SHOT-NOISE SPATIAL BIRTH-and-DEATH PROCESSES

F. Baccelli, UT Austin

Simons Center for Communication, Information & Network Mathematics

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Structure of the Lecture

- **Background and Motivation**
- **Methodology**
- **Part 1**
  Birth and Death Wireless Queues
  Joint work with A. Sankararaman and S. Foss
- **Part 2**
  Wireless Queues under Multihop Routing and Motion
  Joint work with S. Rybko, S. Shlosman and S. Vladimirov

Motivations in Wireless Networks

- **Lack of understanding and analysis of**

  Space-time interactions

  – **Static spatial setting well understood: Stochastic Geometry [FB, Blaszczyszyn 01]**

  – **Churn partly taken into account in flow-based queuing [Bonald, Proutiere 06], [Shakkottai, De Veciana 07]**

- **Contents of this lecture:**

  First models with such dynamics in stochastic geometry

Methodology

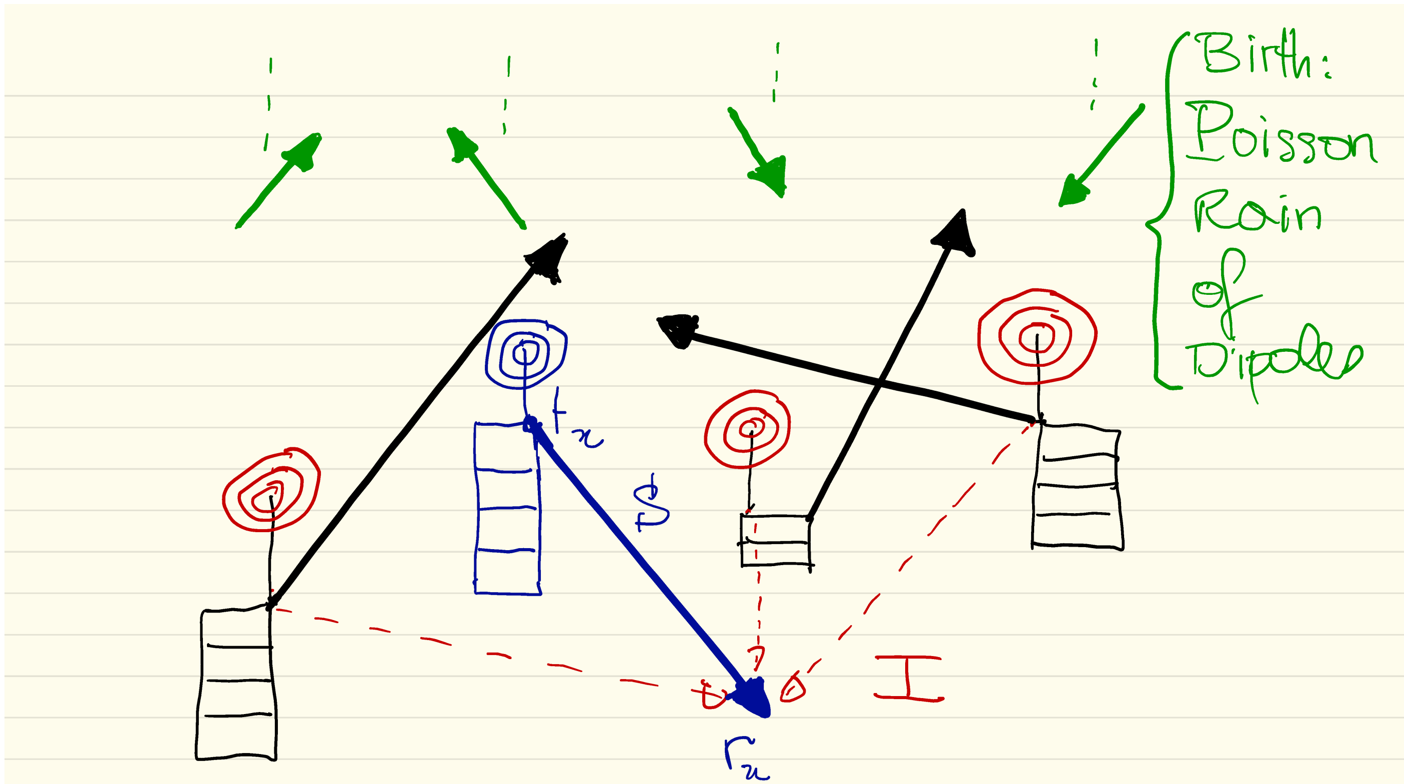
Everything Should Be Made as Simple as Possible,
But Not Simpler
A. E.

Setting

- **Infrastructureless Wireless Network**:
  Ad-hoc Networks, **D2D Networks**, **IoT**
- **Markov Models**:
  Poisson, **Exponential**
- **Mathematical tools**:
  Stochastic Geometry, **Fluid**, **Mean-Field**

I: Wireless Birth and Death

- **Problem Statement**
- **Summary of Results**
- **Proof Overview**



Death Rate : $\frac{B}{L}\log_2\left(1+\frac{S}{N+I}\right)$

Stochastic Network Model

- $\mathbf{S} = [-Q, Q] \times [-Q, Q]$: torus where the wireless links live
- **Links**: (Tx-Rx pairs)
- Links: **arrive** as a PPP on $\mathbb{R} \times \mathbf{S}$ with intensity λ:
  Prob. of a point arriving in space $\mathbf{dx}$ and time $\mathbf{dt}$: $\lambda \mathbf{dxdt}$
- Each Tx has an **i.i.d. exponential file size** of mean $\mathbf{L}$ bits to transmit to its Rx
- A point **exits** after the Tx finishes transmitting its file
- $\mathbf{\Phi_t}$: set of locations of links present at time t:

$$\mathbf{\Phi_t} = \{\mathbf{x_1}, \ldots, \mathbf{x_{N_t}}\}, \quad \mathbf{x_i} \in \mathbf{S}$$

Interference and Service Rate

- **Interference seen at point x due to configuration Φ**

$$\mathbf{I}(\mathbf{x}, \boldsymbol{\Phi}) = \sum_{\mathbf{x_i} \in \boldsymbol{\Phi} \neq \mathbf{x}} \mathbf{l}(||\mathbf{x} - \mathbf{x_i}||)$$

  – **Distance on the torus**

  – $\mathbf{l}(\cdot)$**:** $\mathbb{R}^+ \to \mathbb{R}^+$**: path loss function**

- **The speed of file transfer by link at x in configuration Φ is**

$$\mathbf{R}(\mathbf{x}, \boldsymbol{\Phi}) = \mathbf{B} \log_2 \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N} + \mathbf{I}(\mathbf{x}, \boldsymbol{\Phi})} \right)$$

- $\mathbf{B}, \mathbf{N}$ **Positive constants**

B& D Master Equation

- A point born at $\mathbf{x_p}$ and time $\mathbf{b_p}$ with file-size $\mathbf{L_p}$ dies at time

$$\mathbf{d_p} = \inf \left\{ \mathbf{t} > \mathbf{b_p} : \int_{\mathbf{u}=\mathbf{b_p}}^{\mathbf{t}} \mathbf{R}(\mathbf{x_p}, \boldsymbol{\Phi}_{\mathbf{u}}) \mathbf{du} \geq \mathbf{L_p} \right\}$$

- **Spatial Birth-Death Process**
  - Arrivals from the Poisson Rain
  - Departures happen at file transfer completion

Properties of the Dynamics

- **The statistical assumptions imply that Φ_t is a Markov Process on the set of simple counting measures on $\mathbf{S}$**
- **Euclidean extension of the flow-level models of [Bonald, Proutiere 06], [Shakkottai, De Veciana 07]**

Questions

- **Existence and uniqueness of the stationary regimes of Φ_t**
- **Characterization of the stationary regime(s) if existence**

Main Stability Results

$$\mathbf{a} := \int_{\mathbf{x}\in\mathbf{S}} \mathbf{l}(||\mathbf{x}||)\mathbf{dx}$$

- **Theorem**
  - If $\lambda > \frac{\mathbf{B}}{\ln(\mathbf{2})\mathbf{La}}$, then Φ_t admits no stationary regime.
  - If $\lambda < \frac{\mathbf{B}}{\ln(\mathbf{2})\mathbf{La}}$, and $\mathbf{r} \to \mathbf{l}(\mathbf{r})$ bounded and monotone, then Φ_t admits a unique stationary regime
- Necessary condition by Palm calculus
- Sufficient condition by fluid limit
- **Corollary**

  For the path-loss model $\mathbf{l}(\mathbf{r}) = \mathbf{r}^{-\alpha}$, $\alpha \geq \mathbf{2}$, for all $\lambda > \mathbf{0}$, and all mean file sizes, the process Φ_t admits no stationary-regime

Main Qualitative Result

- Φ stationary point-process on S with Palm distribution $\mathbb{P}^{\mathbf{0}}$
- Clustering

  Φ is clustered if for all bounded, positive, non-increasing functions $\mathbf{f}(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, the shot noise;

$$\mathbf{F}(\mathbf{x}, \boldsymbol{\Phi}) := \sum_{\mathbf{y} \in \boldsymbol{\Phi} \setminus \{\mathbf{x}\}} \mathbf{f}(||\mathbf{y} - \mathbf{x}||)$$

  satisfies

$$\mathbb{E}^{\mathbf{0}}[\mathbf{F}(\mathbf{0}, \boldsymbol{\Phi})] \geq \mathbb{E}[\mathbf{F}(\mathbf{0}, \boldsymbol{\Phi})]$$

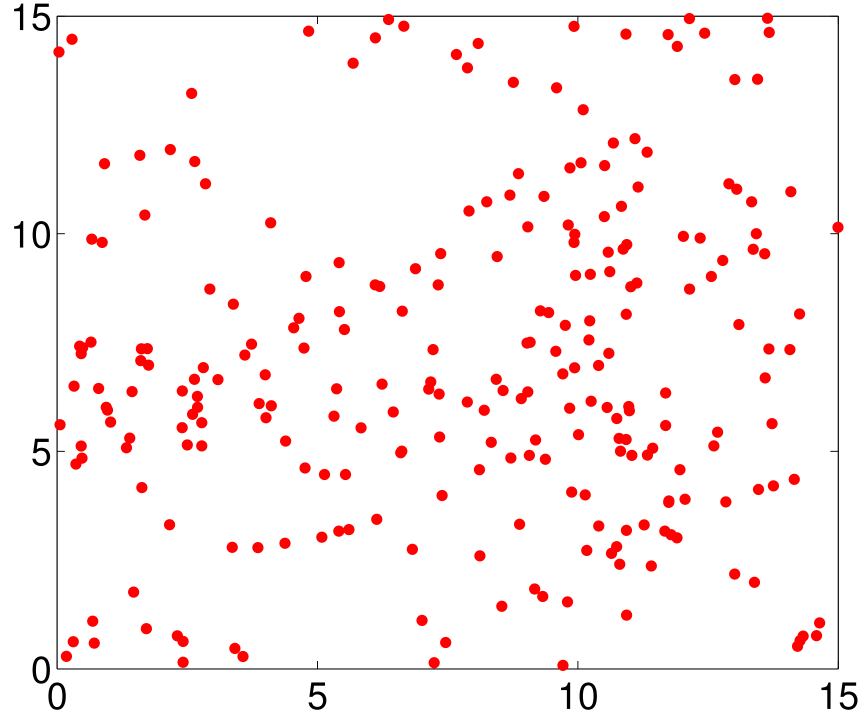
Main Qualitative Result (*continued*)

- **Theorem**
  The steady-state point process, when it exists, is clustered

- **Follows from Palm calculus + the FKG inequality**

- **Interpretation of the result**
  The steady-state interference measured at a uniformly randomly chosen point of is larger in mean than that at an uniformly random location of space.

- **Key Observation**
  - **Dynamics Shapes Geometry**
  - **Geometry Shapes Dynamics**



A sample of Φ when $\lambda = 0.99$ and $\mathbf{l}(\mathbf{r}) = (\mathbf{r}+\mathbf{1})^{-4}$.

First Quantitative Results

- **Mean-field approximations for the intensity of the steady-state process**
  1. **Poisson heuristic $\beta_{\mathbf{f}}$ - derived by neglecting clustering and assuming Poisson**
  2. **Second-order heuristic $\beta_{\mathbf{s}}$ based on a second-order cavity approximation of the dynamics**

Poisson Heuristic

Exact Rate Conservation Law:

$$\lambda \mathbf{L} = \beta \mathbb{E}^{\mathbf{0}}_{\mathbf{\Phi}} \left[\log_{\mathbf{2}} \left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N} + \mathbf{I}(\mathbf{0})} \right) \right].$$

Poisson Heur.: Largest solution to the fixed point equation:

$$\lambda \mathbf{L} = \frac{\beta_{\mathbf{f}}}{\ln(\mathbf{2})} \int_{\mathbf{z}=\mathbf{0}}^{\infty} \frac{\mathbf{e}^{-\mathbf{N}\mathbf{z}}(\mathbf{1} - \mathbf{e}^{-\mathbf{z}})}{\mathbf{z}} \mathbf{e}^{-\beta_{\mathbf{f}} \int_{\mathbf{x} \in \mathbf{S}} (\mathbf{1} - \mathbf{e}^{-\mathbf{z}\mathbf{l}(||\mathbf{x}||)}) \mathbf{d}\mathbf{x}} \mathbf{d}\mathbf{z}$$

Ignores the Palm effect and uses that if $\mathbf{X}, \mathbf{Y}$ are non-negative and independent,

$$\mathbb{E} \left[\ln \left(\mathbf{1} + \frac{\mathbf{X}}{\mathbf{Y} + \mathbf{a}} \right) \right] = \int_{\mathbf{z}=\mathbf{0}}^{\infty} \frac{\mathbf{e}^{-\mathbf{a}\mathbf{z}}}{\mathbf{z}} (\mathbf{1} - \mathbb{E}[\mathbf{e}^{-\mathbf{z}\mathbf{X}}]) \mathbb{E}[\mathbf{e}^{-\mathbf{z}\mathbf{Y}}] \mathbf{d}\mathbf{z}.$$

Second Order Heuristic

The intensity β_s is given by

$$\beta_{\mathbf{s}} = \frac{\lambda \mathbf{L}}{\mathbf{B}\log_{\mathbf{2}}\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N}+\mathbf{I_s}}\right)}$$

where $\mathbf{I_s}$ is the smallest solution of the fixed-point equation

$$\mathbf{I_s} = \lambda \mathbf{L} \int_{\mathbf{x}\in\mathbf{S}} \frac{\mathbf{l}(||\mathbf{x}||)}{\mathbf{B}\log_{\mathbf{2}}\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{N}+\mathbf{I_s}+\mathbf{l}(||\mathbf{x}||)}\right)} \mathbf{dx}$$

Second Order Heuristic (*continued*)

- **Rationale based on $\rho_2(\mathbf{x},\mathbf{y})$: second moment measure of Φ**
- **Rate Conservation for ρ_2: when considering $\mathbf{I_s}$ as a constant**

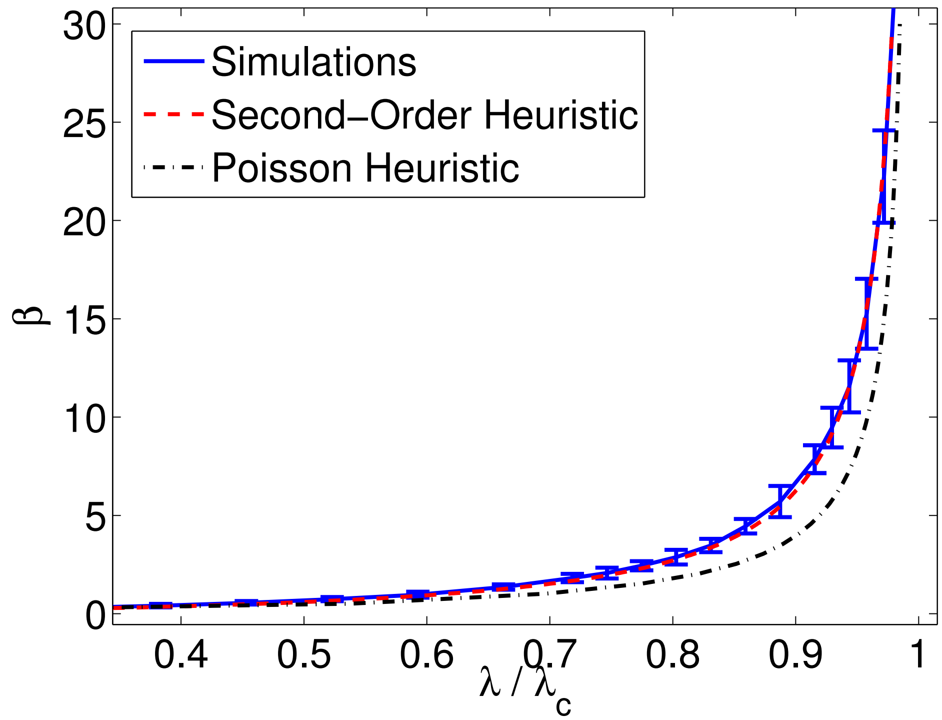
$$\rho_2(\mathbf{x},\mathbf{y})\frac{\mathbf{1}}{\mathbf{L}}\mathbf{B}\log_2\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{N}+\mathbf{I_s}+\mathbf{l}(||\mathbf{x}-\mathbf{y}||)}\right)=\lambda\beta_{\mathbf{s}}$$

- **From the definition of second moment measure,**

$$\mathbf{I_s}=\int_{\mathbf{x}\in\mathbf{S}}\mathbf{l}(||\mathbf{x}||)\frac{\rho_2(\mathbf{0},\mathbf{x})}{\beta_{\mathbf{s}}}\mathbf{dx}$$

which gives the fixed point equation for $\mathbf{I_s}$

- **The formula for $\beta_{\mathbf{s}}$ follows from Rate Conservation for $\rho_1=\beta_{\mathbf{s}}$**



95% **confidence interval when** $\mathbf{l(r) = (r+1)^{-4}}$

Tightness Results & Extensions

- The **Poisson heuristic** is **tight** in **heavy and light traffic**
- **Recent Extensions** obtained with **S. Foss**:
  - **Exact expression** for the intensity β of Φ in the **Low SINR** regime when replacing the death rate by

$$\frac{\mathbf{B}}{\ln(\mathbf{2})\mathbf{L}} \frac{\mathbf{S}}{\mathbf{N}+\mathbf{I}(\mathbf{x},\mathbf{\Phi})}$$

  - **Scalability result**: extension to dynamics on $\mathbb{R}^2$ using **Coupling from the Past** techniques.
- **Future**: introduction of **scheduling or multi-user IT**

Summary

- A new basic representation of space time interactions in wireless networks
- A generative model for clustering as assumed in 3GPP simulation standards
- A new dynamic notion of capacity involving both queuing and IT
- First analytical results in the Low SINR case and good heuristics in general

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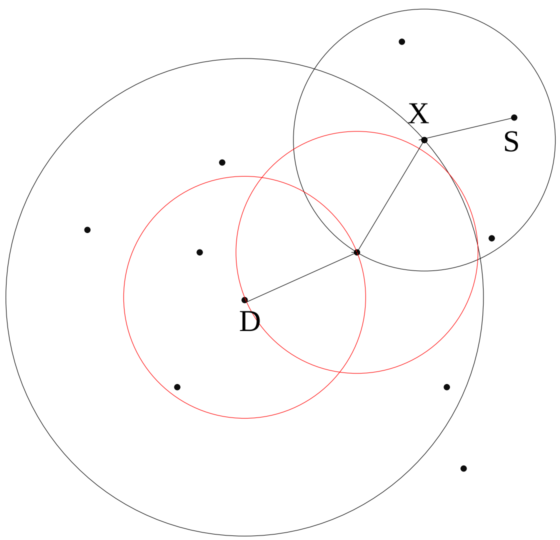
II: Motion and Multi-Hop Routing

- **Problem Statement**
- **Summary of Results**

Problem Statement

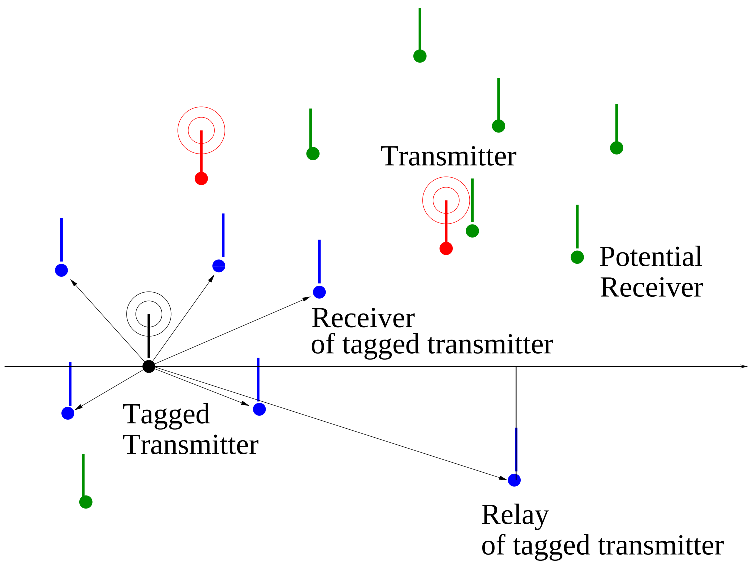
- **Setting: Grossglauser & Tse 02 scaling law problem**
  - **Multihop relaying**
  - **Opportunistic geographic routing**
  - **Motion of nodes**
- **New SG+QT view of the problem**

Example of Geographic Routing



- **Nearest Neighbor Geographic Routing**
  The next hop on the route from S to D is the nearest among the nodes which are closer from D than X.
- **On a Poisson P.P., a.s.**
  - **No ties**
  - **Converges in finite number of steps**

Wireless Geographic Routing



- Each node uses Aloha to split the Poisson p.p. into transmitters and potential receivers
- Potential relays of a transmitters: receivers with a large enough SINR
- Geographic Routing: next hop:= potential relay nearest to destination

Traffic and Relaying

- **DTN type assumptions:**
  - **Wireless nodes move**
  - **Each moving node generates packets**
  - **Each generated packet has a destination, e.g. another node at finite distance**
  - **Multihop relaying: each node transmits**
    - **its own packets**
    - **those of other nodes on their route to destination**
  - **Contention: on each node, packets are queued FIFO and are served as above (SINR condition)**

Reduction (as per Methodology :)

- **Wireless nodes move randomly on a grid or a graph G (e.g. $\mathbb{Z}$ or $\mathbb{Z}/\mathbf{K}\mathbb{Z}$, $\mathbb{Z}^2$, d-regular graph)**
- **Traffic:**
  - **Each moving node generates packets at rate λ**
  - **Each generated packet has a destination (e.g. a point of the grid, vertex of the graph)**
- **Contention: on each node packets are queued FIFO**

Reduction (as per Methodology :) (*continued*)

- **Motion**: neighbor nodes swap their positions with rate β
- **Wireless**: communication to neighbors only
- **Opportunistic multihop routing**:
  upon service at a node, a packet:
  - is routed to the neighbor the closest to its destination
  - leaves the system if the destination is distance 1 or 0

Finite Network Markovization

- **Assumptions**
  - **Poisson arrivals with intensity λ**
  - **exponential service times with mean 1**
  - **finite connected graph with K nodes**
- **Markov representation with discrete non compact state:**
  - **Permutation on $[1, \ldots, K]$**
    (locations of wireless nodes)
  - **Ordered queue at each node**
    (finite ordered list of destinations)

Finite Network Instability Result

- **Maximal degree in the graph: d**
- **Motion rate β**
- **Theorem**
  For all $\beta > 0$, this Markov process is transient when
  $$\mathbf{K} > \mathbf{K}^* = \frac{\mathbf{d}+\mathbf{1}}{\lambda}$$
- **For all load factors, a finite network is unstable when the diameter of the network is large enough!**

Finite Network Instability Result (*continued*)

- **Sketch of Proof in the d-regular case**
  - **nodes mix to the uniform distribution on $[1, \ldots, K]$**
  - **the proportion of time the server harboring a packet is a neighbor of its destination vertex is order $\frac{d+1}{K}$.**
  - **if $\lambda > \frac{d+1}{K}$, drift inside orthant is positive on all coordinates**
  - **supermartingale argument**

Replicated Version

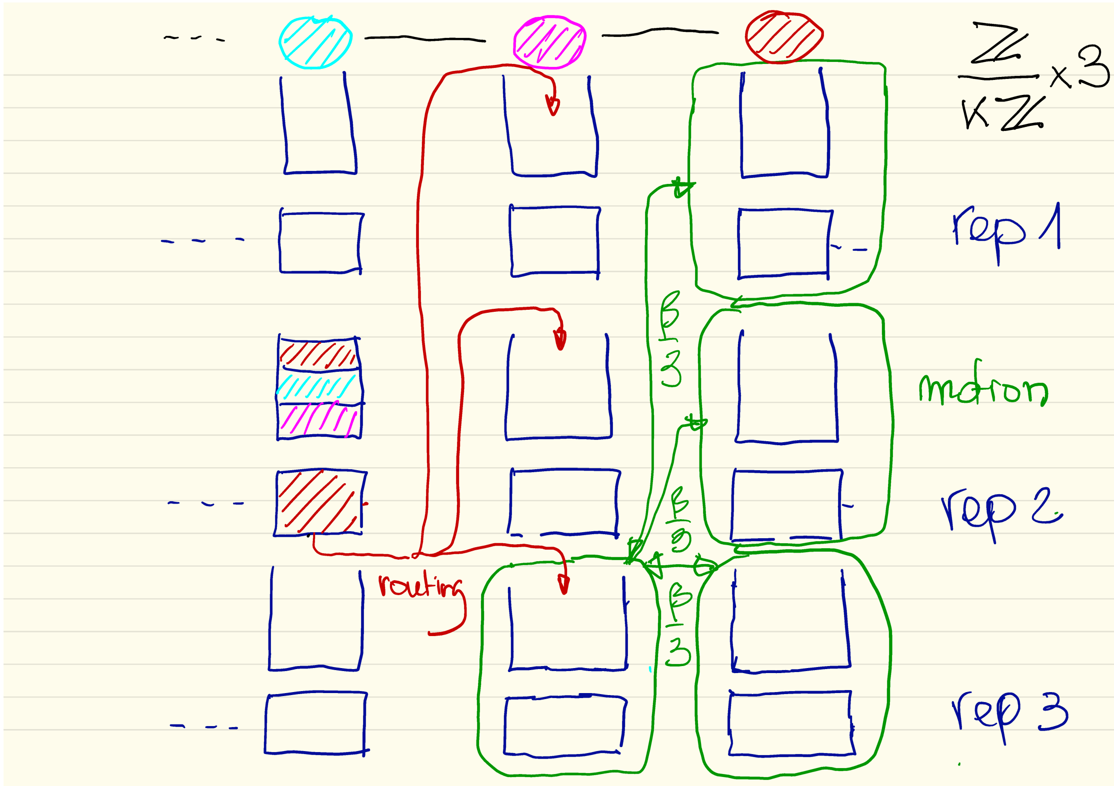
- **N-Replica version of the network on G**
- **New graph $\mathbf{G_N}$ with**

  – $\mathbf{V_N = V \times \{1, \ldots, N\}}$

  – **edge between $\mathbf{(u, i)}$ and $\mathbf{(v, j)}$ if $\mathbf{(u, v)}$ edge in G**
- **Routing**

  – **destination is any replica of the destination vertex in G**

  – **routing to one of the N replicas closest to destination at random**
- **Swaping: between a replica and any neighboring replica at random**



The 3-Replica Version of $\mathbb{Z}/\mathbf{K}\mathbb{Z}$

Instability of the N-Replica Version of a Finite Network

- **Same instability result for $\mathbf{G_N}$ as for G for fixed N**
- **Example The N-replica version of the network on $\frac{\mathbb{Z}}{\mathbf{K}\mathbb{Z}}$ is unstable under the same condition as the network on $\frac{\mathbb{Z}}{\mathbf{K}\mathbb{Z}}$ namely if $\mathbf{K > 3/\lambda}$**

Mean-Field Version

- **Mean-Field version of the network on G:**
  weak limit of the latter when N tends to infinity.
- **Notation: network on G^∞**
- **Existence: tightness arguments**

Mean Field Networks on $\mathbb{Z}$

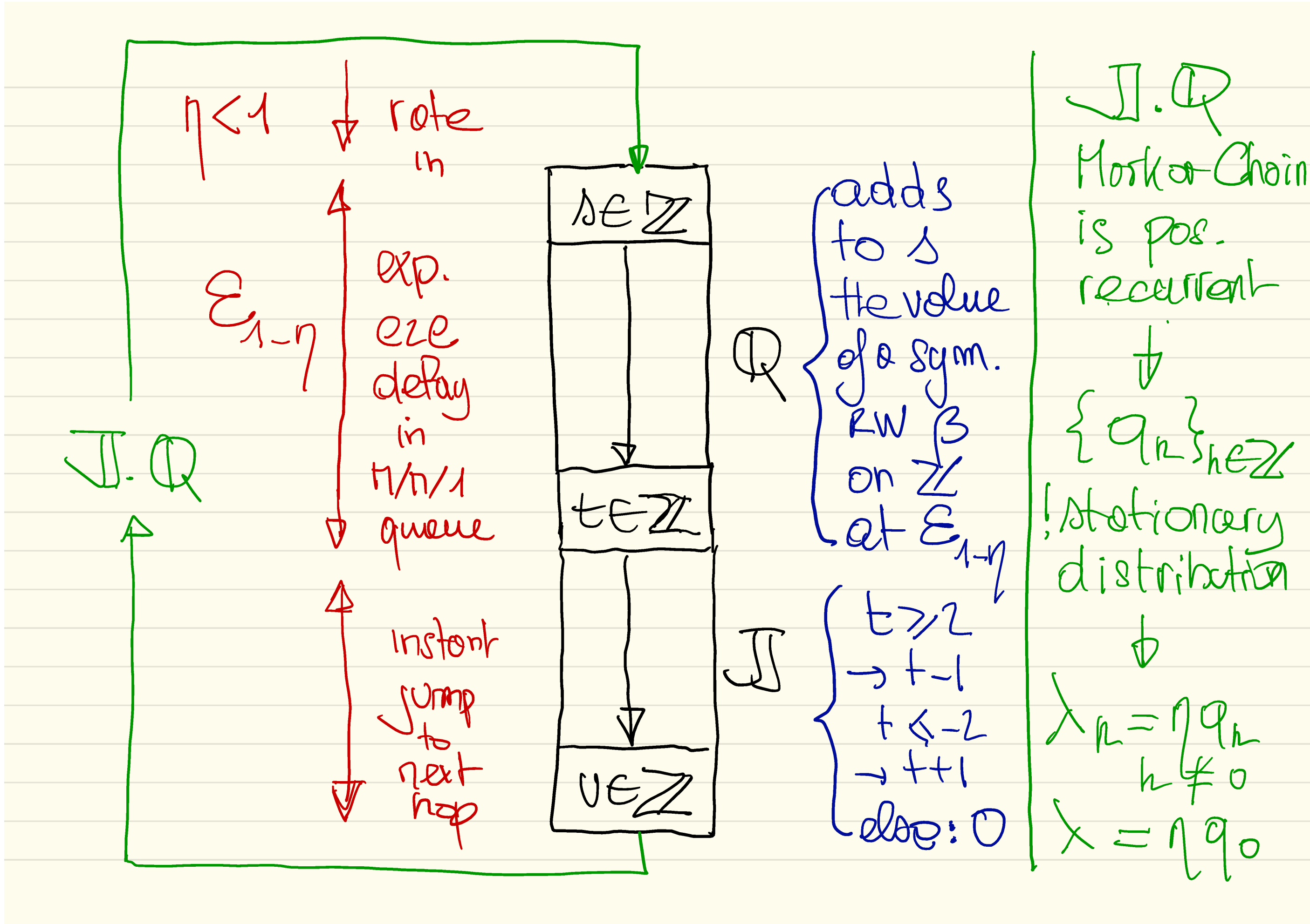
- **Non Linear Markov Process roughly, dynamical system on probability measures μ on queue states**

$$\mu(\mathbf{q}) = \mu(\mathbf{n_1}, \ldots, \mathbf{n_l}), \quad \mathbf{n_k} : \textbf{relative location of dest}(\mathbf{c_k})$$

- **Functional equation for fixed points $\mu(\mathbf{q})$ of this dynamical system**

Mean Field Networks on $\mathbb{Z}$ (*continued*)

- **Theorem** For all $0 < \eta < 1$, there exists
  a unique $0 < \lambda = \lambda_\eta < 1$
  a unique probability distribution $\mu = \mu_\eta$ on the queue state
  such that, for the exogenous arrival λ,
  – μ is solution of the functional equation and
  – the total rate in a node under μ is η
- **Sketch of Proof**
  Special case where the destination is the vertex of birth


J.Q
η<1
rate in
$\varepsilon_{1-\eta}$
exp. e2e delay in M/M/1 queue
instant jump to next hop
$s \in \mathbb{Z}$
$t \in \mathbb{Z}$
$v \in \mathbb{Z}$
Q
adds to s the value of a sym. RW β on $\mathbb{Z}$ at $\varepsilon_{1-\eta}$
J
$t \geq 2 \to t-1$
$t \leq -2 \to t+1$
else: 0
J.Q Markov Chain is pos. recurrent
$\{q_n\}_{n \in \mathbb{Z}}$! stationary distribution
$\lambda_n = \eta q_n$
$n \neq 0$
$\lambda = \eta q_0$

Existence of Multiple Solutions

- **Theorem** **For the mean-field version of the network on $\mathbb{Z}$, there exists a λ_* such that for all $\lambda < \lambda_*$, there are at least two different values $\eta = \eta_-(\lambda)$ and $\eta = \eta_+(\lambda)$ s.t.**
  - $\lambda(\eta) = \lambda$
  - $\eta_-(\lambda) \to \mathbf{0}$ **as** $\lambda \to \mathbf{0}$
  - $\eta_+(\lambda) \to \mathbf{1}$ **as** $\lambda \to \mathbf{0}$
- **Sketch of Proof**
  - **When** η **tends to 0,** $\lambda(\eta) = \lambda \mathbf{q_0}$ **tends to 0 by M/M/1**
  - **When** η **tends to 1,** $\lambda(\eta) = \lambda \mathbf{q_0}$ **tends to 0 by M/M/1 as well**

Illustration

- MC of the **distance to destination:**

$$\mathbf{p(n, n+1)} = \beta, \quad \mathbf{p(n, n-1)} = \beta + \gamma,$$

with $\gamma = 1 - \eta$,

$$\mathbf{p(1, 2)} = \beta, \quad \mathbf{p(1, 0)} = \beta, \quad \mathbf{p(1, *)} = \gamma$$

and

$$\mathbf{p(0, 1)} = \mathbf{2}\beta, \quad \mathbf{p(0, *)} = \gamma,$$

where $*$ is absorbing

Illustration (*continued*)

- **Mean # nodes visited by a customer till absorption:**

$$\mathbb{E}[\mathbf{N}] = \mathbf{1} + \frac{\mathbf{2}\beta^{\mathbf{2}}}{\gamma(\mathbf{3}\beta + \gamma)}$$

- **Flow equation:**

$$\eta = \mathbf{1} - \gamma = \lambda\left(\mathbf{1} + \frac{\mathbf{2}\beta^{\mathbf{2}}}{\gamma(\mathbf{3}\beta + \gamma)}\right)$$

When β is large, there are two roots

$$\eta^- = \lambda + \frac{\mathbf{1} - \lambda - \sqrt{(\mathbf{1} - \lambda)^{\mathbf{2}} - \frac{\mathbf{8}}{\mathbf{3}}\lambda\beta}}{\mathbf{2}}$$

$$\eta^+ = \lambda + \frac{\mathbf{1} - \lambda + \sqrt{(\mathbf{1} - \lambda)^{\mathbf{2}} - \frac{\mathbf{8}}{\mathbf{3}}\lambda\beta}}{\mathbf{2}}$$

with

$$0 < \eta^- < \eta^+ < 1, \qquad \nu^- \to 0, \quad \nu^+ \to 1, \quad \textbf{when } \lambda \to 0$$

Generalization

- **The multiple fixed point result can be extended to the Cayley graph of any group G s.t.**
  - **G has a finite generating set**
  - **G is infinite**
  - **the arrival rate, swap rate, swap rule, destination rule are G-invariant**
- **Example: the network of $(\mathbb{Z})^\infty$ has at least to fixed points stationary regimes for every $\lambda < \lambda_*$ with $\lambda_* > 0$.**

Meta-Stability

- **Finitely many replicas–Infinitely many replicas stability difference.**
- **No contradiction with the fact that, for $\mathbf{N} < \infty$, the network of $(\mathbb{Z})^{\mathbf{N}}$ has no stationary regimes**
  the time - replica diagram does not commute here!

Current and Future Extensions

- **Wireless primitives: (SINR, MAC) to assess transmission between nearby nodes**
- **Motion primitives: Brownian, random waypoint**
- **Queuing primitives: beyond Poisson arrivals and exponential service times**
- **Scheduling: beyond FIFO: needed to see why motion increases capacity**

Summary

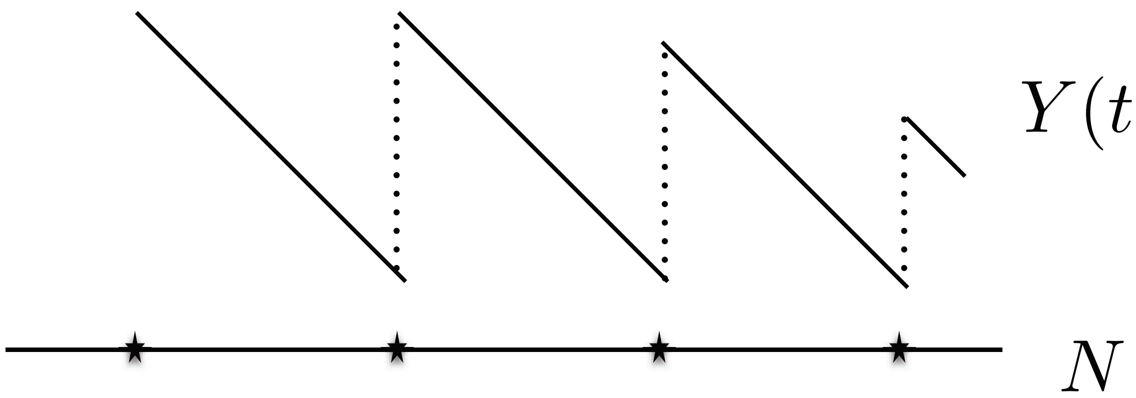
- **Way to go but first step of a mean-field representation of mutihop routing in wireless networks beyond scaling laws**
- **Meta-stability is encouraging news**
- **When fully interconnected with SG, new analytical handle for optimization in this class of problems.**

References

- Part 1
  - with A. Sankararaman,
    Spatial birth-death wireless networks,
    ArXiv 1604.07884, IEEE Tr. IT, 63(6) 2017
  - with S. Foss and A. Sankararaman,
    Infinite spatial birth-death wireless networks,
    in preparation
- Part 2
  - with S. Rybko, S. Shlosman and S. Vladimirov,
    Metastability of Queuing Networks with Mobile Servers,
    ArXiv 1704.02521, submitted

Proof Sketch - Necessary Condition

Assume stability and write down 'Rate Conservation Equations'.
Then find a contradiction.



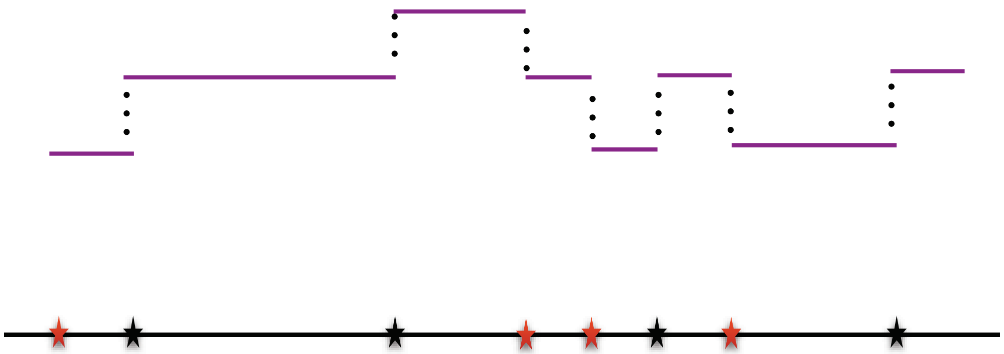
Both process are stationary

If $$Y(t) = Y(0) + \int_{s=0}^{t} D(s)ds + \int_{s=0}^{t} (Y(s) - Y(s^-))N(ds)$$

Implies $$\mathbb{E}[D(0)] + \lambda_N \mathbb{E}_N^0[Y(0) - Y(0^-)] = 0$$

Proof Sketch - Necessary Condition

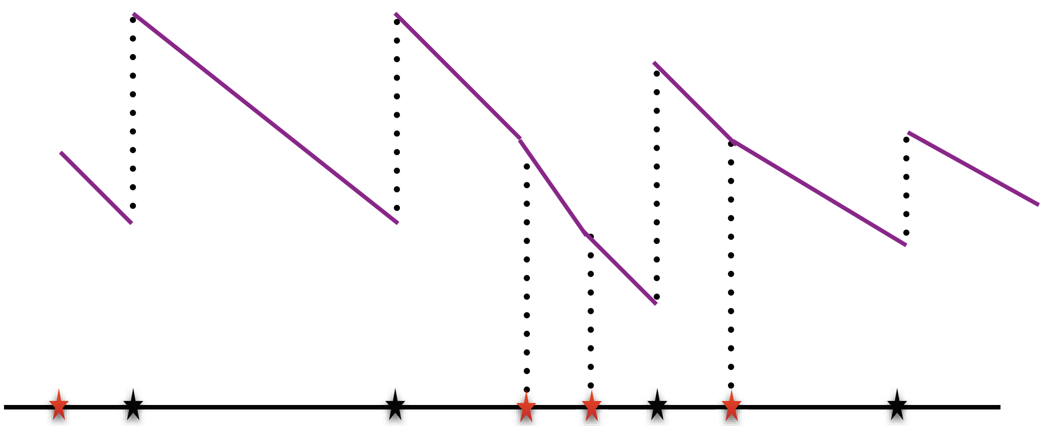
$\phi_t(\mathbf{S})$



Red - Epochs of Death
Black - Epochs of Arrivals

RCL implies $\lambda|S| = \lambda_d$

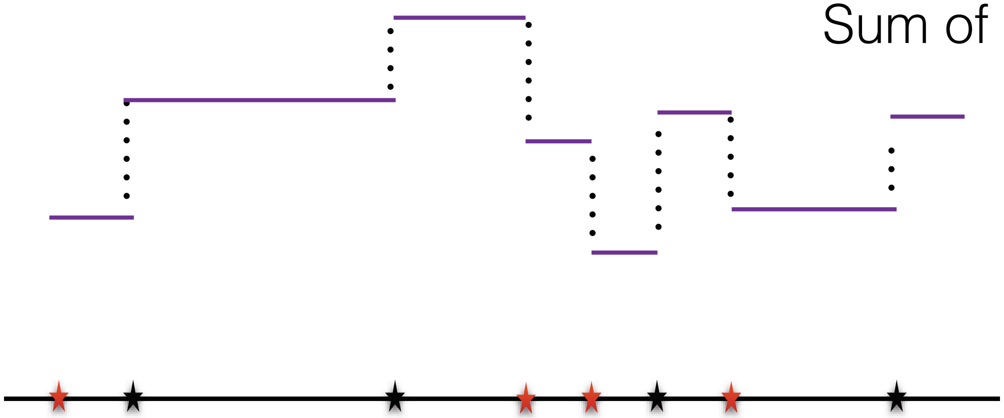
Total bits left in the network i.e. remaining 'workload'



RCL implies

$$\lambda|S|L = \mathbb{E}\left[\sum_{x \in \phi_0} R(x, \phi_0)\right]$$

Proof Sketch - Necessary Condition



Sum of interference seen at all points

$$Y(t) = \sum_{x \in \phi_t} I(x, \phi_t)$$

Red - Epochs of Death
Black - Epochs of Arrivals

RCL implies $\lambda |S| \mathbb{E}_B^0[\mathcal{I}] = \lambda_d \mathbb{E}_D^0[\mathcal{D}]$

PASTA and RCL for $\phi_t(\mathbf{S})$ $\mathbb{E}[\mathcal{I}] = \mathbb{E}_D^0[\mathcal{D}]$

Campbell's Theorem $\mathbb{E}[\mathcal{I}] = 2\frac{\mathbb{E}[\phi_0(\mathbf{S})]}{|S|} \int_{x \in \mathbf{S}} l(||x||)dx$

Handle $\mathbb{E}_D^0[\mathcal{D}]$ through Papangelou's Stochastic Intensity formula

Proof Sketch - Necessary Condition

We have $$\mathbb{E}[\mathcal{I}] = \mathbb{E}^0_D[\mathcal{D}] = 2\frac{\mathbb{E}[\phi_0(\mathbf{S})]}{|S|}\int_{x\in\mathbf{S}} l(||x||)dx$$

The Death Point process admits as stochastic intensity - $\mathbf{R}_t = \sum_{x\in\phi_t} R(x,\phi_t)$
with respect to the filtration $\mathcal{F}_t = \sigma(\phi_s : s \leq t)$

Papangelou's theorem implies $$\frac{d\mathbb{P}^0_D}{d\mathbb{P}}|_{\mathcal{F}_{0-}} = \frac{\mathbf{R}_0}{\mathbb{E}[\mathbf{R}_0]}$$

This gives

$$\mathbb{E}^0_D[\mathcal{D}] = \mathbb{E}\left[\frac{\mathbf{R}_0}{\mathbb{E}[\mathbf{R}_0]}\mathcal{D}\right] = 2\mathbb{E}\left[\frac{\mathbf{R}_0}{\mathbb{E}[\mathbf{R}_0]}\sum_{x\in\phi_0}\frac{R(x,\phi_0)}{\mathbf{R}_0}I(x,\phi_0)\right]$$

Proof Sketch - Necessary Condition

We have $$\mathbb{E}[\mathcal{I}] = \mathbb{E}_D^0[\mathcal{D}] = 2\frac{\mathbb{E}[\phi_0(\mathbf{S})]}{|S|}\int_{x\in\mathbf{S}} l(||x||)dx$$

and

$$\mathbb{E}_D^0[\mathcal{D}] = \mathbb{E}\left[\frac{\mathbf{R}_0}{\mathbb{E}[\mathbf{R}_0]}\mathcal{D}\right] = 2\mathbb{E}\left[\frac{\mathbf{R}_0}{\mathbb{E}[\mathbf{R}_0]}\sum_{x\in\phi_0}\frac{R(x,\phi_0)}{\mathbf{R}_0}I(x,\phi_0)\right]$$

Algebra -

$$2\frac{\mathbb{E}[\phi_0(\mathbf{S})]}{|S|}\int_{x\in\mathbf{S}} l(||x||)dx = 2\frac{\mathbb{E}_{\phi_0}^0[R(0,\phi_0)I(\phi_0)]\mathbb{E}[\phi_0(\mathbf{S})]}{\lambda L|S|}$$

$$\int_{x\in\mathbf{S}} l(||x||)dx = \frac{\mathbb{E}_{\phi_0}^0[R(0,\phi_0)I(0,\phi_0)]}{\lambda L}$$

Noticing that $R(x,\phi)I(x,\phi) \leq C\log_2(e)$ yields the necessary condition !

"*Clustering*" in Steady State

Assuming there is a stationary regime,

$$\lambda|S|L = \mathbb{E}\left[\sum_{x\in\phi_0} R(x,\phi_0)\right] = \mathbb{E}^0_{\phi_0}[R(0,\phi_0)]\beta|S|$$

From RCL arguments

$$\int_{x\in\mathbf{S}} l(||x||)dx = \frac{\mathbb{E}^0_{\phi_0}[R(0,\phi_0)I(0,\phi_0)]}{\lambda L}$$

Negative Association yields $\mathbb{E}^0_{\phi_0}[R(0,\phi_0)I(0,\phi_0)] \leq \mathbb{E}^0_{\phi_0}[R(0,\phi_0)]\mathbb{E}^0_{\phi_0}[I(0,\phi_0)]$

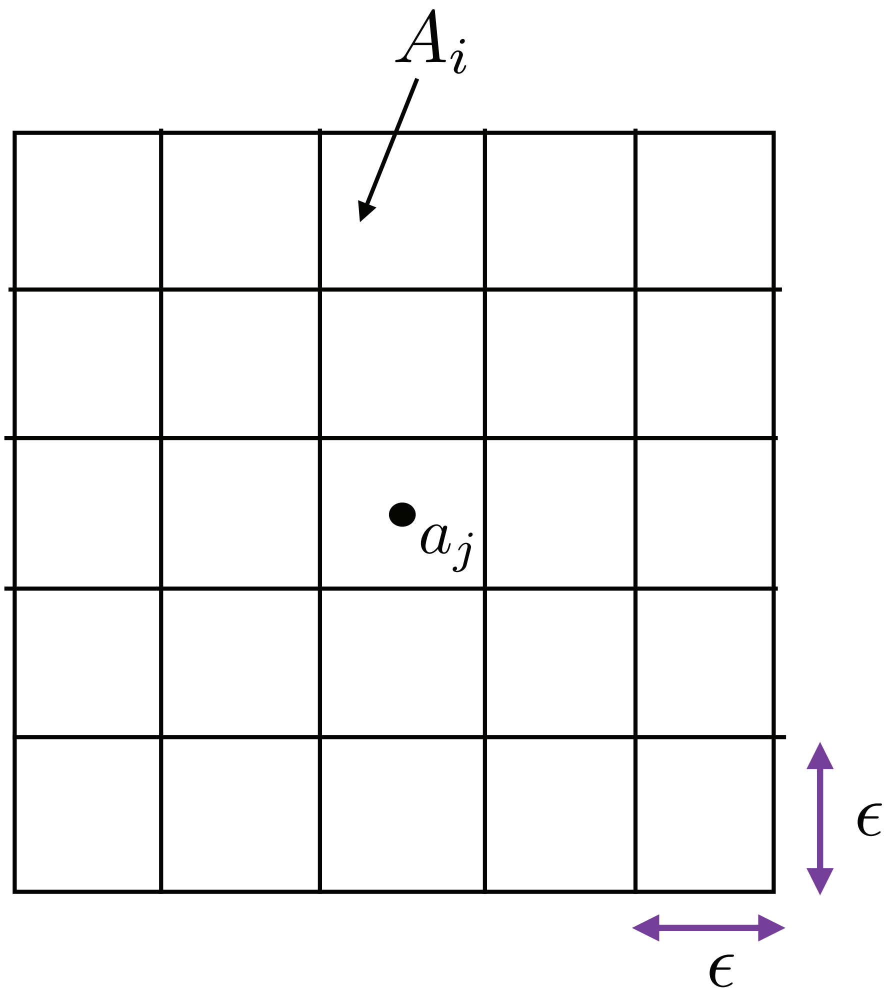
Putting the above together $\mathbb{E}[I(0,\phi_0)] \leq \mathbb{E}^0_{\phi_0}[I(0,\phi_0)]$

Implies Clustering if path-loss is non-increasing and implies repulsion if path-loss is non-decreasing

Proof Sketch - Sufficient Condition

Consider an *approximate* birth-death process on the ϵ width lattice which is easy to study.

Tesselate into grids of side length at-most ϵ which results in N_ϵ grids.



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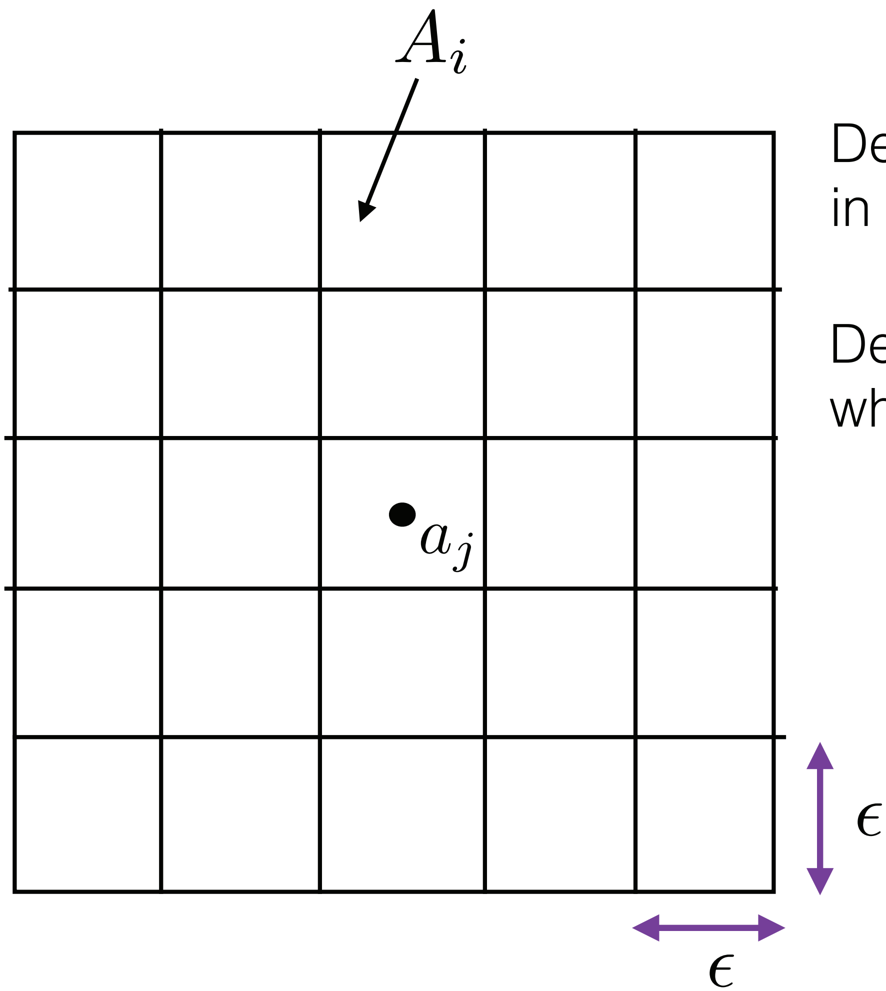

Denote by ϕ_t^ϵ as the configuration at time t in this ϵ approximate system

Denote by the vector $\mathbf{X}(t) \in \mathbb{N}^{N_\epsilon}$ where $X_i(t) = \phi_t^\epsilon(A_i)$.

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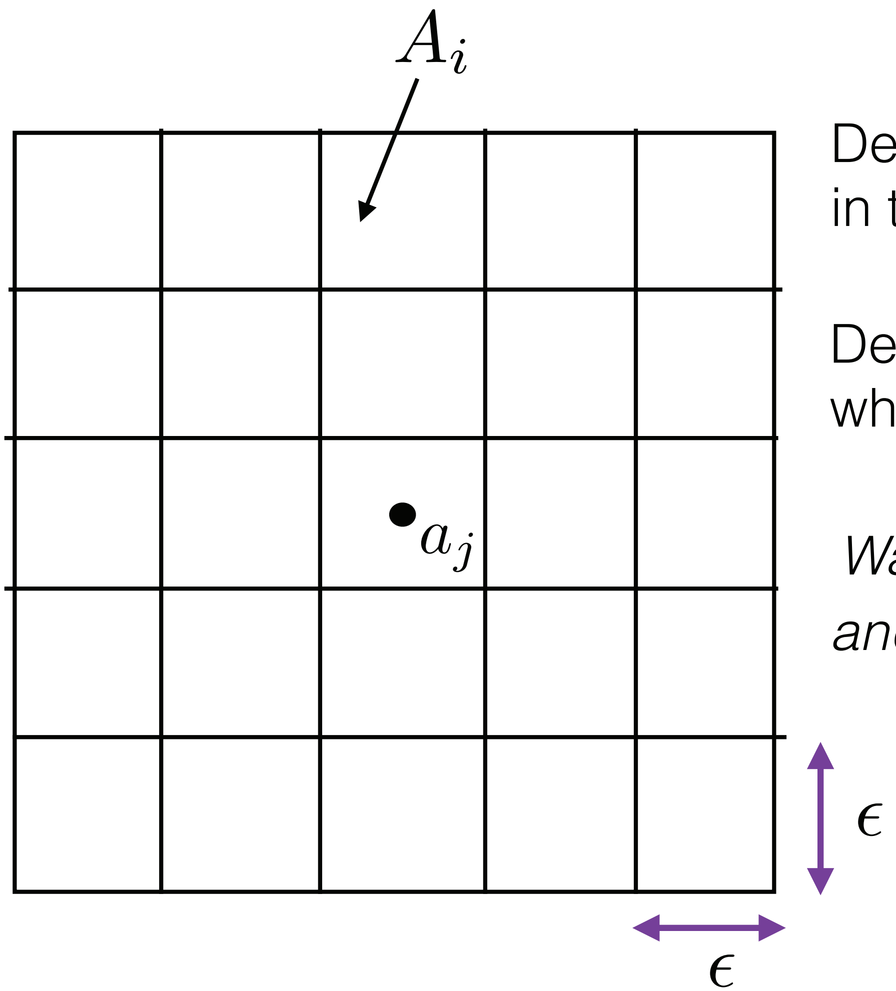

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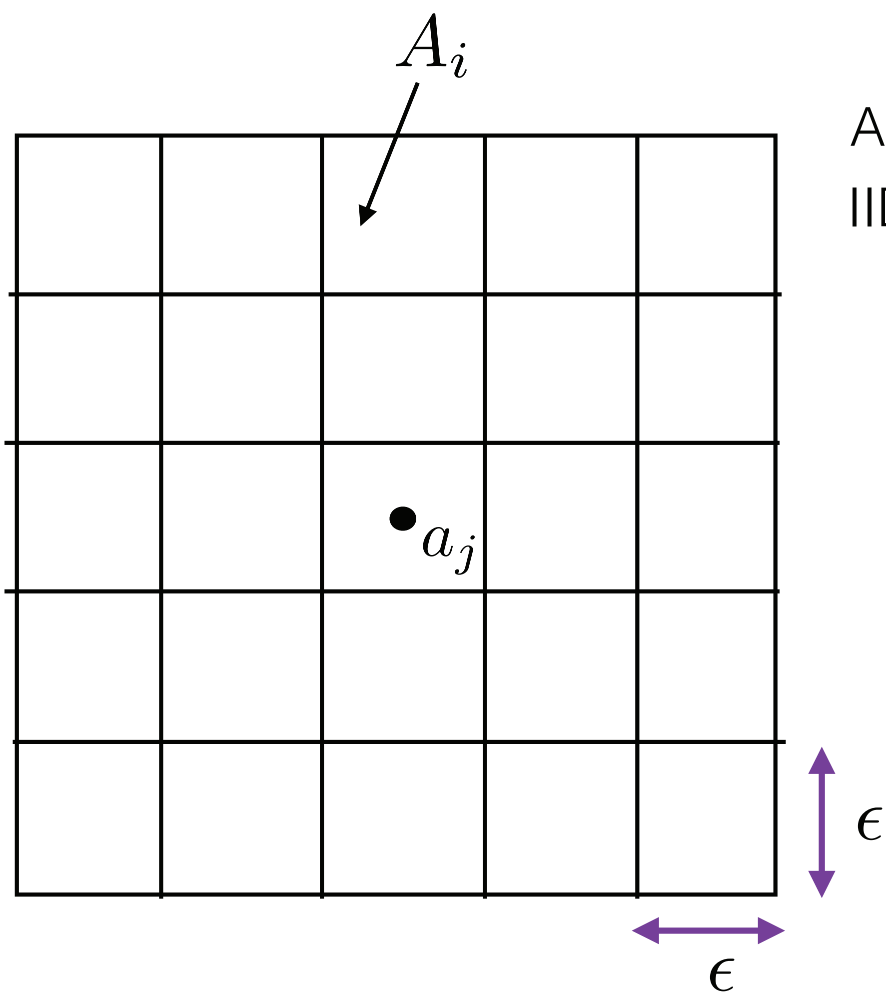

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IID exponential File Sizes of mean L

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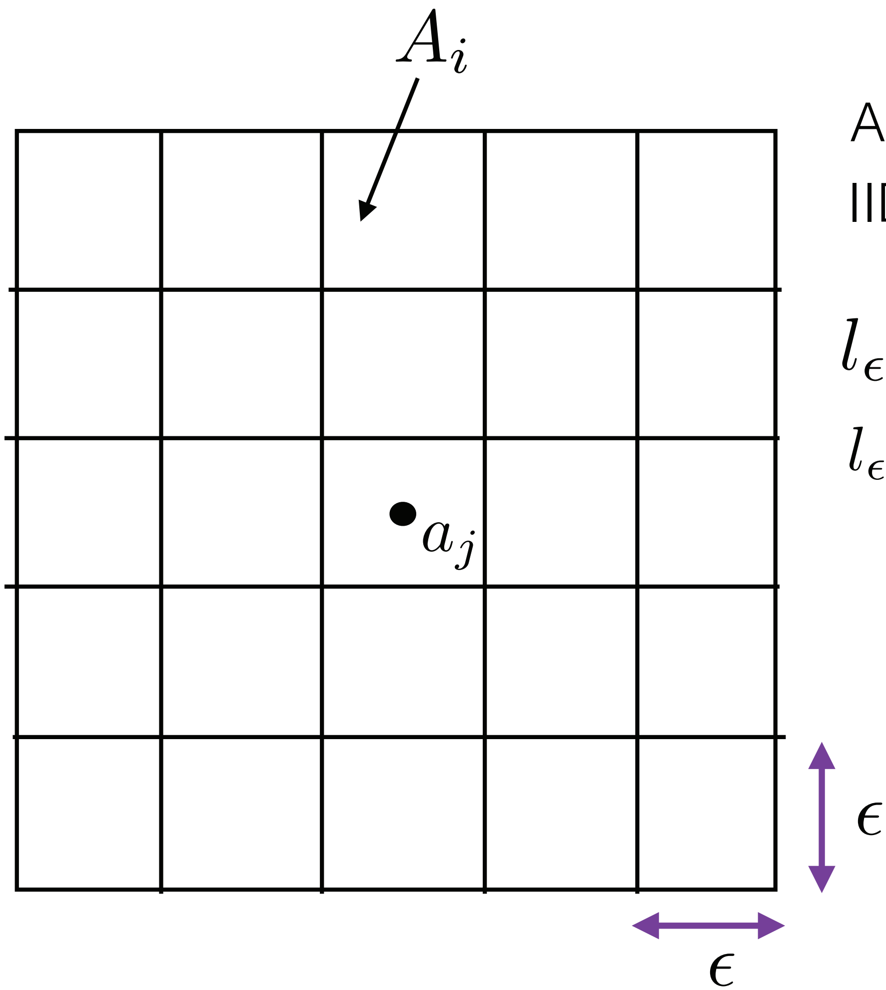

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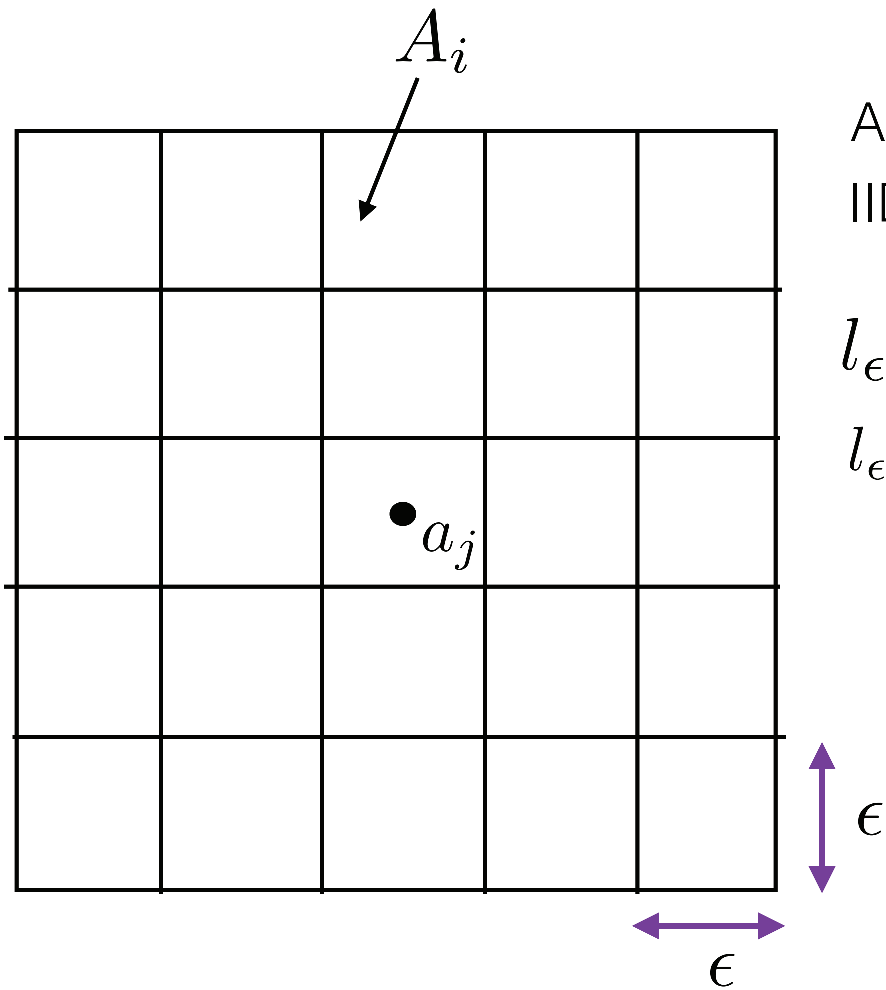

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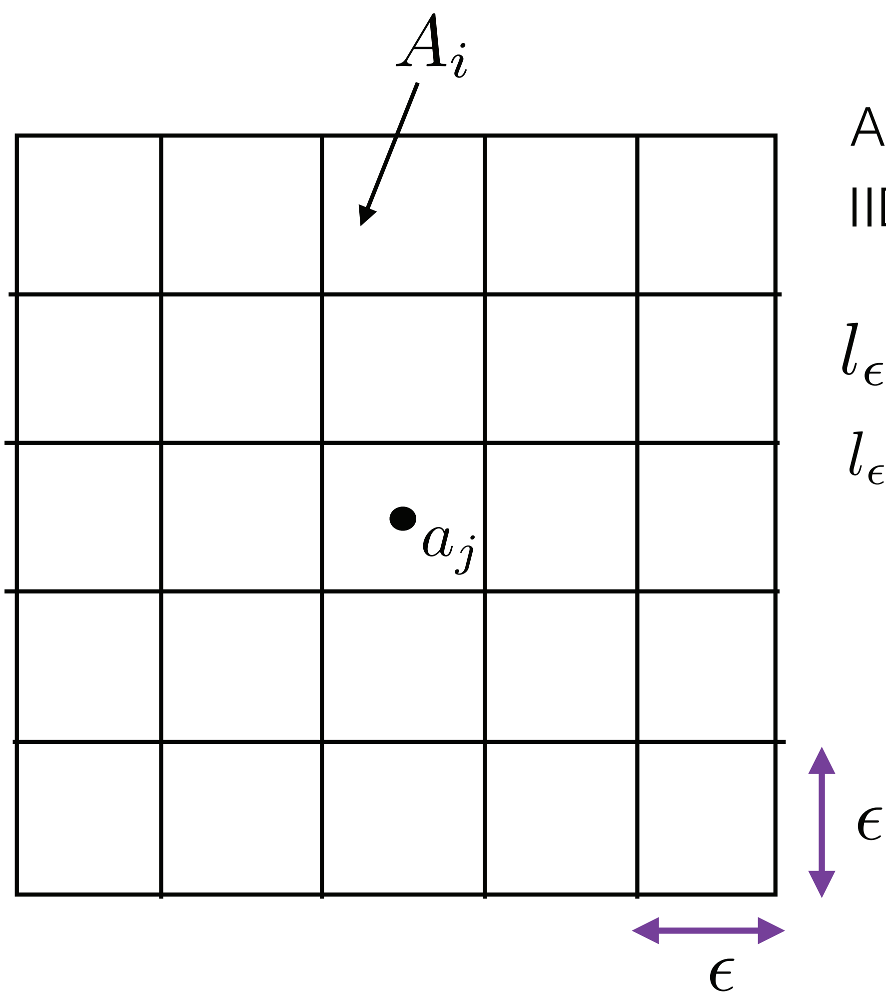

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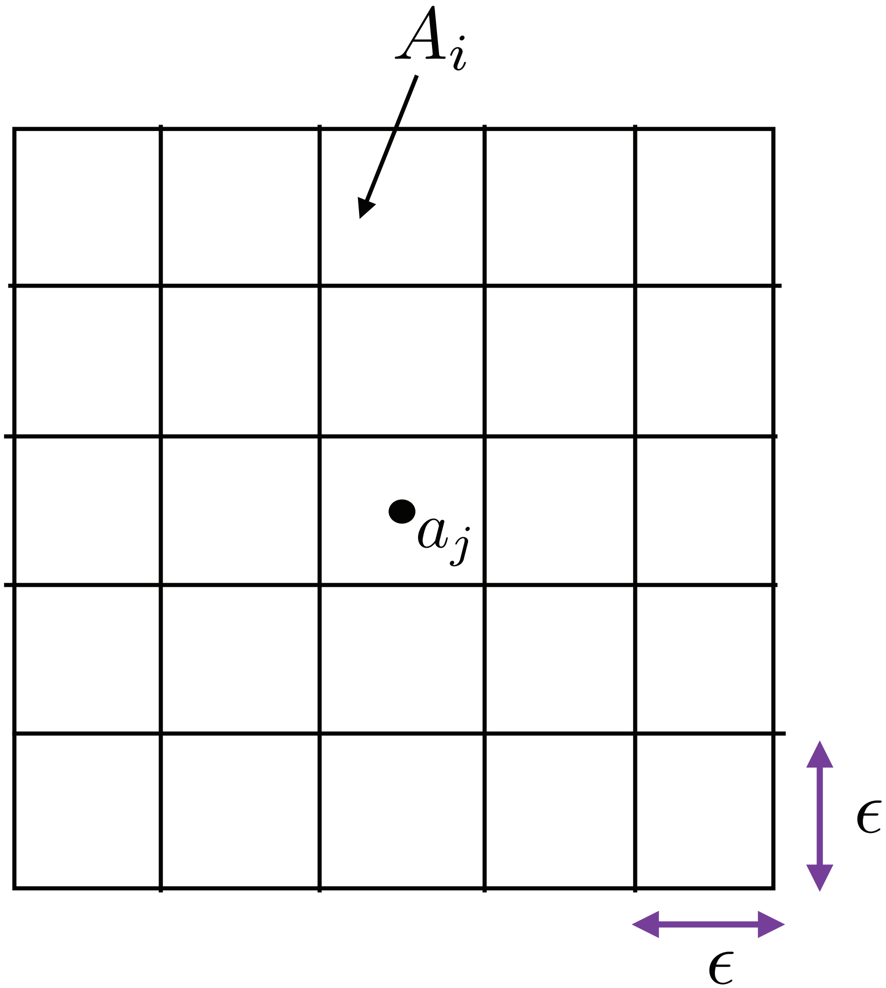
Need monotonicity of $l(r)$!!

Proof Sketch - Sufficient Condition



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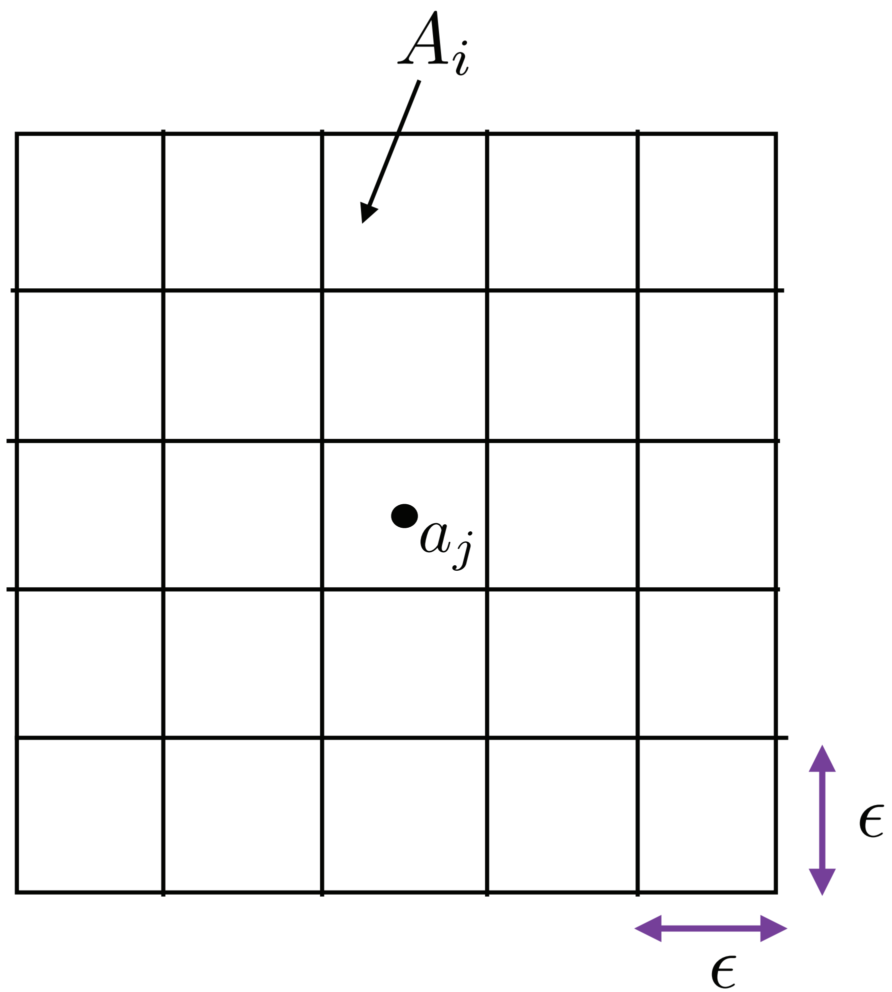

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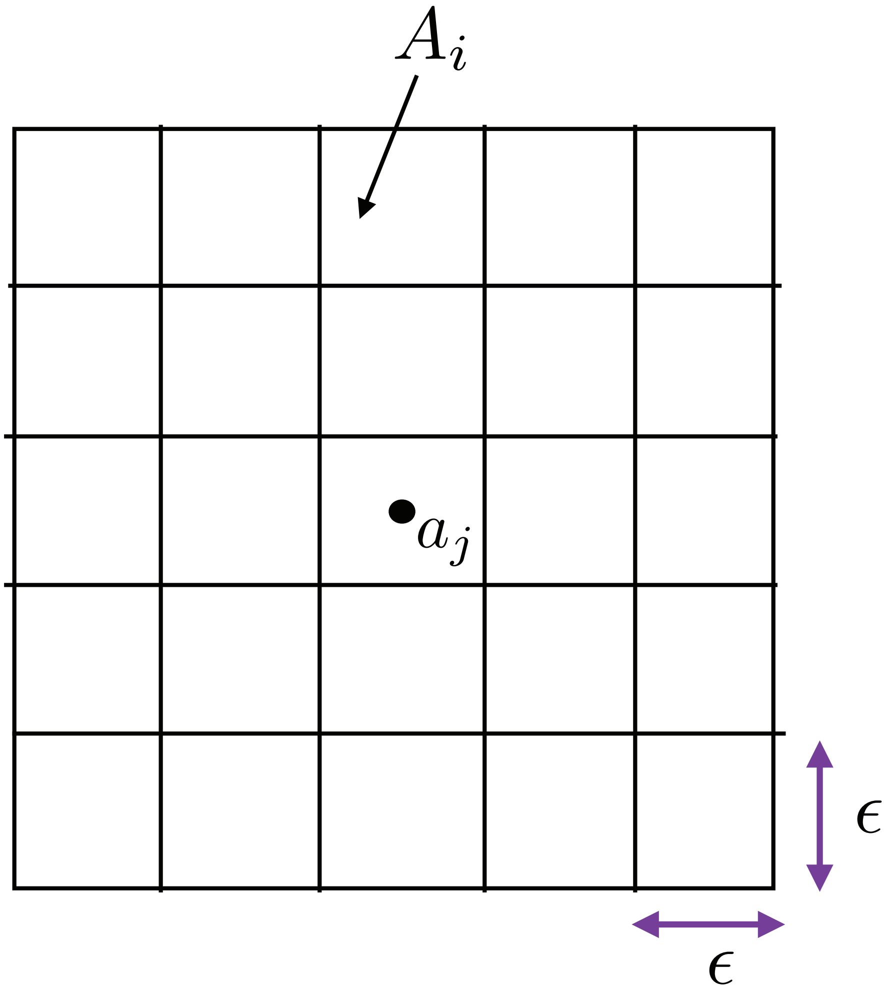

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Obtaining the best possible bound (by optimizing over ϵ) gives that

$$\liminf_{\epsilon \downarrow 0} \lambda L \int_{x \in \mathbf{S}} l_\epsilon(x,0)dx < \log_2(e)$$ as the stability region of ϕ_t

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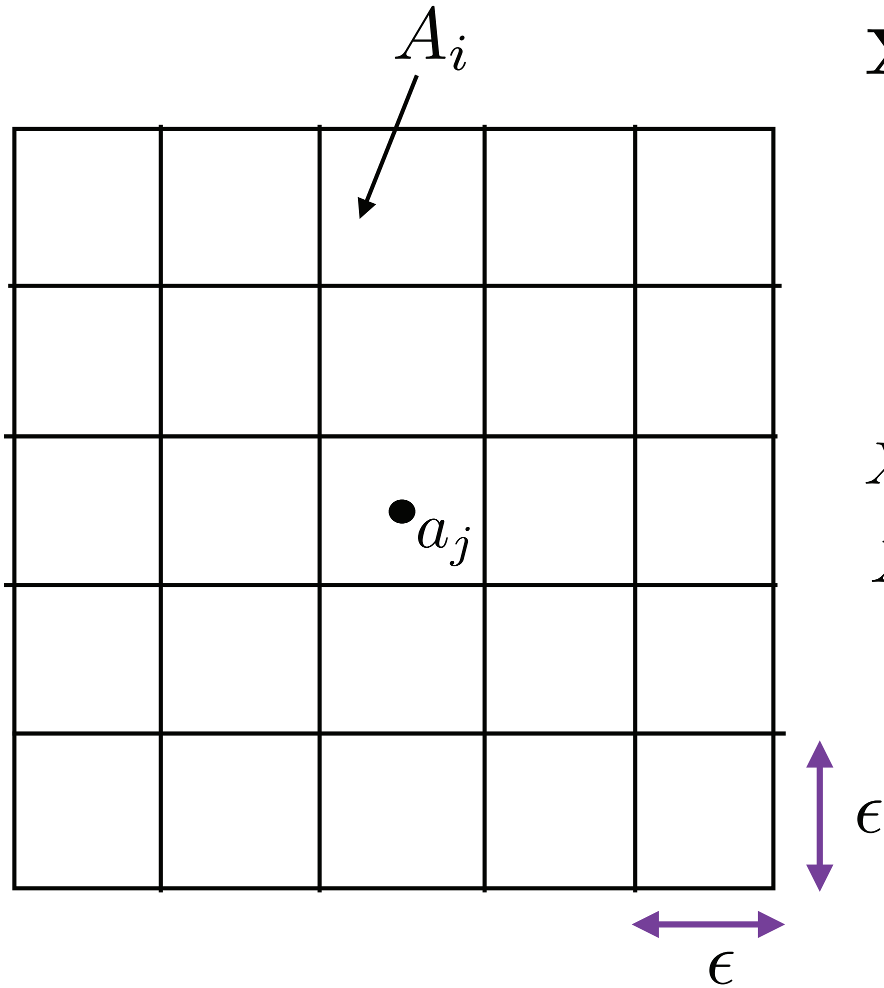

$\mathbf{X}(t) \in \mathbb{N}^{N_\epsilon}$ with $X_i(t) = \phi_t^\epsilon(A_i)$

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The Evolution

$X_i \to X_i + 1$ at rate $\lambda\epsilon^2$

$X_i \to X_i - 1$ at rate $\frac{1}{L} C X_i \log_2\left(1 + \frac{1}{N_0 + I_i^\epsilon(X)}\right)$

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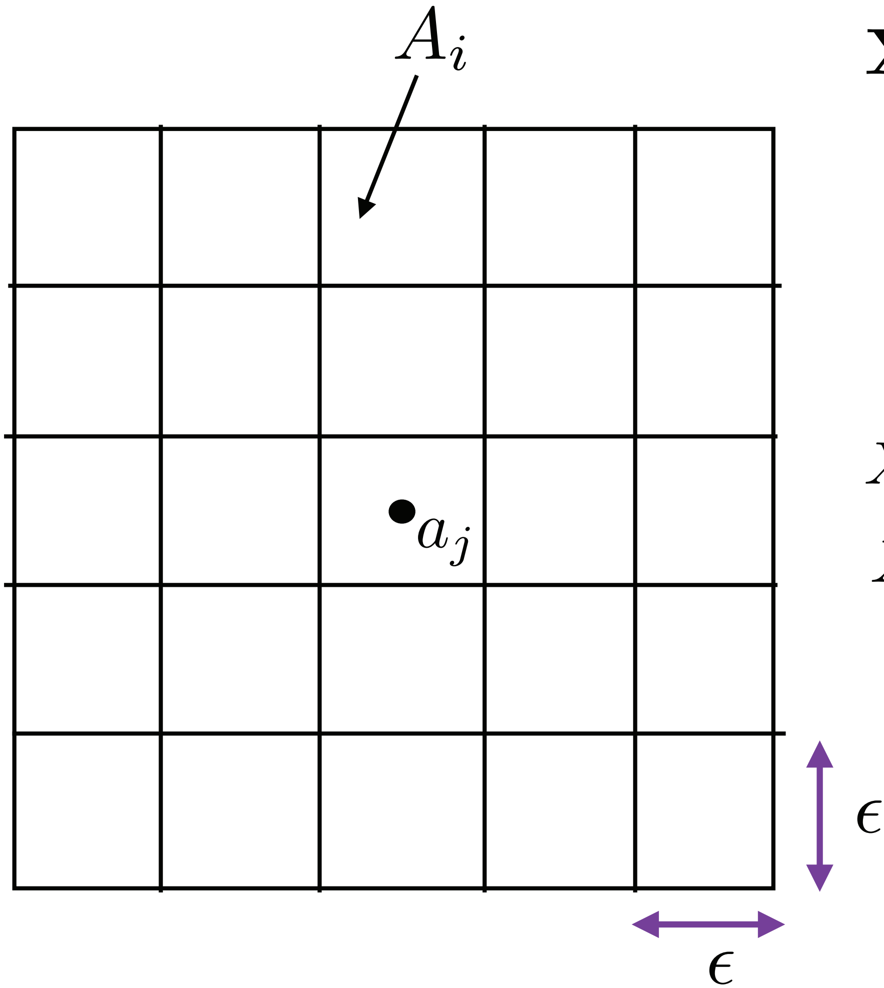

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Analyze this evolution through Fluid Limit techniques of [Dai 95], [Massoulié 07].

J.G. Dai , “On Positive Harris Recurrence of Multi-class Queuing Networks: A Unified Approach through Fluid Limit Models”
L. Massoulie´, “Structural Properties of Proportional Fairness: Stability and Insensitivity“

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The Fluid model of the above evolution

$$x_i(t) := x_i(0) + \lambda\epsilon^2 t - D_i(t)$$

with the derivative of the interference satisfying $\dot{D}_i(t) = \frac{x_i(t)}{I_i^{\epsilon,f}(t)}$

Interference in the fluid scale $I_i^{\epsilon,f}(t) = \sum_k x_k(t) l_\epsilon(a_k, a_i)$

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Or equivalently

$$\frac{d}{dt} x_i(t) = \lambda\epsilon^2 - \frac{x_i(t) C \log_2(e)}{L \sum_k x_k(t) l_\epsilon(a_k, a_i)}$$

The fluid model arises as a result of appropriate space-time scaling

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More precisely, one can show that for a sequence of initial conditions $\{X^{(k)}(0)\}_{k\geq 1}$ and sequence of numbers $\{z_k\}_{k\geq 1}$ such that $z_k \to \infty$ and $\lim_{k\to\infty} \frac{X^{(k)}(0)}{z_k} = x(0)$, one has $\frac{X^{(k)}(z_k t)}{z_k} \xrightarrow{\mathbb{P}} x(t)$ u.o.c i.e $\forall \epsilon > 0$ and $\forall T \in (0, \infty)$

$$\lim_{k\to\infty} \mathbb{P}\left(\inf_{f \in S(x(0))} \sup_{t \in [0,T]} |z_k^{-1} X^{(k)}(z_k t) - f(t)| > \epsilon \right) = 0$$

Need $l(r)$ to be bounded

Idea borrowed from [Massoulié, 07]

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- Proof technique for ergodicity when $\mathbf{S} = \mathbb{R}^2$