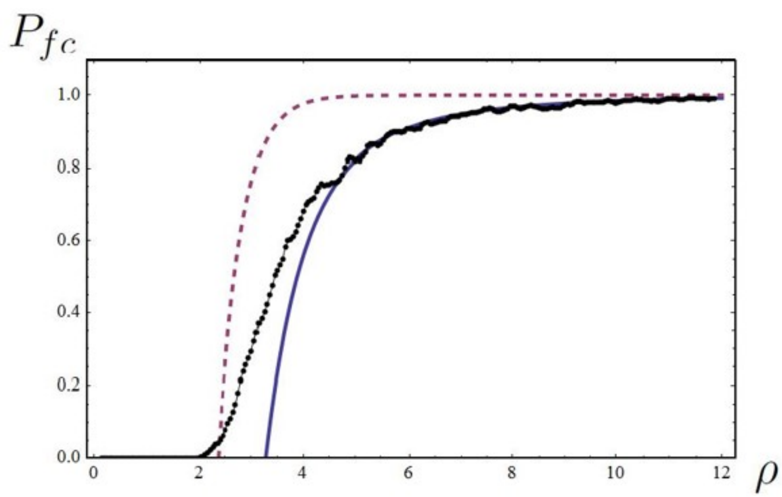
Notation: Mean degree $\mathcal{K}$, (full) connection probability P_{fc}.

We see two main transitions as density increases:

Percolation Formation of a cluster comparable to system size:
Largely independent of geometry. $\mathcal{K} = 4.5122\ldots$ in 2D

Connectivity All nodes connected in multi-hop fashion:
Strongly dependent on geometry. $\mathcal{K} \approx \ln N$.

P_{fc} as a function of density and geometry?



Mathematics of connectivity in RGG

Rigorous results are for $N \to \infty$, scaling at least two of r_0, ρ and L.

For the random geometric graph in dimension $d \geq 2$, it was shown by Penrose, and by Gupta & Kumar, that the r_0 threshold for **connectivity** is almost always the same as for **isolated nodes**.

In turn, isolated nodes are local events, so described by a limiting Poisson process: The probability of a node having degree k is given by

$$P(k) = \frac{\bar{\mathcal{K}}^k}{k!} e^{-\bar{\mathcal{K}}}$$

where $\bar{\mathcal{K}}$ is the mean degree, equal to $\rho \pi r_0^2$ for the 2D RGG. This leads to

$$P_{fc} \approx \exp\left[-\rho V e^{-\rho \pi r_0^2}\right]$$

where V is the "volume" (ie area) of the domain.

Remarks: At fixed probability and connection range, V increases exponentially with ρ; also most isolated nodes are in the bulk when $d = 2$. The number of isolated nodes at corners cannot be Poisson.

Connectivity in soft random geometric graphs

Penrose (2016) showed that for connection functions that are symmetric, positive at the origin and stretched exponentially decaying (also radially symmetric and monotonic for $d > 2$), the number of isolated nodes is asymptotically Poisson distributed. Further, if its support is sufficiently small, the (full) connection probability is asymptotically that of there being no isolated nodes. (See also Mao & Anderson 2013, Iyer arxiv 2015).

Here we assume the resulting formula is approximately valid for finitely many nodes, including for connection functions with unbounded support:

$$P_{fc} \approx \exp\left[-\int \rho e^{-\int \rho H(r_{12}) d\mathbf{r}_1} d\mathbf{r}_2\right]$$

where ρ is the density, $H(r)$ is the iid probability of connection between nodes with mutual distance r and the integrals are over the domain $\mathcal{V} \subset \mathbb{R}^d$.

We want to approximate P_{fc} for finite ρ, taking into account boundaries.

In progress: $d = 1$, eg vehicular networks!

Connectivity and boundaries

For large ρ, P_{fc} is dominated by the regions of small (reduced) connectivity mass; recall

$$M(\mathbf{r}_2) = \int H(r_{12}) d\mathbf{r}_1$$

Exactly on the boundary, this is given by

$$M_B = H_{d-1}\omega_B$$

where (recall)

$$H_m = \int_0^\infty H(r) r^m dr$$

is the mth moment, and ω_B is the (solid) angle associated with the boundary component B, eg $\pi/2$ for a right angled corner, π for an edge.

Analysing the vicinity of boundaries more carefully...

Boundary effects by Laplace's method

In the following, boundary components are labelled by (d,i), the dimension of the whole space, and the codimension of the boundary component.

Step 1: Integration on a non-centred line

$$F(x)=\int_0^\infty H(\sqrt{x^2+t^2})dt$$

Expanding in powers of x, taking care with any discontinuities, we find

$$F(x)=H_0+\frac{x^2}{2}\left(H'_{-1}+\Delta_{-1}\right)+\dots$$

where H_0 is the zeroth moment, and

$$H'_{-1}=\int_0^\infty\frac{H'(r)}{r}dr=H_{-2}$$

using integration by parts, if the latter converges.

$$\Delta_{-1}=\sum_k\frac{H(r_k+)-H(r_k-)}{r_k}$$

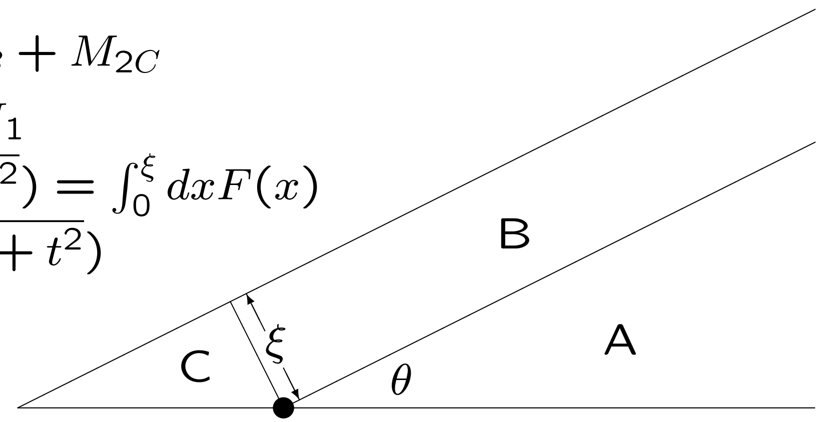
where the sum is over discontinuities (as in the unit disk model). It is convenient to combine these in the notation to write

$$\tilde{H}_{-2}=H'_{-1}+\Delta_{-1}$$

Step 2: Connectivity mass of a wedge

Define $M^{\omega}_{2,2}(r,\theta)$ to be connectivity mass of a wedge of angle ω from a point at polar coordinates (r,θ).

$$M^{\theta}_{2,2}(\xi \csc\theta, 0) = M_{2A} + M_{2B} + M_{2C}$$
$$M_{2A} = \int_0^{\theta} d\phi \int_0^{\infty} H(r)dr = \theta H_1$$
$$M_{2B} = \int_0^{\xi} dx \int_0^{\infty} dt H(\sqrt{x^2+t^2}) = \int_0^{\xi} dx F(x)$$
$$M_{2C} = \int_0^{\xi} dx \int_0^{x\cot\theta} dt H(\sqrt{x^2+t^2})$$
$$\approx \tfrac{1}{2} H(0)\xi^2 \cot\theta$$



Putting it together we have for this wedge

$$M^{\theta}_{2,2}(\xi\csc\theta,0) = \theta H_1 + \xi H_0 + \frac{\xi^2}{2} H(0)\cot\theta + \frac{\xi^3}{6}\tilde{H}_{-2} + \ldots$$

From this we can find a general wedge, edge and bulk:

$$\begin{aligned}
M^{\omega}_{2,2}(r,\theta) &= M^{\theta}_{2,2}(r,0) + M^{\theta'}_{2,2}(r,0) \qquad (\theta' = \omega - \theta) \\
&= \omega H_1 + r H_0 (\sin\theta + \sin\theta') + \frac{r^2}{4} H(0)(\sin 2\theta + \sin 2\theta') \\
&\quad + \frac{r^3}{6}\tilde{H}_{-2}(\sin^3\theta + \sin^3\theta') + \ldots \\
M_{2,1}(r) &= 2M^{\pi/2}_{2,2}(r,0) = \pi H_1 + 2rH_0 + \frac{r^3}{3}\tilde{H}_{-2} + \ldots \\
M_{2,0} &= 2\pi H_1
\end{aligned}$$

Step 3: Calculation of the outer integral

Here, we use Laplace's method, treating ρ as the large parameter. For example, a wedge of angle ω:

$$\begin{aligned}
P^{\omega}_{2,2} &= \rho \int_{\text{wedge}} e^{-\rho M^{\omega}_{2,2}(r,\theta)} r dr d\theta \\
&= \rho \int_0^{\omega} d\theta \int_0^{\infty} r dr e^{-\rho\left[\omega H_1 + r H_0(\sin\theta+\sin\theta') + \frac{H(0)r^2}{4}(\sin 2\theta + \sin 2\theta') + \frac{\tilde{H}_{-2}r^3}{6}(\sin^3\theta + \sin^3\theta') + \ldots\right]} \\
&= \rho e^{-\rho\omega H_1} \int_0^{\omega} d\theta \int_0^{\infty} r dr e^{-\rho r H_0(\sin\theta+\sin\theta')} \\
&\quad \left[1 - \frac{\rho H(0) r^2}{4}(\sin 2\theta + \sin 2\theta') - \frac{\rho \tilde{H}_{-2} r^3}{6}(\sin^3\theta + \sin^3\theta') + \ldots\right] \\
&= e^{-\rho\omega H_1} \int_0^{\omega} d\theta \\
&\quad \left[\frac{1}{\rho H_0^2(\sin\theta + \sin\theta')^2} - \frac{3H(0)(\sin 2\theta + \sin 2\theta')}{2\rho^2 H_0^4(\sin\theta + \sin\theta')^4} - \frac{4\tilde{H}_{-2}(\sin^3\theta + \sin^3\theta')}{\rho^3 H_0^5(\sin\theta + \sin\theta')^5} + \ldots\right] \\
&= e^{-\rho\omega H_1} \left[\frac{1}{\rho H_0^2 \sin\omega} - \frac{H(0)(2\cos\omega + 1)}{\rho^2 H_0^4 \sin^2\omega} - \frac{2\tilde{H}_{-2}}{\rho^3 H_0^5 \sin\omega} + \ldots\right]
\end{aligned}$$

General formula

$$P_{fc} = \exp\left[-\sum_B \rho^{1-i_B} G_B V_B e^{-\rho\omega_B H_{d-1}}\right]$$

where i_B is the boundary codimension, V_B is its $d-i$ dimensional volume, and G_B is the geometrical factor

G_B	$i=0$	$i=1$	$i=2$	$i=3$
$d=2$	1	$\frac{1}{2H_0}$	$\frac{1}{H_0^2 \sin\omega}$	
$d=3$	1	$\frac{1}{2\pi H_1}$	$\frac{1}{\pi^2 H_1^2 \sin(\omega/2)}$	$\frac{4}{\pi^2 H_1^3 \omega \sin\omega}$

where the 3D corner has a right angle.

Curved boundaries? To leading order, modification of the exponential but not the geometrical factor:

$$P_{2,1} = \dots e^{-\rho(\pi H_1 - \kappa H_2)}$$

$$P_{3,1} = \dots e^{-\pi\rho(2H_2 - \kappa H_3)}$$

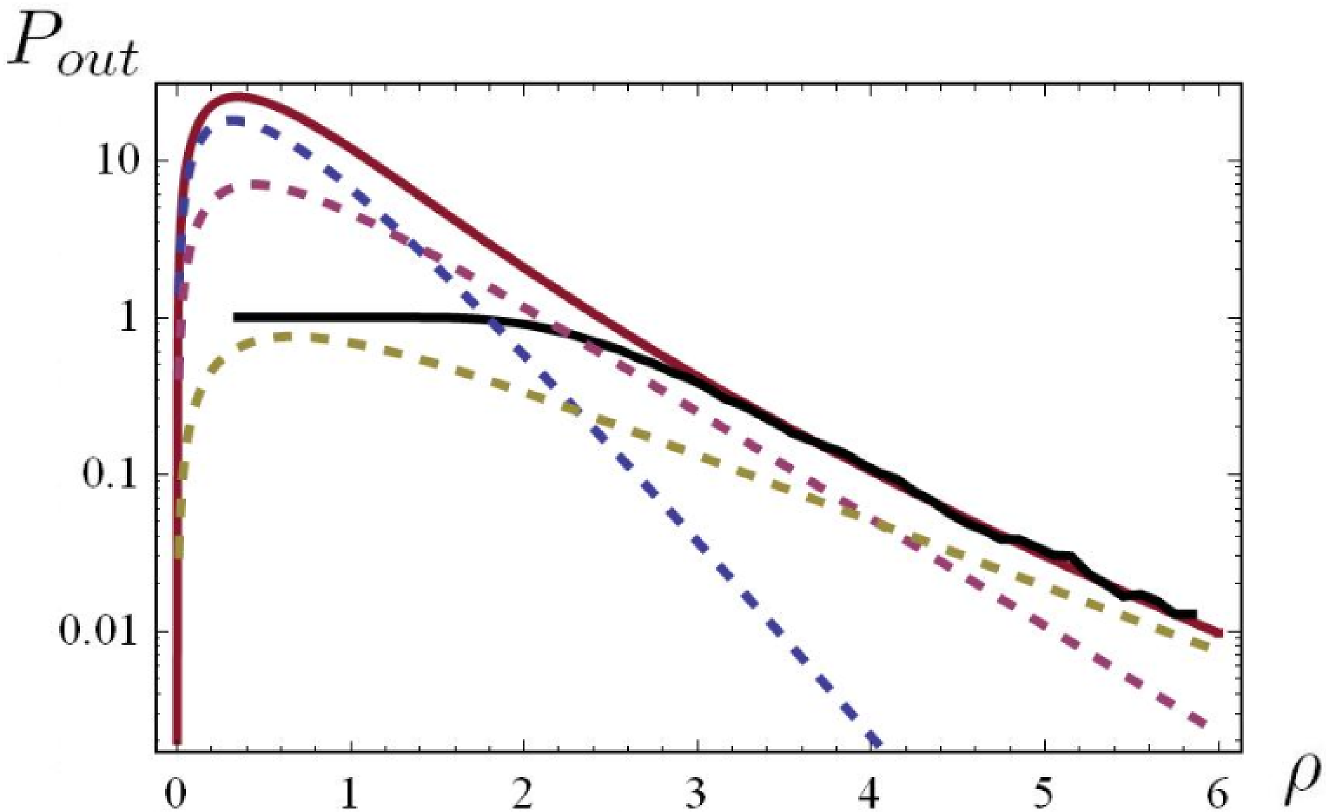
where κ is (mean) curvature.

Summary: We can do arbitrary convex geometries with piecewise smooth boundaries; $H(r)$ appears only via a few moments.

Example: A square

The previous formula gives

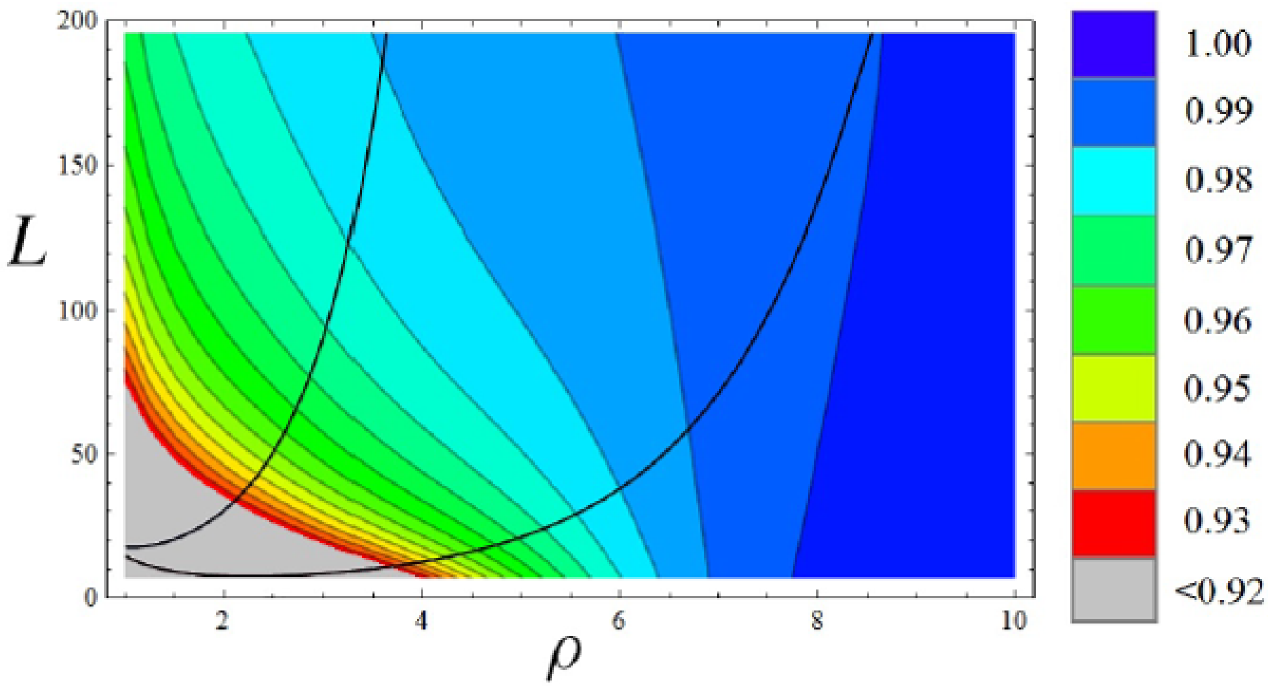
$$1 - P_{fc} \approx L^2 \rho e^{-\pi\rho} + \frac{4L}{\sqrt{\pi}} e^{-\frac{\pi\rho}{2}} + \frac{16}{\pi\rho} e^{-\frac{\pi\rho}{4}}$$



Phase diagram

Testing convergence of

$$\frac{1 - P_{fc}}{\sum_B \cdots}$$



Conclusion

We have shown how to construct pair connection functions and use these to calculate mean degrees and connectivity of networks in general convex domains.

More on pair connection functions and their moments can be found in Ref. [DG16].

From this point, we can branch out in several directions:

Link correlations In the RGG, nodes that are very close are likely to have exactly the same environment. This is less often true for the SRGG. These correlations can be quantified using graph entropy.

Asymmetry Transmission power diversity leads naturally to spatial digraphs (directed networks).

Interference Behaviour of a network depends in practice on signals from many other nodes transmitting simultaneously, leading naturally to spatial hypergraphs.

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