2^nd Symposium on Spatial Networks
13-14 September 2017

GROWING SIMPLICIAL COMPLEXES AND EMERGENT NETWORK GEOMETRY

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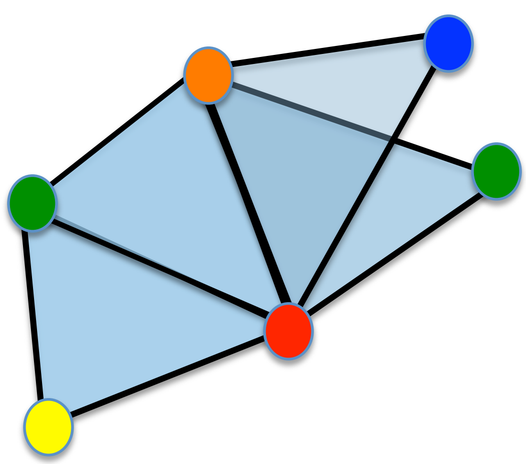
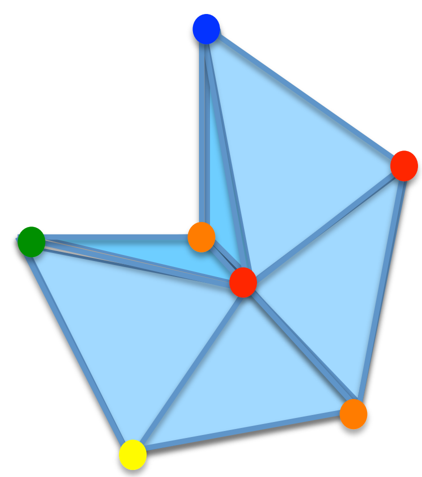
Simplicial Complexes

Simplicial complexes are characterizing the interaction between two ore more nodes and are formed by nodes, links, triangles, tetrahedra etc.

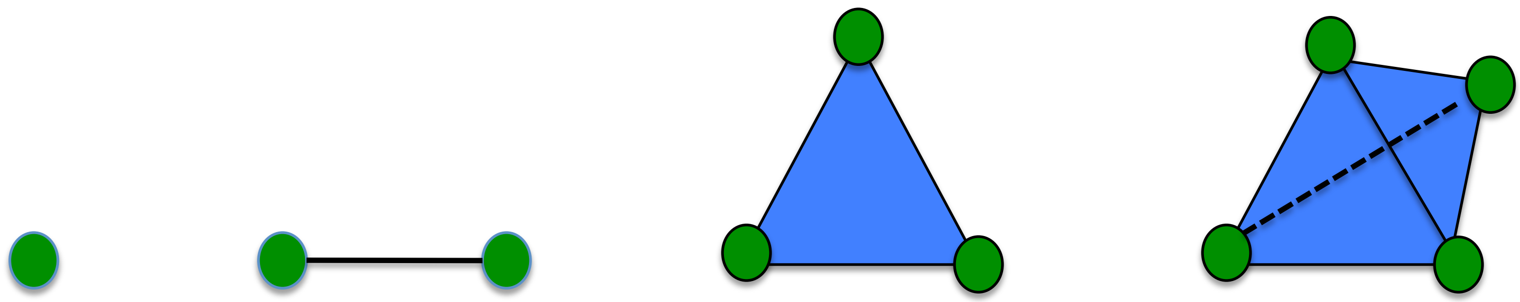


d=2 simplicial complex



d=3 simplicial complex

Simplices



0-simplex 1-simplex 2-simplex 3-simplex

A simplex of dimension d is a set of d+1 nodes

-it indicates the interactions between the nodes

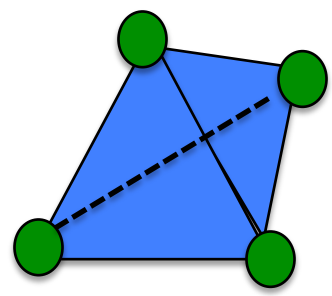
-it admits a geometrical interpretation

Faces of a simplex

A face of a d-dimensional simplex

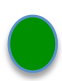
is a δ-dimensional simplex (with $\delta < d$)

formed by a non-empty subset of its nodes



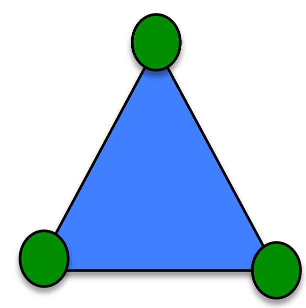
3-simplex

Faces



4 0-simplices

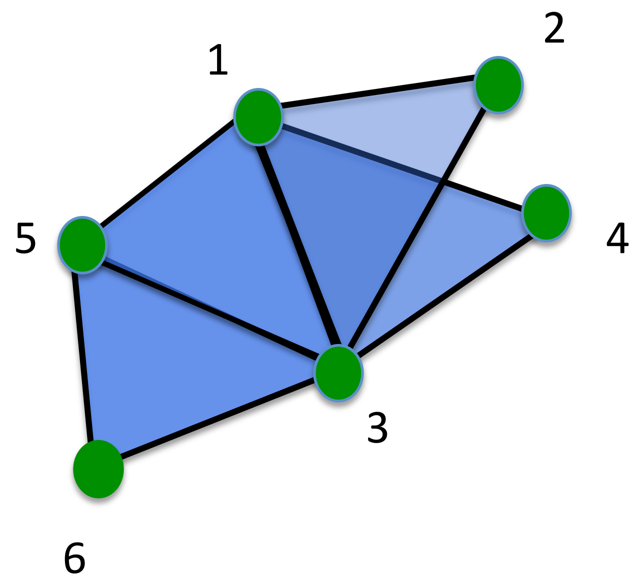
6 1-simplices



4 2-simplices

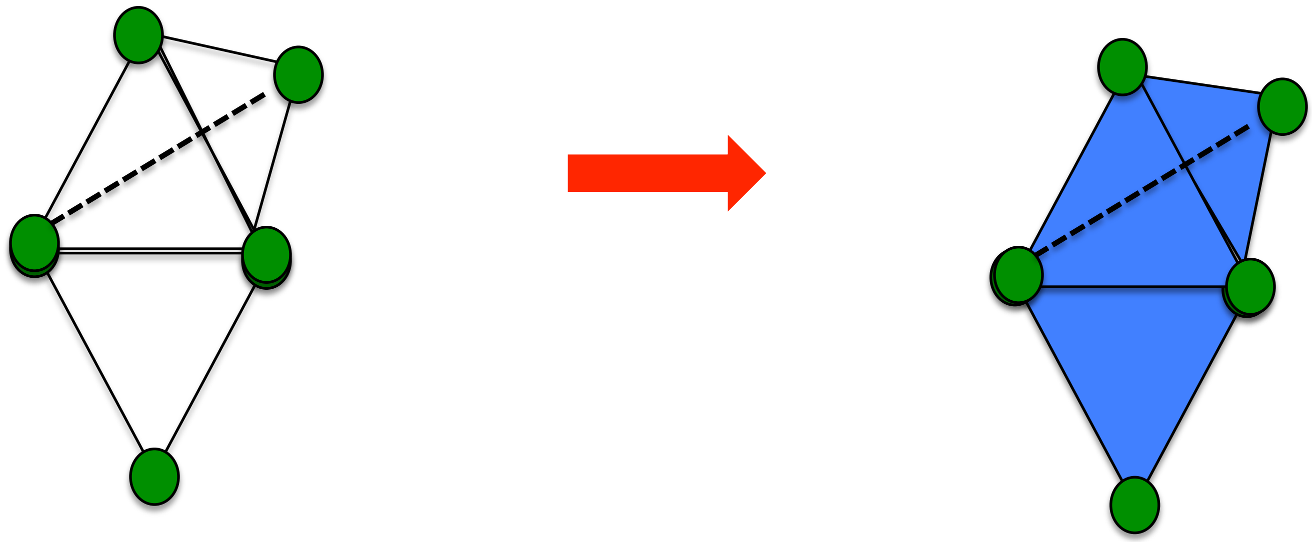
Simplicial complex

A simplicial complex K is a set of simplices with the following condition:



if a simplex μ belongs to K every face of μ must belong to K

Clique complex



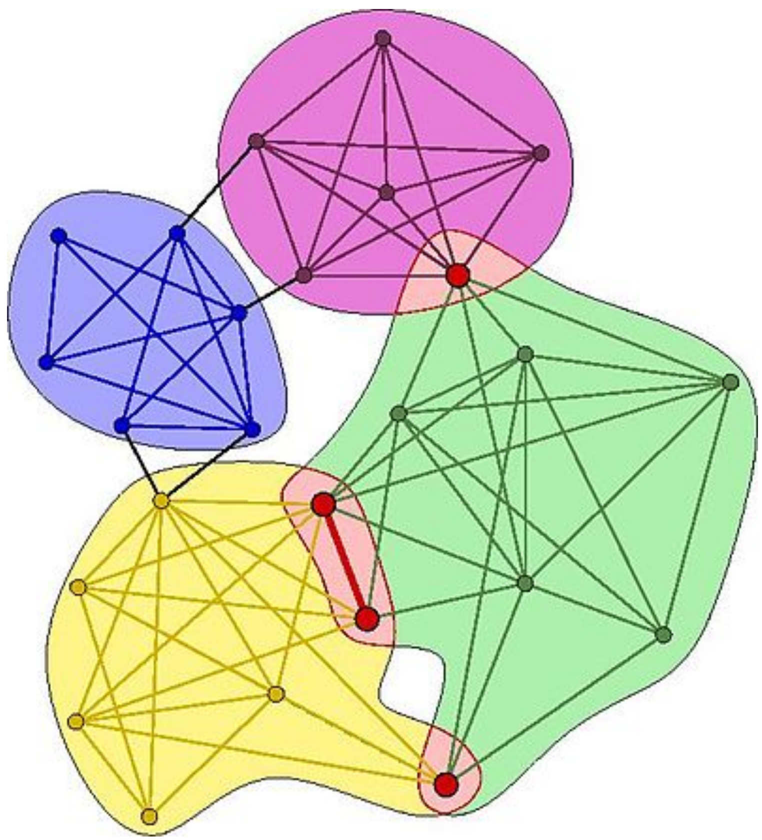
From every network we can extract a simplicial complex called

clique complex

by associating a simplex to every clique

(fully connected subgraph)

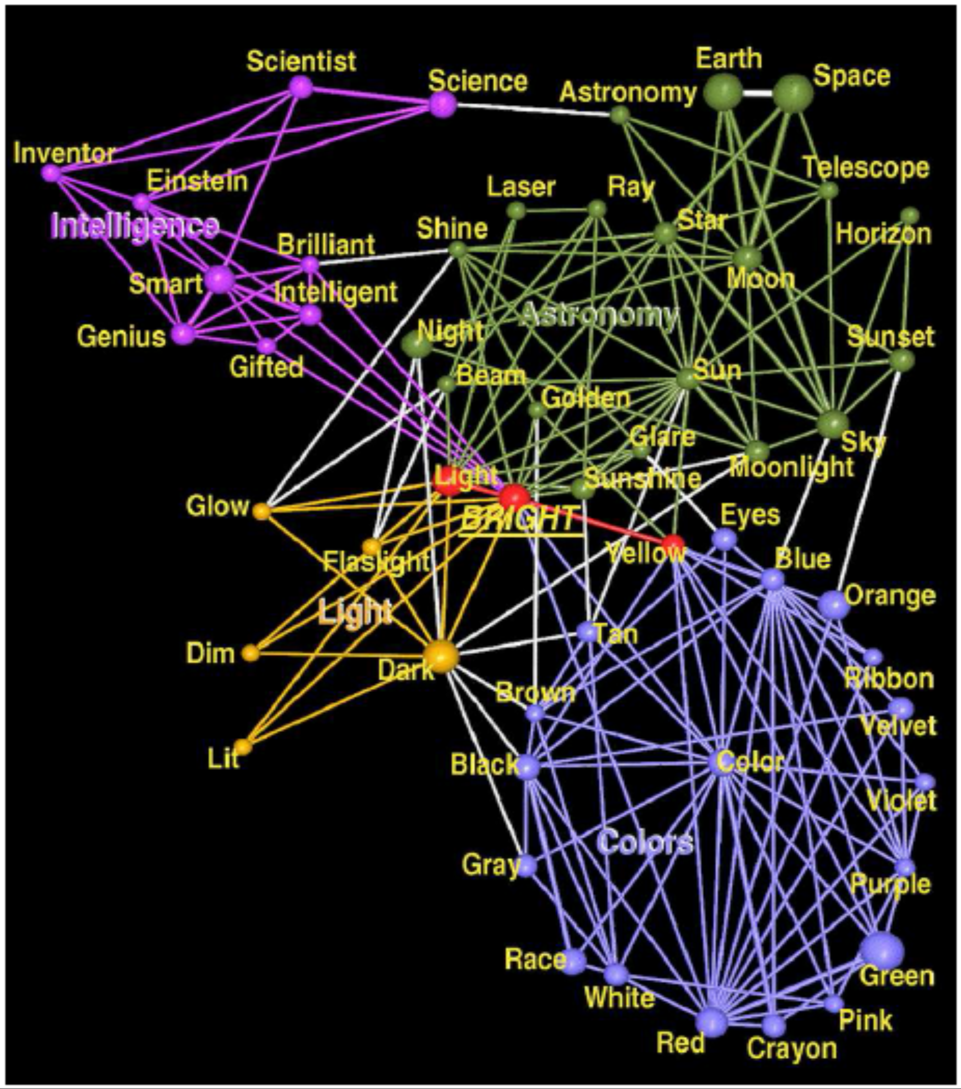
Clique percolation



Clique percolation reveals overlapping communities by characterizing how cliques percolate in networks

Palla et al. Nature (2005)

World association network

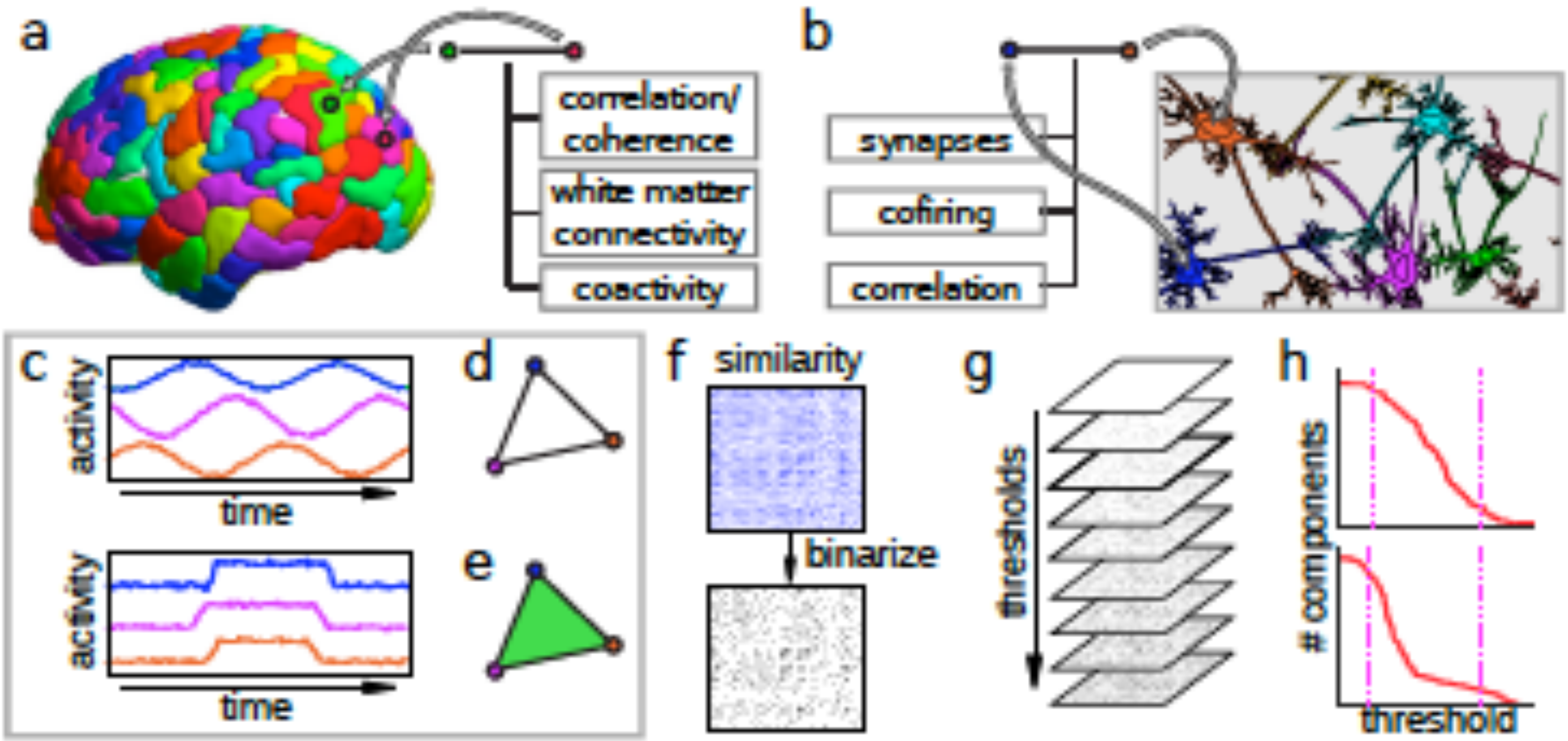


Palla et al.
Nature (2005)

From a simplicial complex to a network

From every simplicial complex
we can extract a network
by considering exclusively
nodes and links

Brain data as simplicial complexes

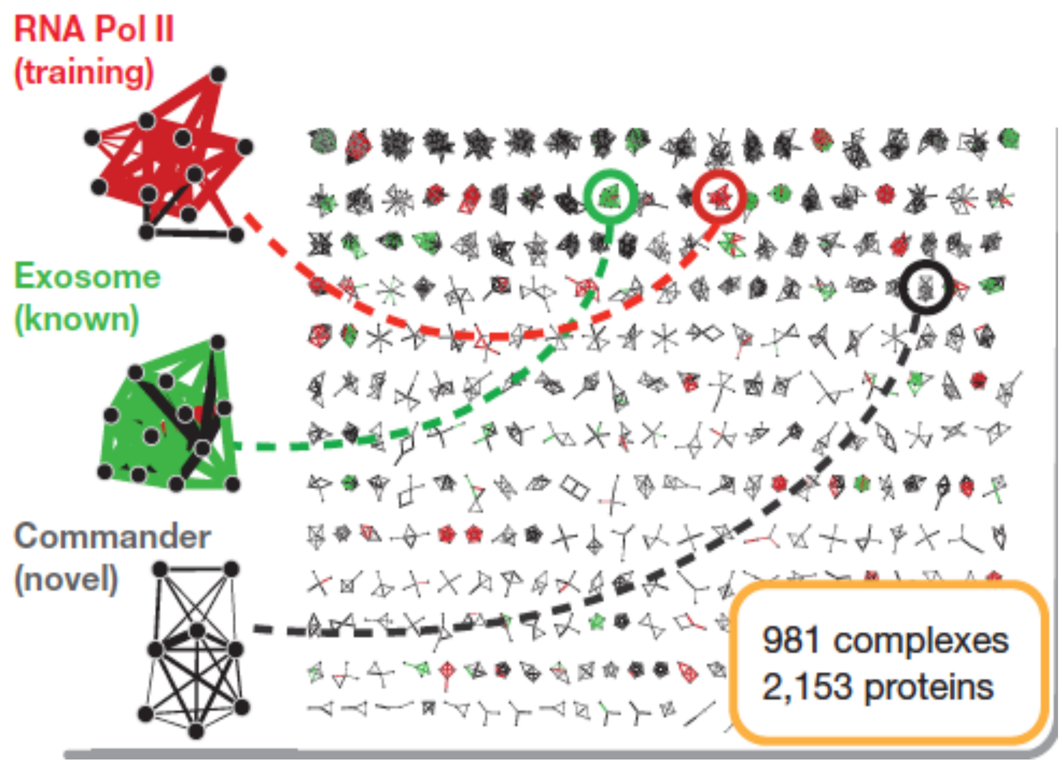


Giusti et al (2016)

Protein interaction networks as simplicial complexes

Protein interaction networks

- Nodes: proteins
- Simplices: protein complexes



Wan et al. Nature 2015

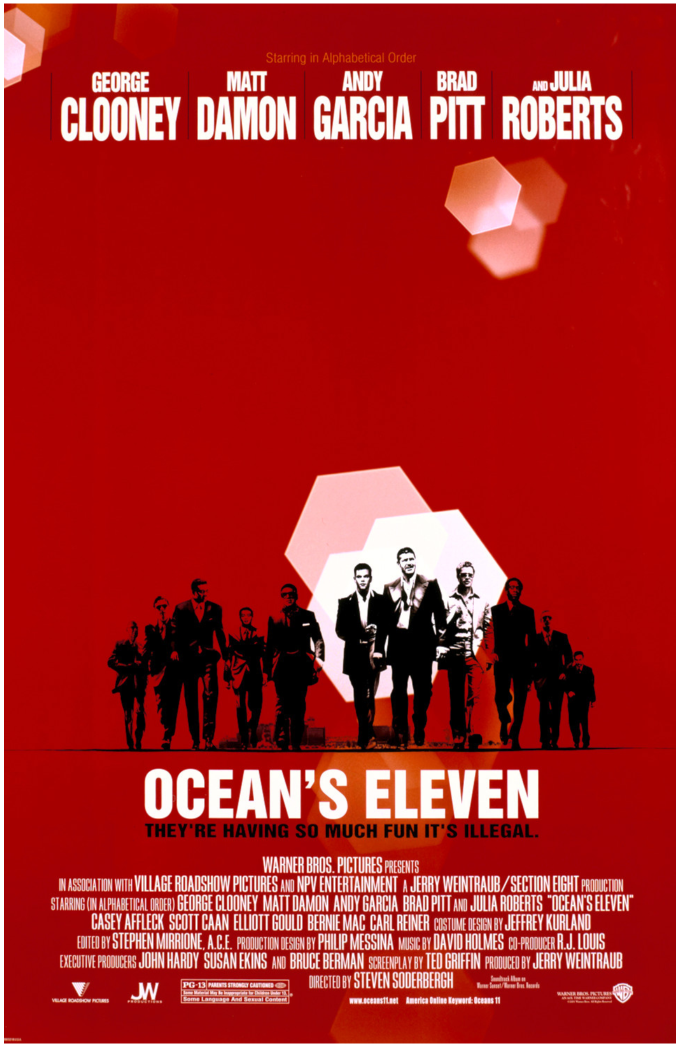
Collaboration networks as simplicial complexes

Actor collaboration networks

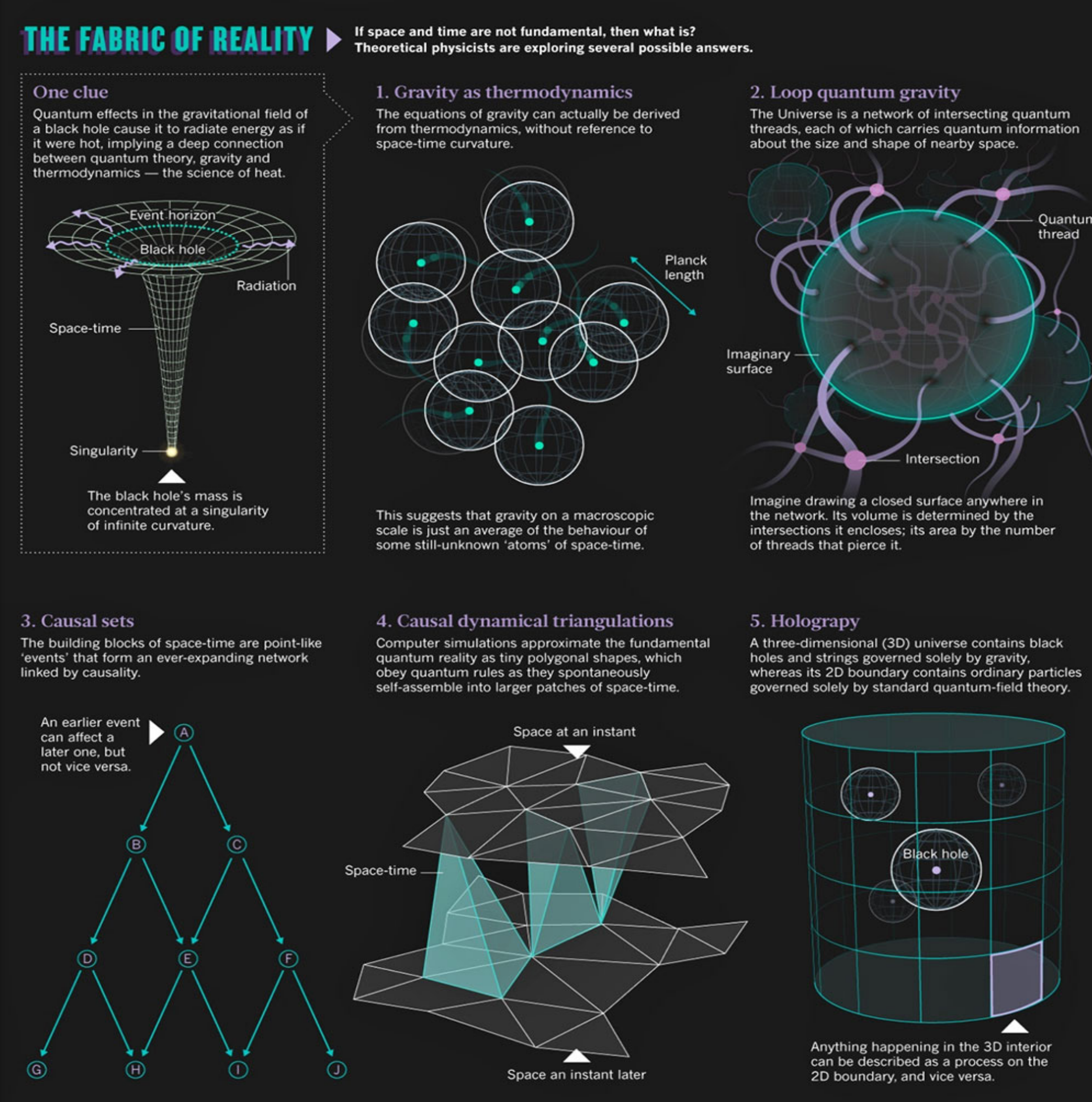
- Nodes: Actors
- Simplicies: Co-actors of a movie

Scientific collaboration networks

- Nodes: Scientists
- Simplicies: Co-authors



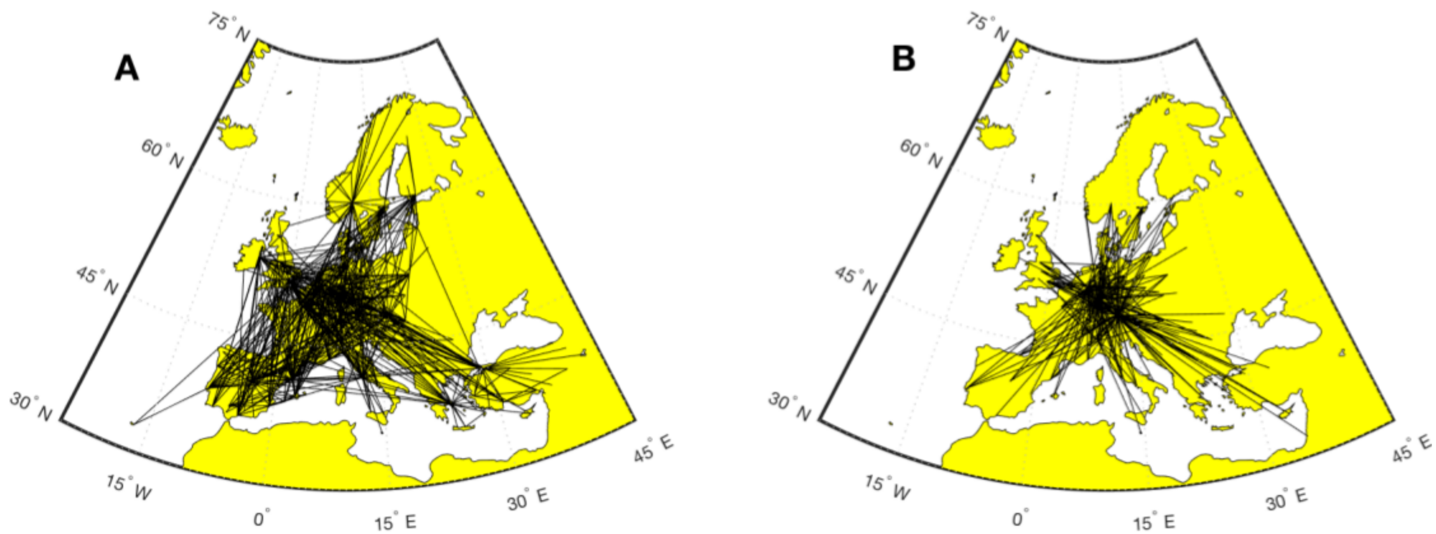
Quantum Spacetime



Quantum gravity approaches
networks and simplicial complexes

- Causal-Dynamical-Triangulations
- Tensor networks
- Group Field Theory
- Causal sets
- Loop Quantum Gravity
- Spin-Foams
- …

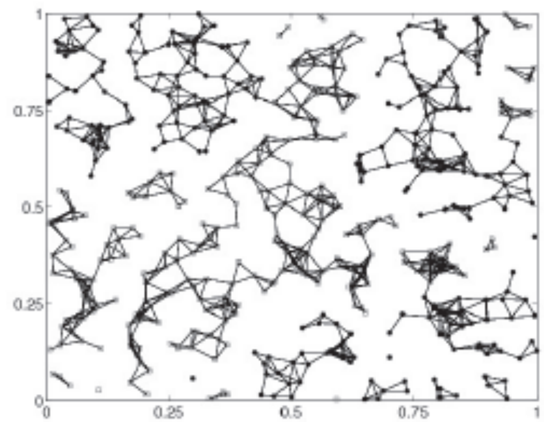
Spatial networks: infrastructures



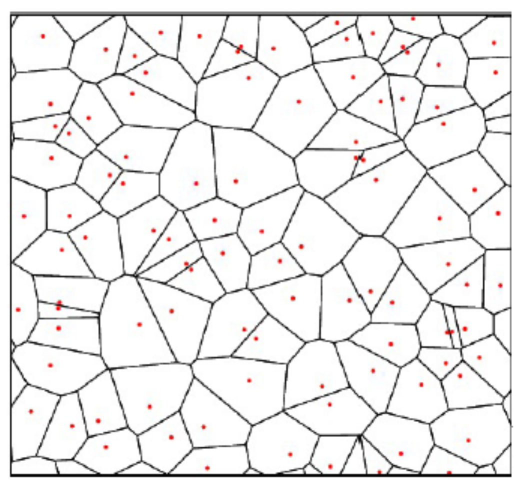
M. Barthelemy Phys. Rep. (2011)

Spatial networks: models and numerical methods

Random Geometric Networks



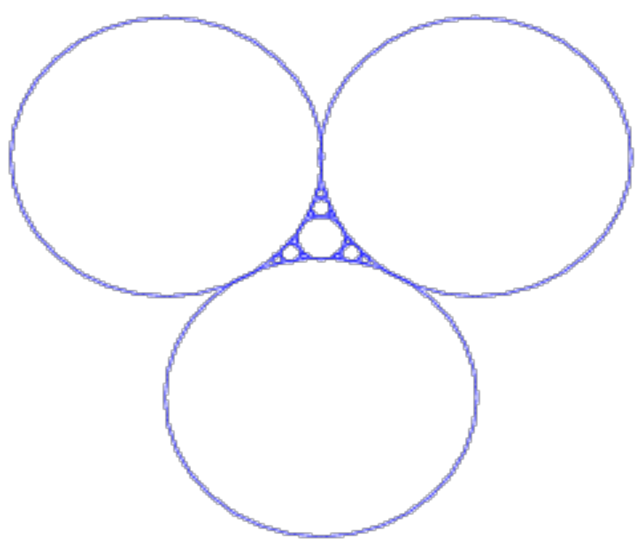
Tesselations

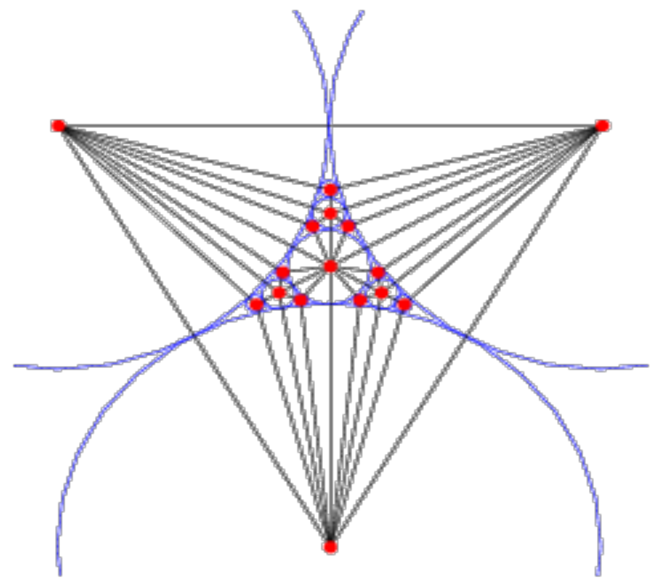


M. Barthelemy Phys. Rep. (2011)

Spatial networks: Apollonian network

- The Apollonian gasket from Apollonius of Perga c.262-c.190 B.C. is formed by triple of circle each one tangent to the other two.
- The Apollonian network is generated by placing a node to each center of the circles and connecting nodes corresponding to tangent circles





The Apollonian network is scale-free with exponent $\gamma = 1 + \frac{\ln 3}{\ln 2} \approx 2.585$

Scale-free networks

Technological networks
Internet
World-Wide Web

Biological networks
Metabolic networks,
Protein-interaction networks,
Transcription networks

Transportation networks
Airport networks

Social networks
Collaboration networks
Citation networks
Facebook

Economical networks
Networks of shareholders
The World Trade Web

Growth by uniform attachment of links

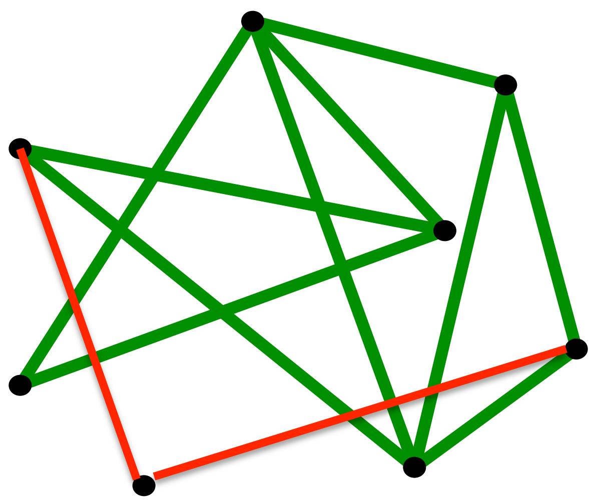
(1) GROWTH :

At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

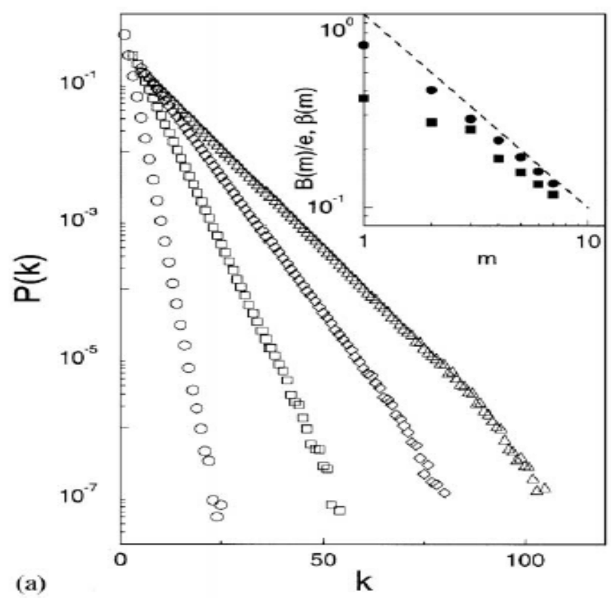
(2) UNIFORM ATTACHMENT :

The probability Π_i that a new node will be connected to node *i is uniform*

$$\Pi_i = \frac{1}{N}$$



Exponential



Barabási & Albert, Physica A (1999)

BA scale-free model

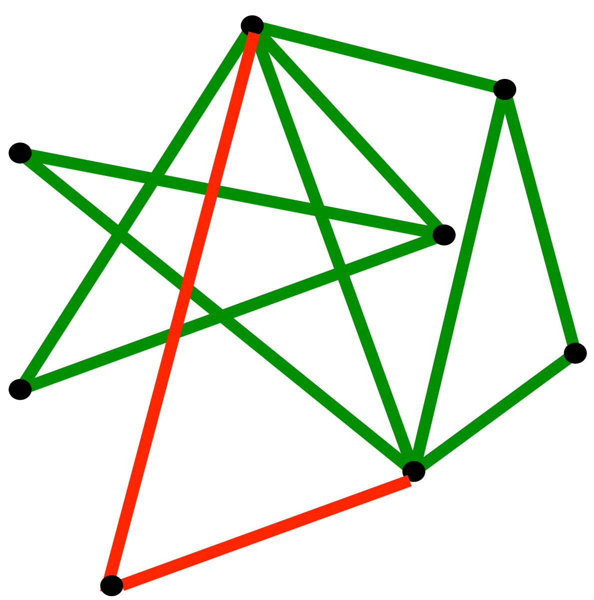
(1) GROWTH :

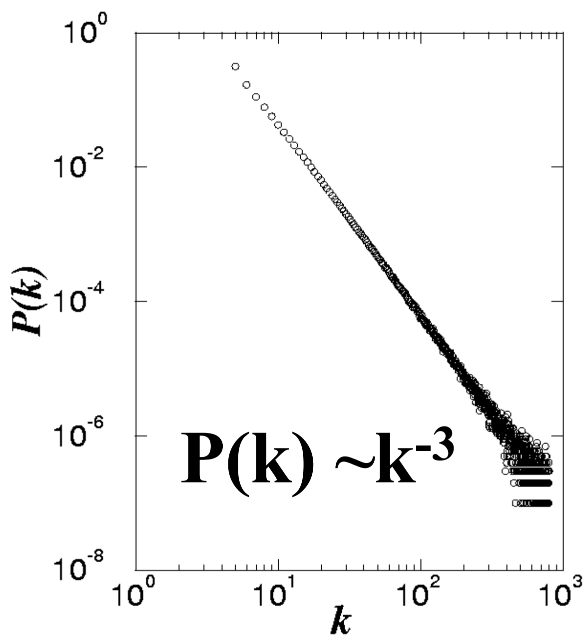
At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

(2) PREFERENTIAL ATTACHMENT :

The probability $\Pi(k_i)$ that a new node will be connected to node *i* depends on the connectivity k_i of that node

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$





Barabási et al. Science (1999)

The Bianconi-Barabasi model

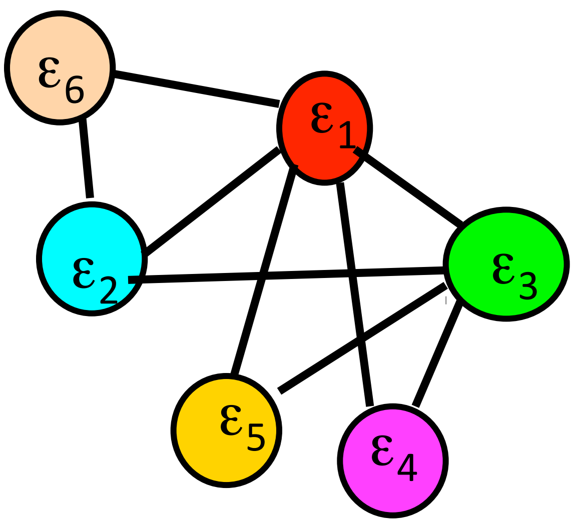
Growth:

–At each time a new node and m links are added to the network.

–To each node i we assign a energy ε_i from a $g(\varepsilon)$ distribution

Preferential attachment towards high degree low energy nodes:

–Each node connects to the rest of the network by m links attached preferentially to well connected, low energy nodes.

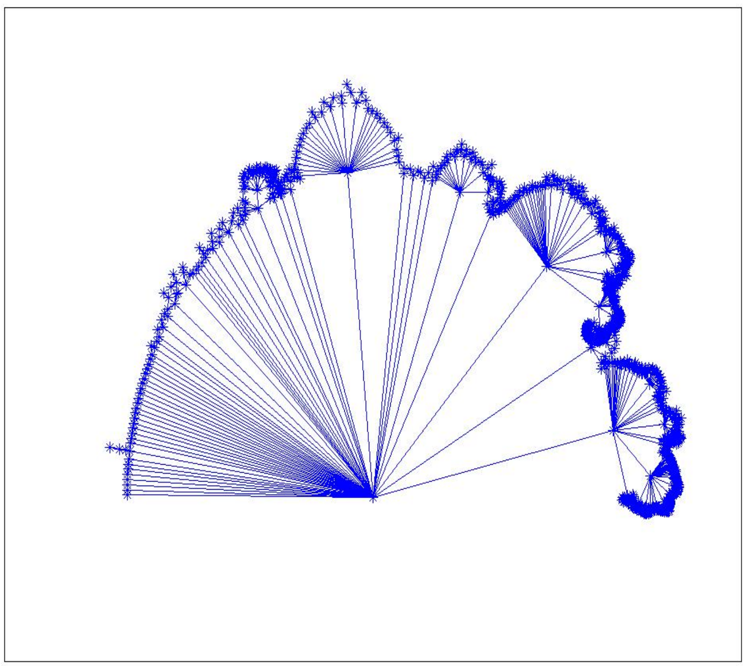


$$\Pi_i = \frac{e^{-\beta\varepsilon_i} k_i}{\sum_j e^{-\beta\varepsilon_j} k_j}$$

Bose-Einstein condensation in complex networks

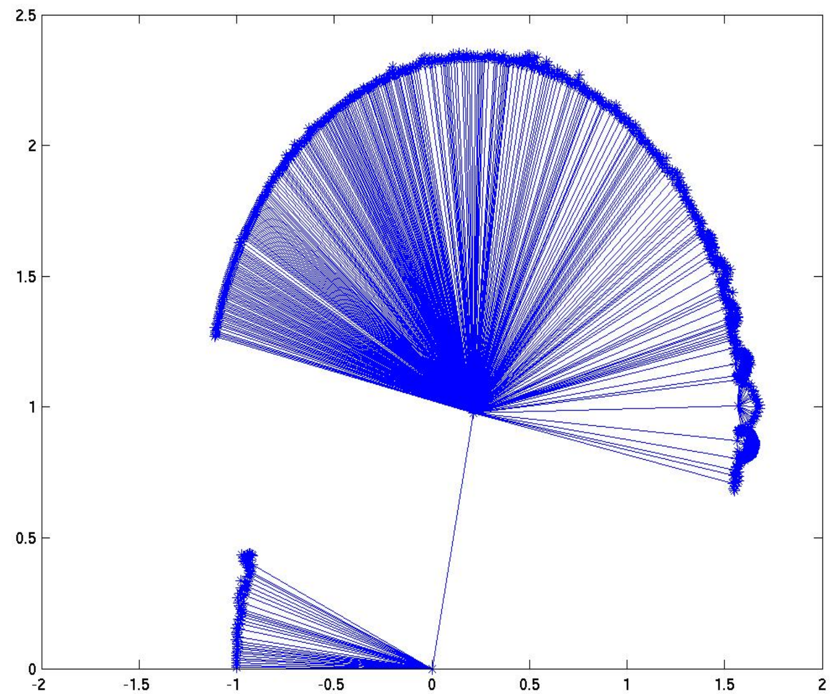
Scale-Free Fit-get-rich Phase

$$\beta < \beta_C$$



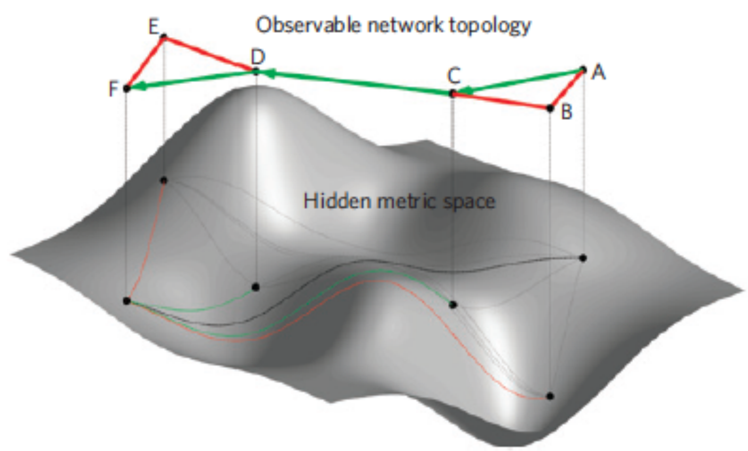
Bose-Einstein condensate Phase

$$\beta > \beta_C$$



G. Bianconi, A.-L. Barabási 2001

The hidden metric of complex networks



Boguna, Krioukov, Claffy
Nature Physics (2008)

It is believed that most complex networks have
an hidden metric
such that the nodes close in the hidden metric
are more likely to be linked to each other.

Hyperbolic geometry of complex networks could contribute to improve routing algorithms

This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE INFOCOM 2007 proceedings.

Geographic Routing Using Hyperbolic Space

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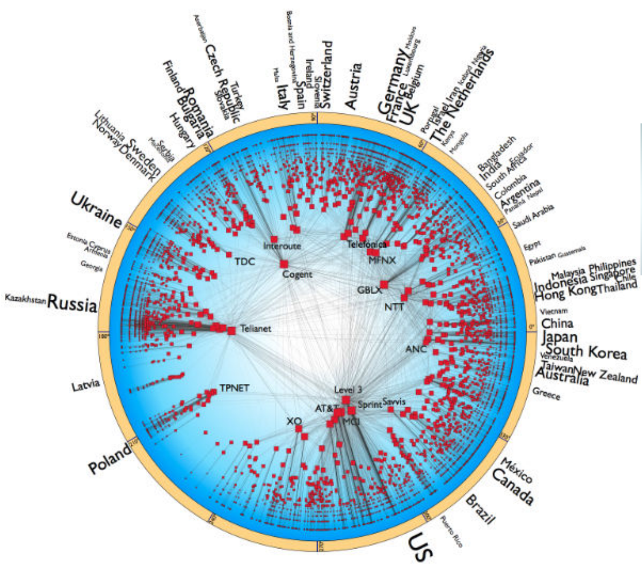
*Supported by an NSF Mathematical Sciences Postdoctoral Research Fellowship. Portions of this work were completed while the author was a postdoctoral fellow at UC Berkeley.



nature COMMUNICATIONS

ARTICLE

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DOI: 10.1038/ncomms1063

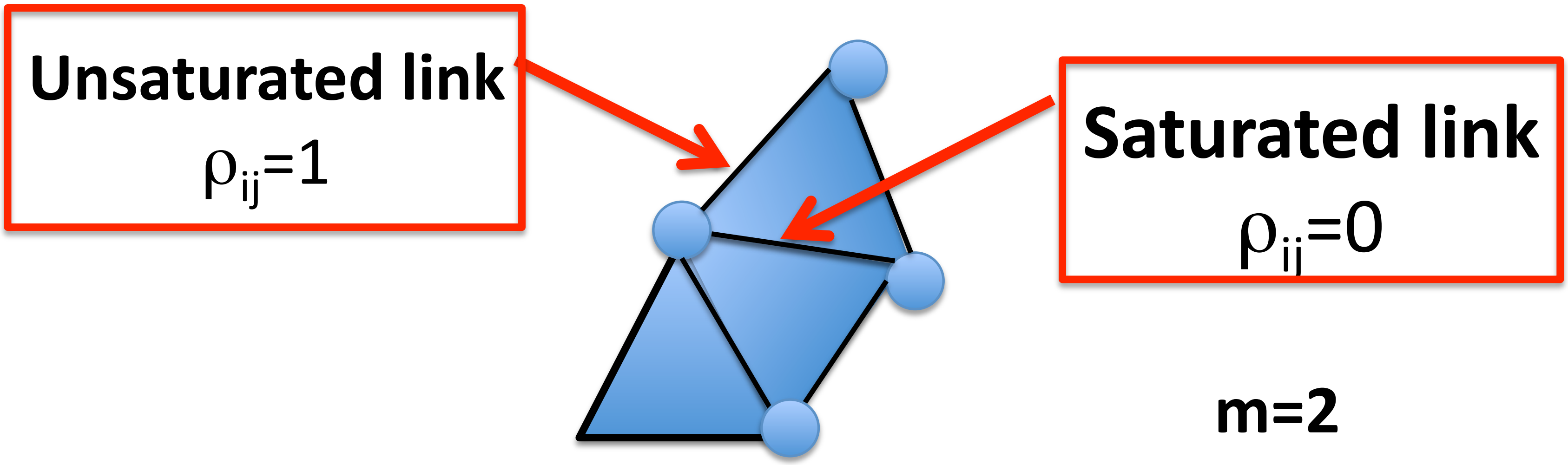
Sustaining the Internet with hyperbolic mapping

Marián Boguñá[1], Fragkiskos Papadopoulos[2] & Dmitri Krioukov[3]

Emergent geometry

In the framework of emergent geometry
networks with hidden geometry
are generated
by equilibrium or non-equilibrium dynamics
that makes no use of the
hidden geometry

Saturated and Unsaturated links



- $\rho_{ij}=1$ if the link is unsaturated, i.e. less than m triangles are incident on it
- $\rho_{ij}=0$ if the link is saturated, i.e. the number of incident triangles is given by m

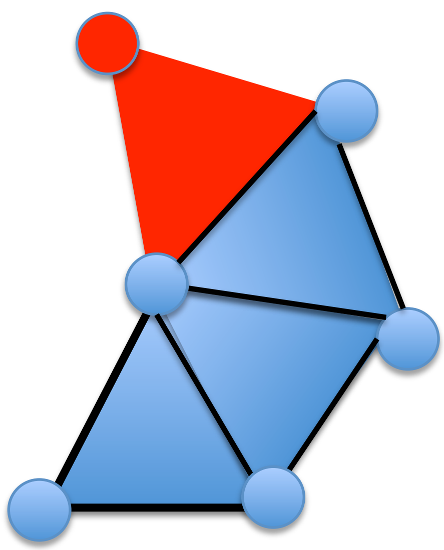
Process (a)

We choose a link (i,j) with probability

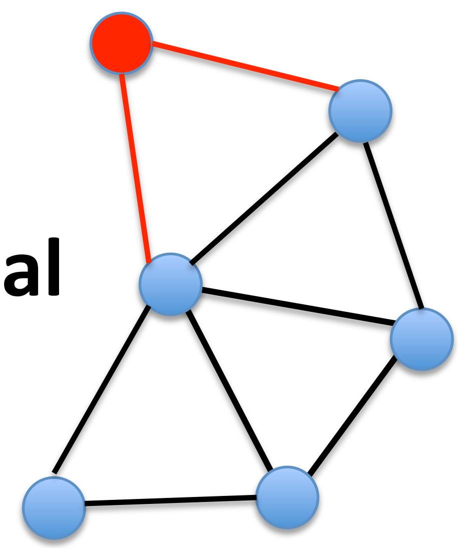
$$\Pi_{i,j}^{[1]} = \frac{a_{ij}\rho_{ij}}{\sum_{r,s} a_{rs}\rho_{rs}}$$

and glue a new triangle the link

Growing Simplicial Complex



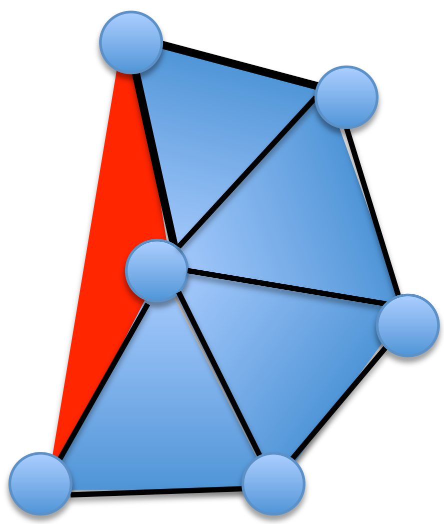
Growing Geometrical Network



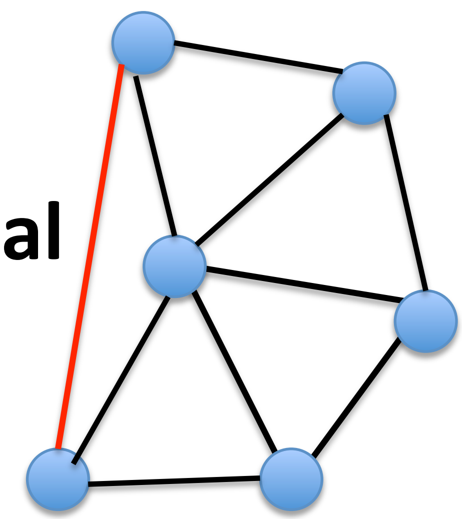
Process (b)

We choose a two adjacent unsaturated links and we add the link between the nodes at distance 2 and all triangles that this link closes as long that this is allowed.

Growing Simplicial Complex



Growing Geometrical Network



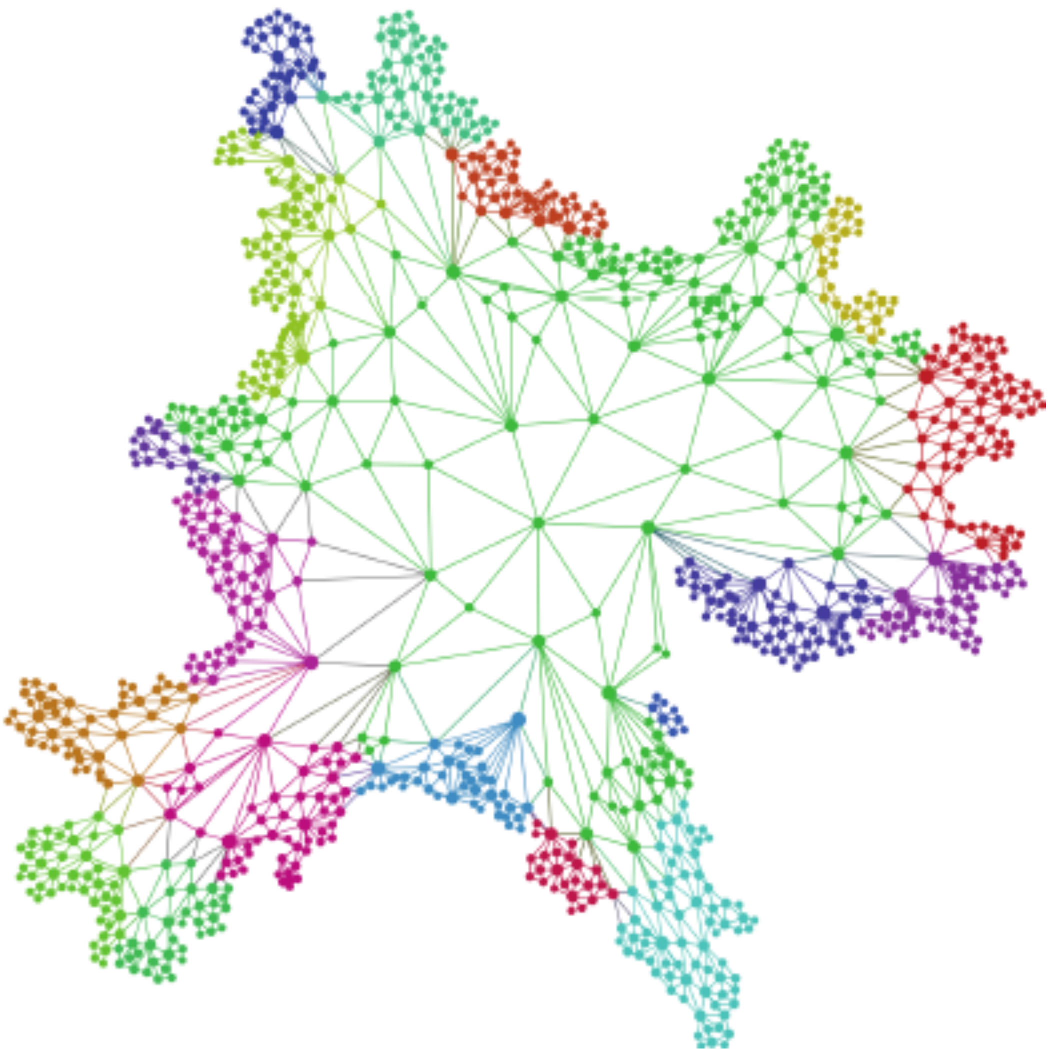
The model

Starting from an initial triangle,

At each time

- process (a) takes place and
- process (b) takes place with probability p.

Discrete Manifolds



A discrete manifold of dimension d=2 is a simplicial complex formed by triangles such that every link is incident to at most two triangles.

Therefore the emergent network geometry for our model with m=2 is a discrete 2d manifold.

Scale-free networks

In the case $m = \infty$

a scale-free network with high clustering, significant community structure, finite spectral dimension is generated.

Planar for p=0.

Degree distribution

- For *m=2* and *p=0* we can calculate the degree distribution given by

$$P(k) = \frac{1}{2}\left(\frac{2}{3}\right)^{k-1}$$

- For $m = \infty$ and *p=0* we can calculate the degree distribution given by

$$P(k) = \frac{12}{(k+2)(k+1)k}$$

Combinatorial Curvature

The combinatorial curvature for a node i of a planar triangulation is

$$R_i = 1 - \frac{k_i}{2} + \frac{T_i}{3}$$

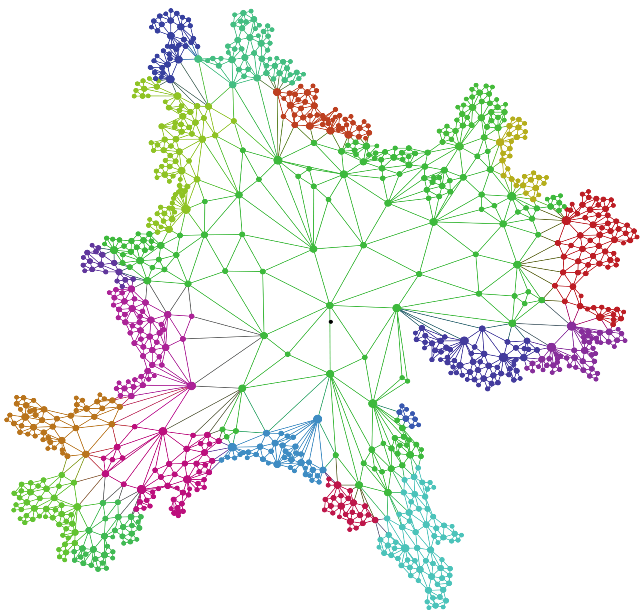
- k_i is the degree of the node i,
- T_i is the number of triangles to which node i belongs

For a node in the bulk $$R_i = \frac{6 - k_i}{6}$$

For a node at the surface $$R_i = \frac{4 - k_i}{6}$$

Emergent network geometry and curvature distribution

$m = 2 \quad p = 0.9$

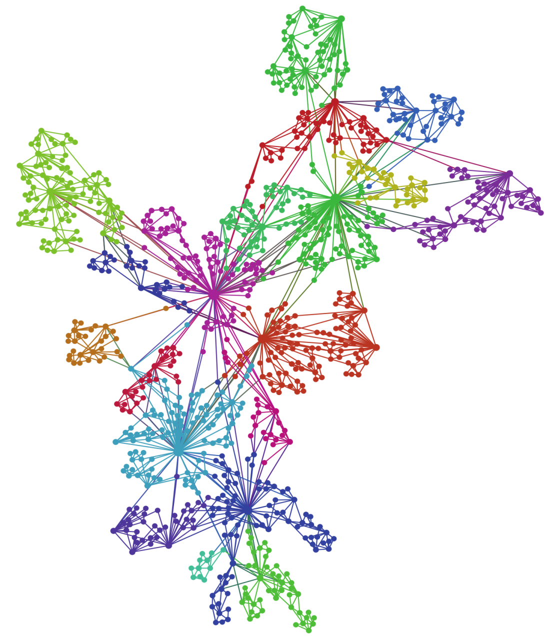


Exponential network

$m = 4 \quad p = 0.9$

Broad degree distribution

$m = \infty \quad p = 0$

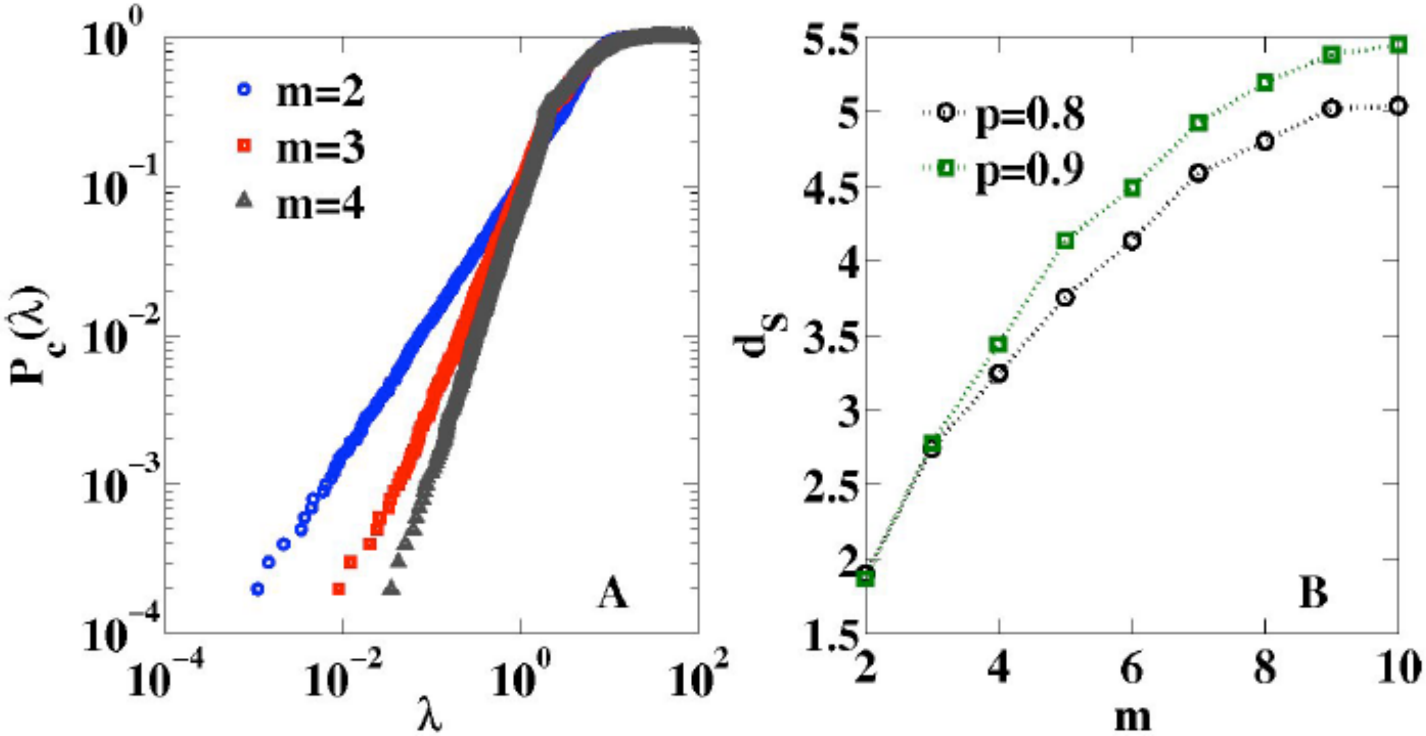


Scale-free network

$$\langle R \rangle = \frac{1}{N}$$
$$\langle R^2 \rangle < \infty$$

$$\langle R \rangle = \frac{1}{N}$$
$$\langle R^2 \rangle = \infty$$

Finite spectral dimension



$$L_{ij} = k_i \, \delta_{ij} - a_{ij}$$

$$\rho(\lambda) \approx \lambda^{(d/2-1)}$$

$$P_c(\lambda) \approx \lambda^{d/2}$$

Properties of emergent network geometries

- **Small world**
- **Finite clustering**
- **High modularity**
- **Finite spectral dimension**

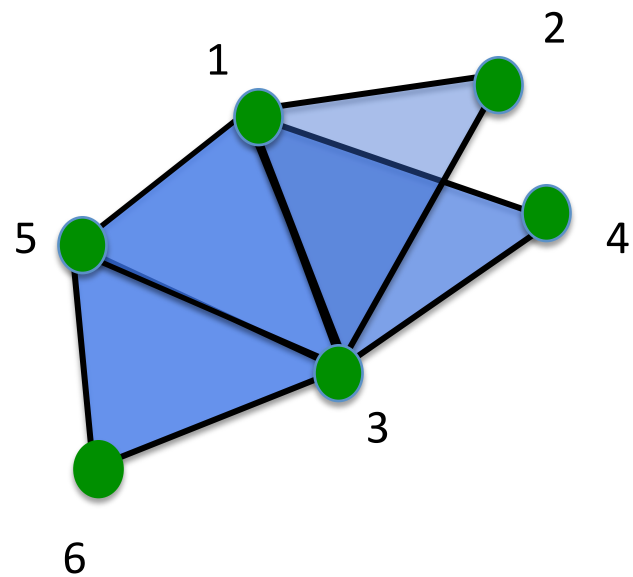
Which are properties of many real network datasets.

Properties of real datasets

Datasets	N	L	$\langle \ell \rangle$	C	M	d_S
1L8W (protein)	294	1608	5.09	0.52	0.643	1.95
1PHP (protein)	219	1095	4.31	0.54	0.638	2.02
1AOP chain A (protein)	265	1363	4.31	0.53	0.644	2.01
1AOP chain B (protein)	390	2100	4.94	0.54	0.685	2.03
Brain-(coactivation) [45]	638	18625	2.21	0.384	0.426	4.25
Internet [46]	22963	48436	3.8	0.35	0.652	5.083
Power-grid[38]	4941	6594	19	0.11	0.933	2.01
Add Health (school61)[47]	1743	4419	6	0.22	0.741	2.97

Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a δ-face μ in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face μ.

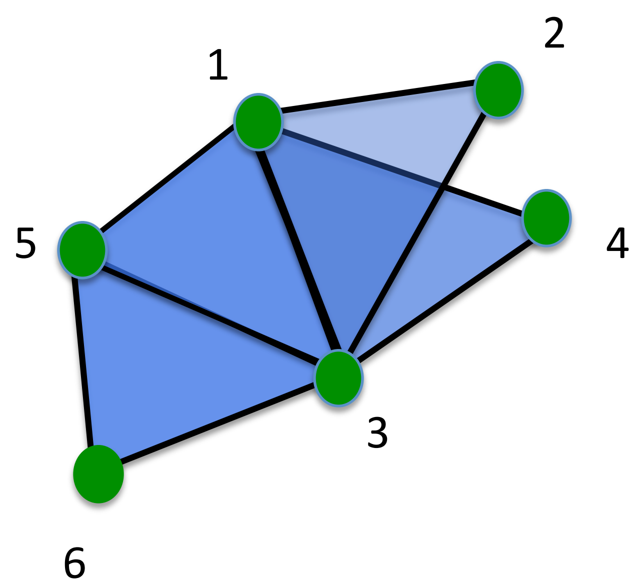


$k_{2,0}(\mu)$ Number of triangles incident to the node μ

$k_{2,1}(\mu)$ Number of triangle incident to the link μ

Generalized degree

The generalized degree $k_{d,\delta}(\mu)$ of a δ-face μ in a d-dimensional simplicial complex is given by the number of d-dimensional simplices incident to the δ-face μ.



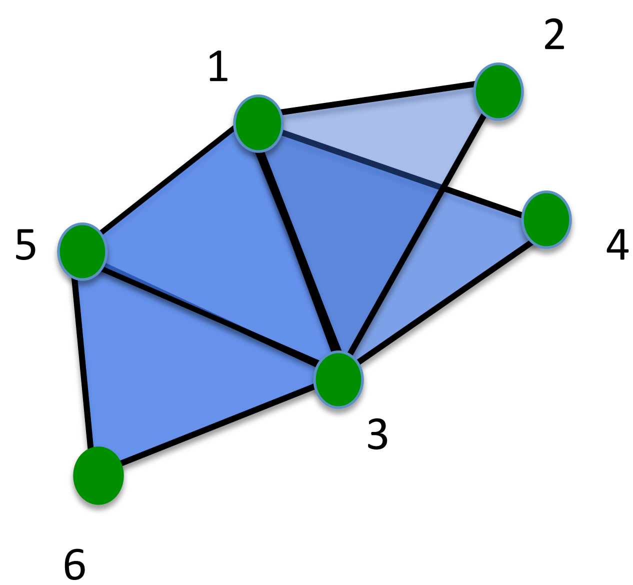
i	$k_{2,0}(i)$
1	3
2	1
3	4
4	1
5	2
6	1

(i,j)	$k_{2,1}(i,j)$
(1,2)	1
(1,3)	3
(1,4)	1
(1,5)	1
(2,3)	1
(3,4)	1
(3,5)	2
(3,6)	1
(5,6)	1

Incidence number

To each (d-1)-face μ we associate the incidence number

$$n_{\mu} = k_{d,d-1}(\mu) - 1$$

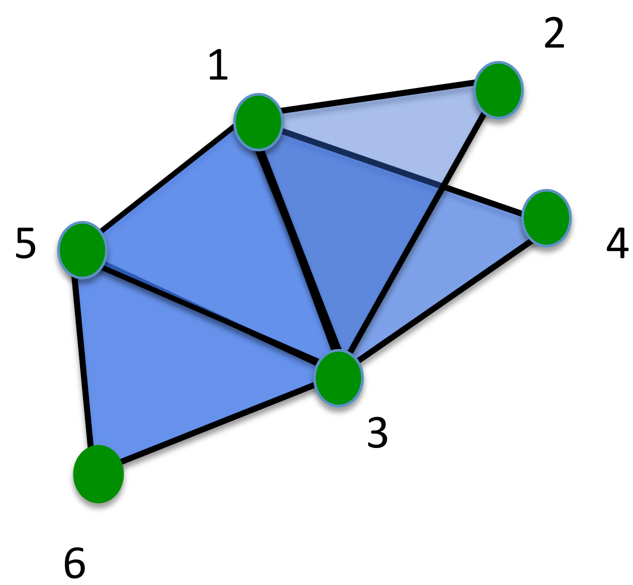


(i,j)	$n_{(i,j)}$
(1,2)	0
(1,3)	2
(1,4)	0
(1,5)	0
(2,3)	0
(3,4)	0
(3,5)	1
(3,6)	0
(5,6)	0

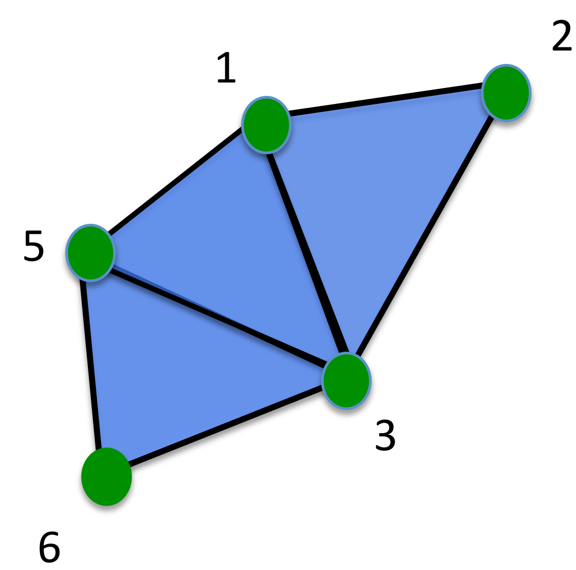
Manifolds

If n_μ takes only values $n_\mu=0,1$ each (d-1)-face is incident at most to two d-dimensional simplices.

In this case the simplicial complex is a discrete manifold.



NOT A MANIFOLD



MANIFOLD

Network Geometry with Flavor

Starting from a single d-dimensional simplex

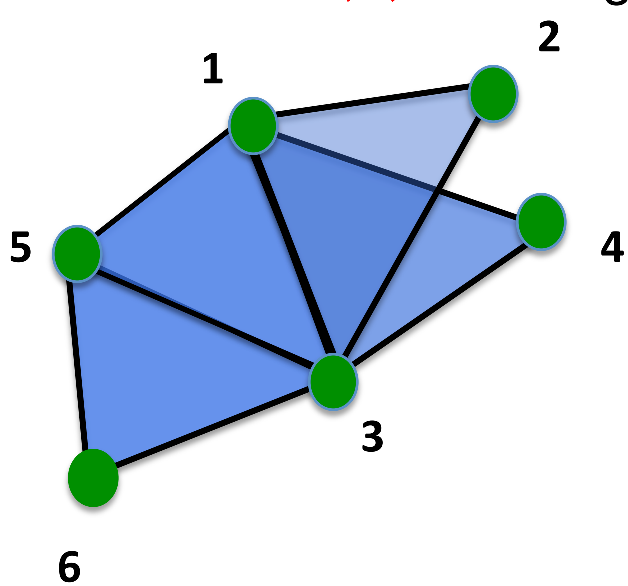
(1) GROWTH :

At every timestep we add a new node d simplex

(formed by one new node and an existing (d-1)-face).

(2) ATTACHMENT:

The probability that a new node will be connected to a face α depends on the flavor s=-1,0,1 and is given by



$$\Pi_{\mu}^{[s]} = \frac{1 + s n_{\mu}}{\sum_{\mu'} (1 + s n_{\mu'})}$$

Bianconi & Rahmede (2016)

Attachment probability

$$\Pi_{\mu}^{[s]} = \frac{(1+s\,n_{\mu})}{\sum_{\mu'\in Q_{d,d-1}}(1+s\,n_{\mu'})} = \begin{cases} \dfrac{(1-n_{\mu})}{Z^{[-1]}}, & s=-1 \\ \dfrac{1}{Z^{[0]}}, & s=0 \\ \dfrac{k_{d,d-1}(\mu)}{Z^{[1]}}, & s=1 \end{cases}$$

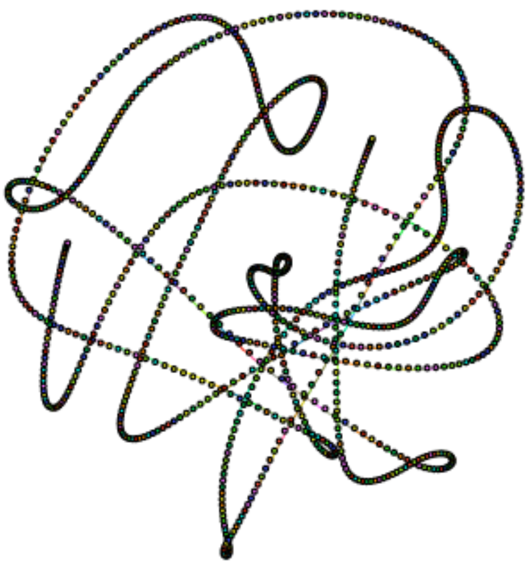
s=-1 Manifold $n_{\mu}=0,1$

s=0 Uniform attachment $n_{\mu}=0,1,2,3,4\ldots$

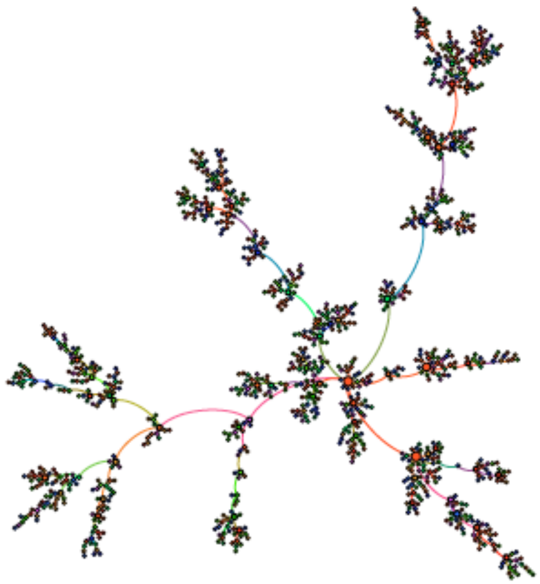
s=1 Preferential attachment $n_{\mu}=0,1,2,3,4\ldots$

Dimension d=1

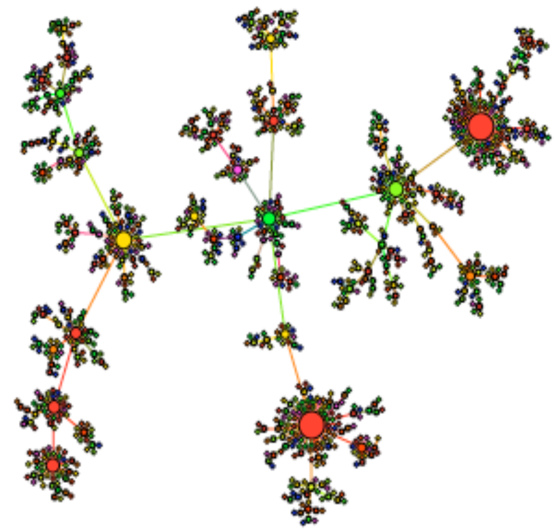
Manifold | **Uniform attachment** | **Preferential attachment**



Chain



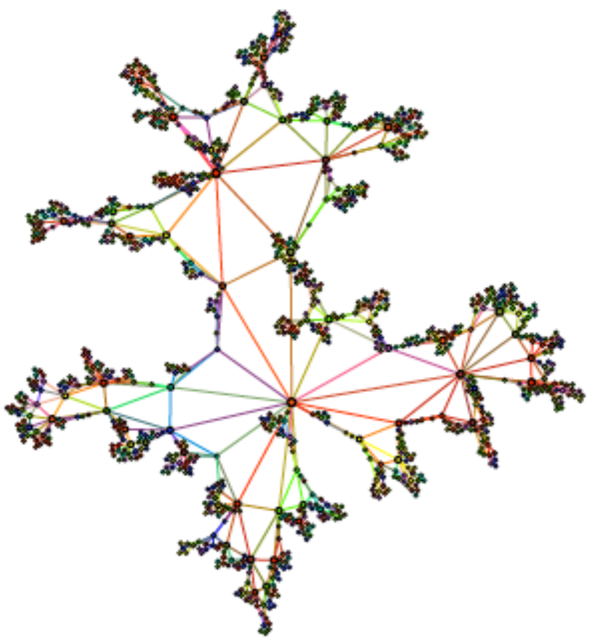
Exponential



Scale-free BA model

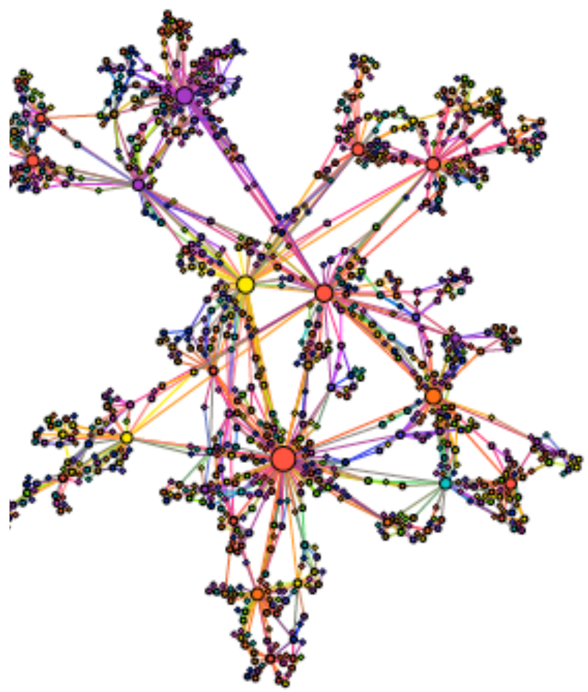
Dimension d=2

Manifold



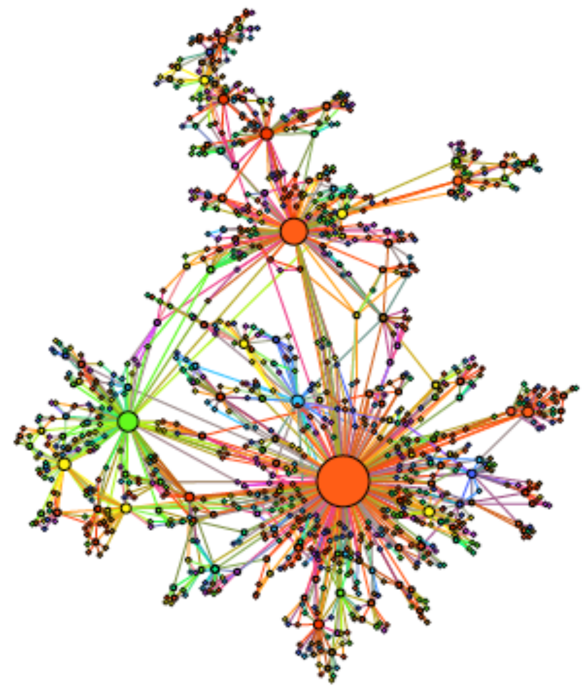
Exponential

Uniform attachment



Scale-free

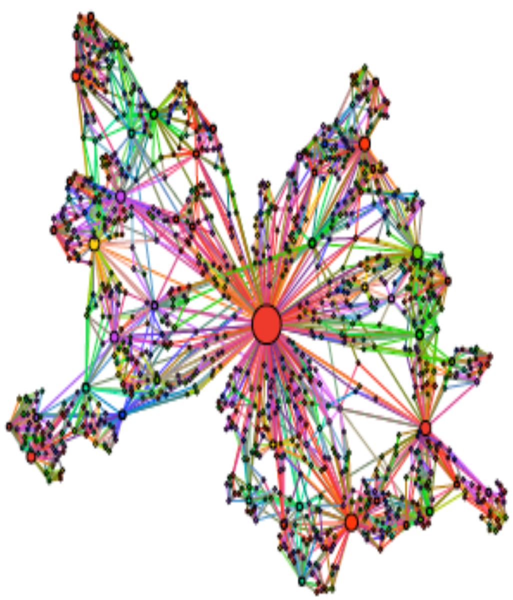
Preferential attachment



Scale-free

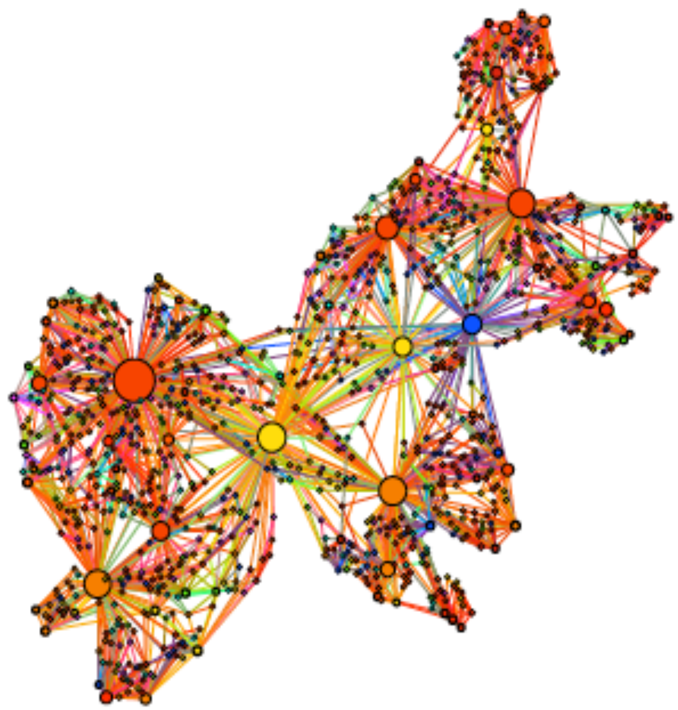
Dimension d=3

Manifold



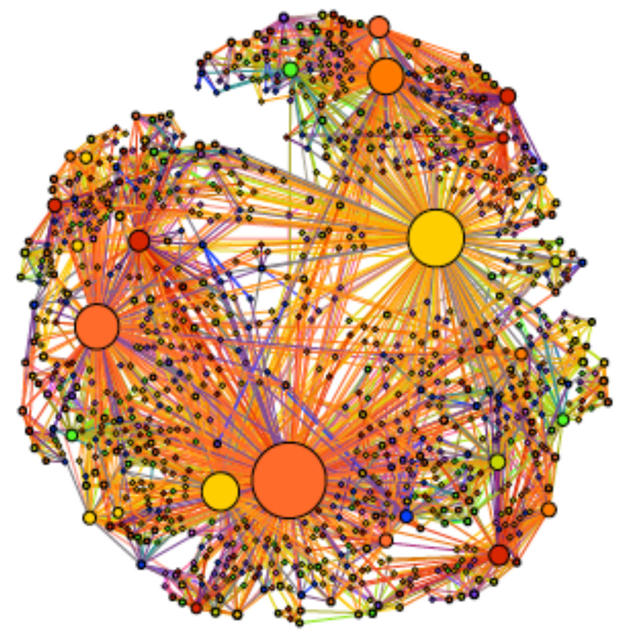
Scale-free

Uniform attachment



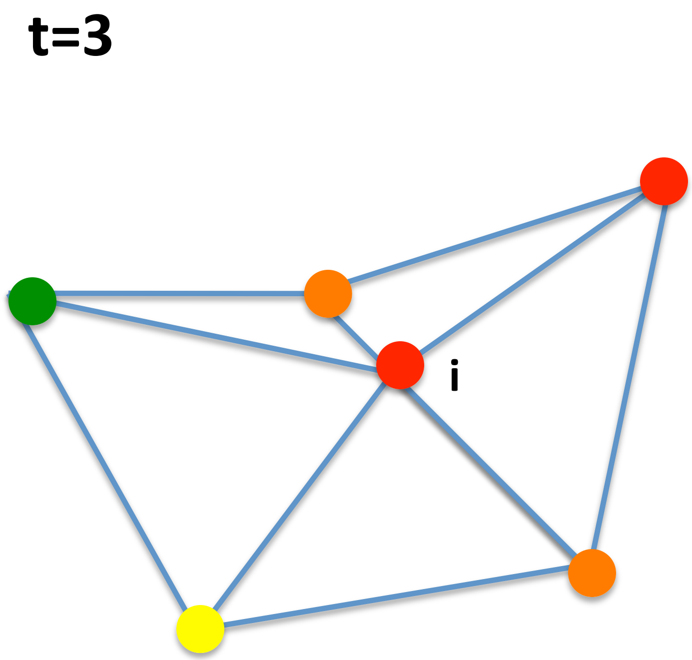
Scale-free

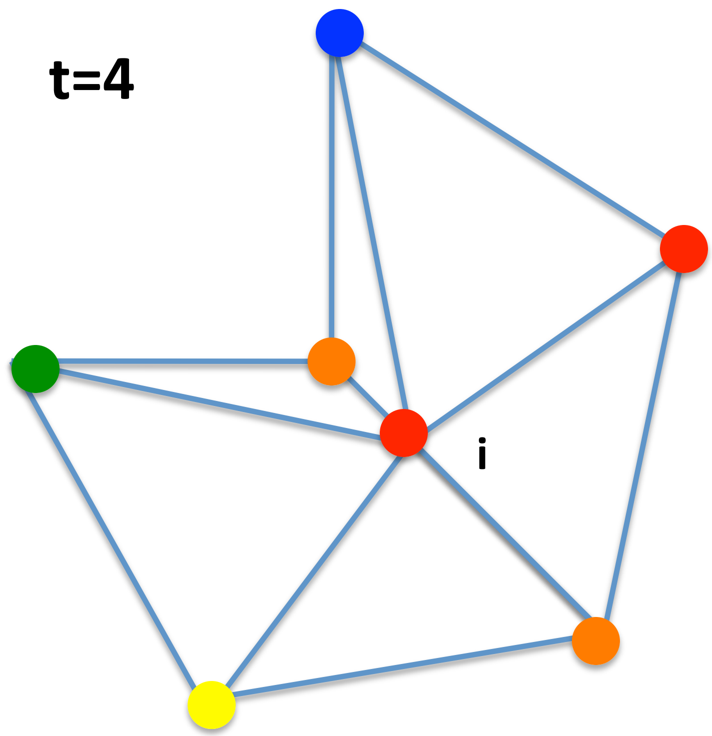
Preferential attachment



Scale-free

Effective preferential attachment in d=3





Node i has generalized degree 3

Node i has generalized degree 4

Node i is incident to 5 unsaturated faces

Node i is incident to 6 unsaturated faces

Degree distribution

For d+s=1

$$P_d(k) = \left(\frac{d}{d+1}\right)^{k-d} \frac{1}{d+1}$$

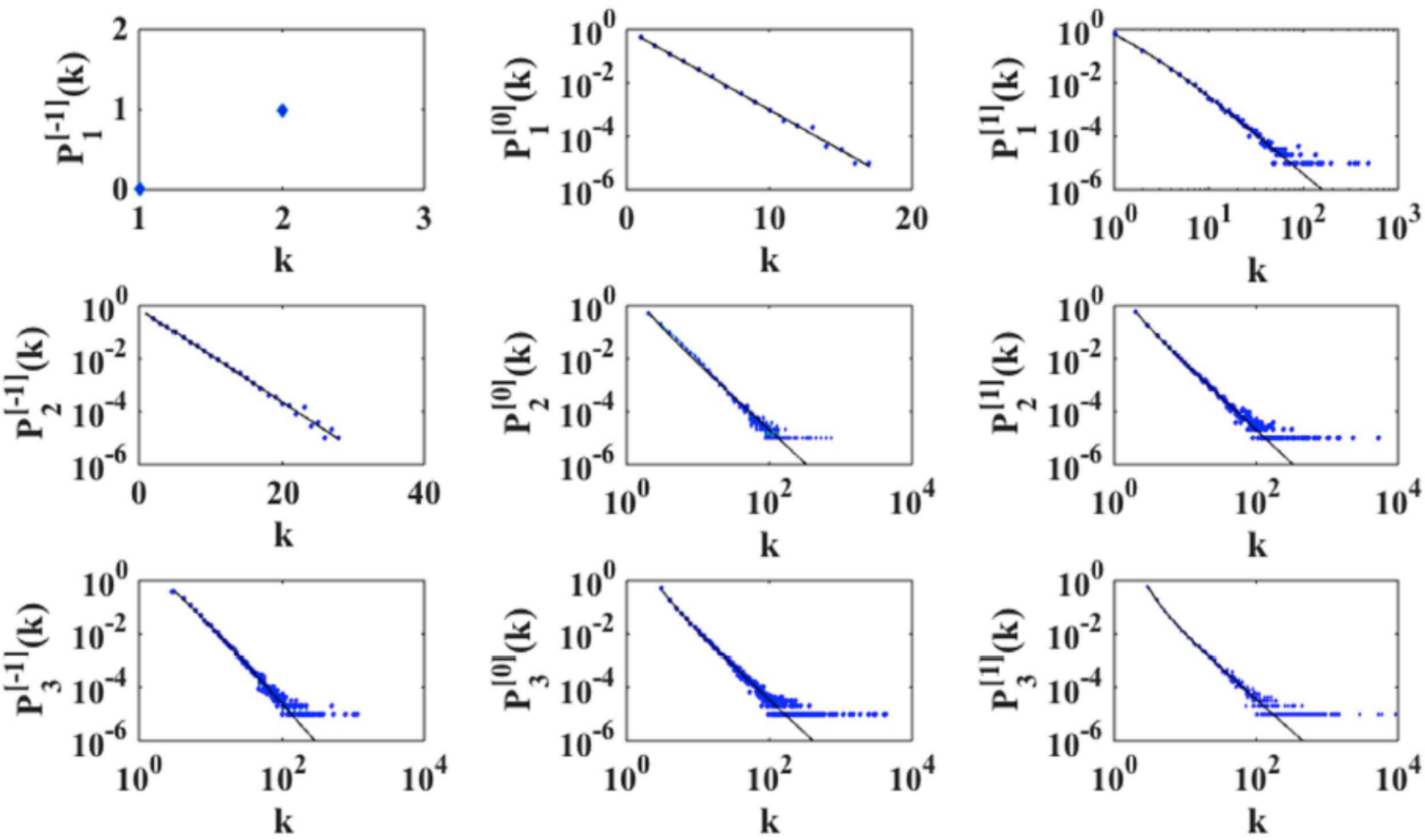
For d+s>1

$$P_d(k) = \frac{d+s}{2d+s} \frac{\Gamma(1+(2s+s)(d+s-1))}{\Gamma(d/(d+s-1))} \frac{\Gamma(k-d+d/(d+s-1))}{\Gamma(k-d+(2d+s)(d+s-1))}$$

NGF are always scale-free for d>1-s

- For s=1 NGF are always scale free
- For s=0 and d>1 the NGF are scale-free
- For s=-1 and d>2 the NGF are scale-free

Degree distribution of NGF



Modularity
and
Clustering coefficient
of NGF

M	s = −1	s = 0	s = 1	C	s = −1	s = 0	s = 1
$d = 2$	0.97	0.94	0.90	$d = 2$	0.65	0.74	0.79
$d = 3$	0.91	0.85	0.80	$d = 3$	0.77	0.81	0.84

Master equation approach

A master equation can be written

for the number of δ faces $N^{t}_{d,\delta}(k)$

that have generalized degree *k* at time *t*

$$N^{t+1}_{d,\delta}(k)=\begin{cases} N^{t}_{d,\delta}(k)+m_{d,\delta}(k-1)N^{t}_{d,\gamma}(k-1)-m_{d,\delta}(k)N^{t}_{d,\delta}(k) & \text{for } k\neq m \\ N^{t}_{d,\delta}(k)-m_{d,\delta}(k)N^{r}_{d,\delta}(k)+1 & \text{for } k=m \end{cases}$$

with

$$m_{d,\delta}(k)=\frac{(1-s)+(d+s-\delta-1)k}{(d+s)t}$$

indicating the probability that a δ face increases its generalized degree at time t

Generalized degree distributions

flavor	$s=-1$	$s=0$	$s=1$
$\delta = d-1$	Bimodal	Exponential	Power-law
$\delta = d-2$	Exponential	Power-law	Power-law
$\delta \leq d-3$	Power-law	Power-law	Power-law

The power-law

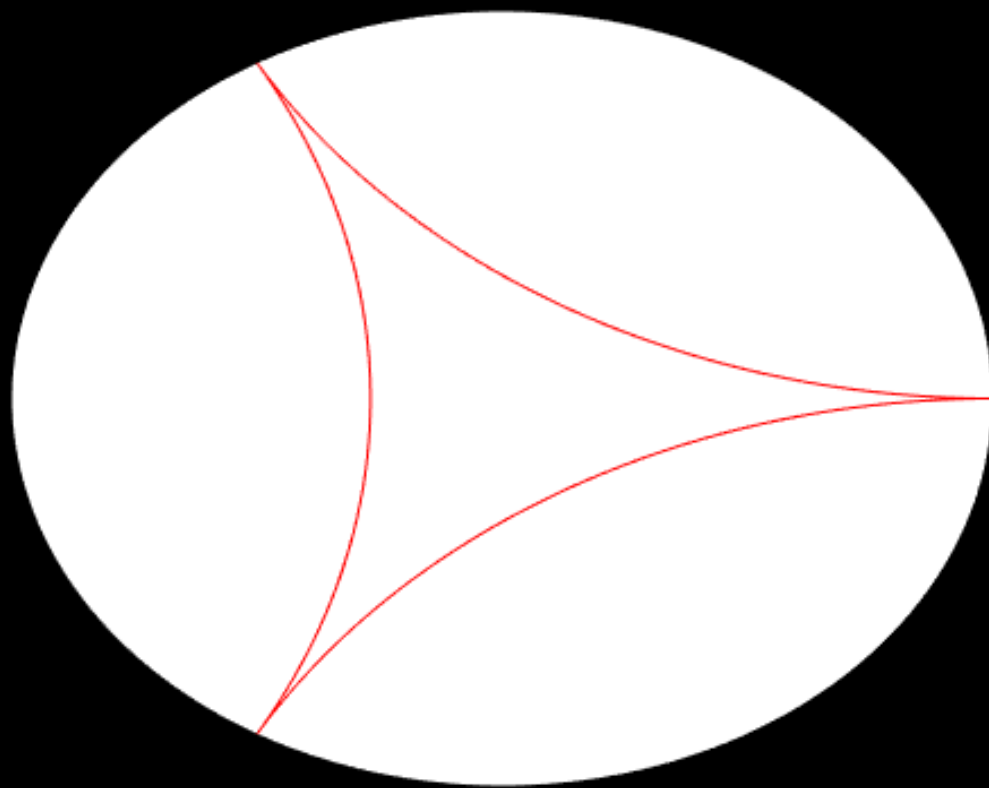
generalized degree distribution

are scale-free for

$$d \geq d_c^{[\delta,s]} = 2(\delta+1)+s$$

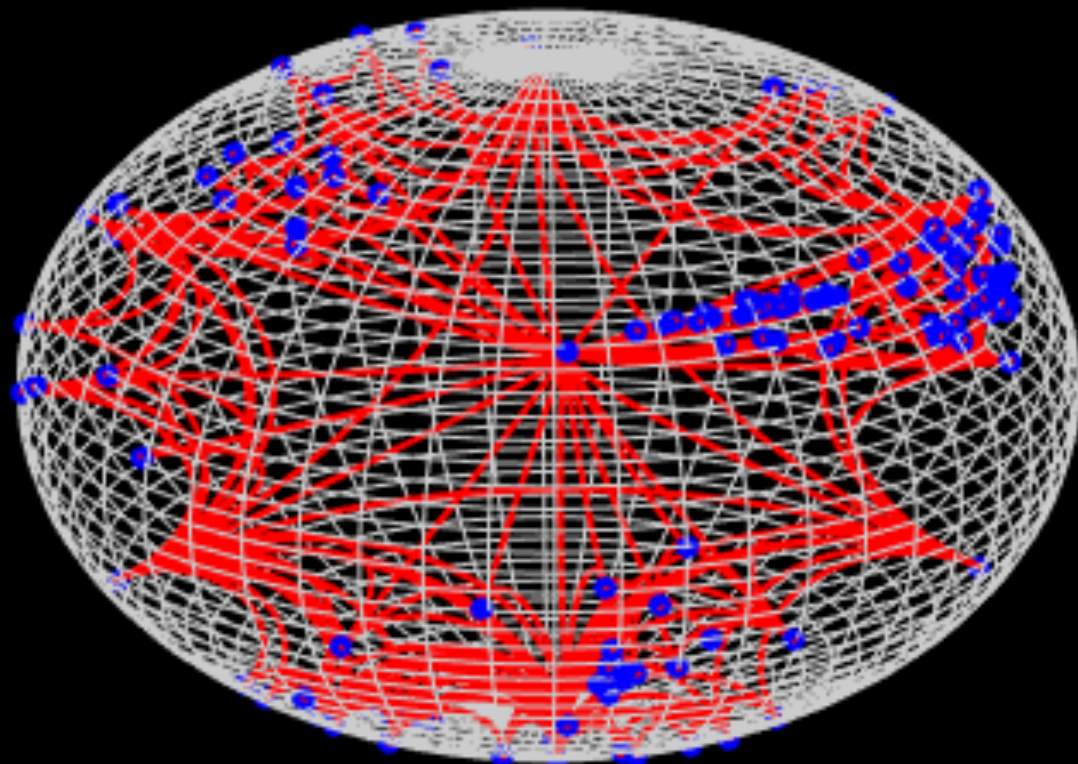
Emergent Hyperbolic geometry

The emergent hidden geometry is the hyperbolic H^d space
Here all the links have equal length



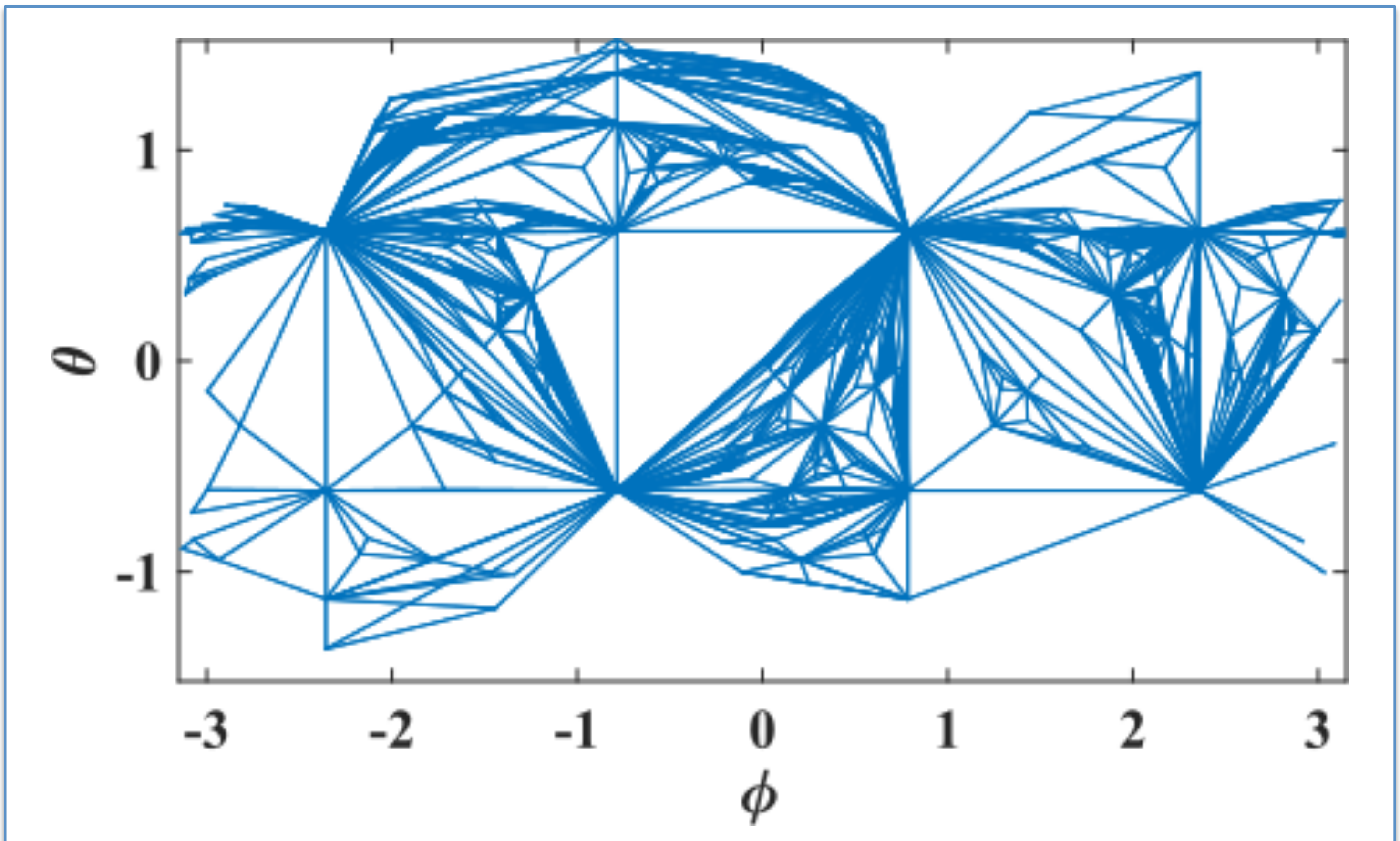
d=2

Emergent hyperbolic geometry



d=3

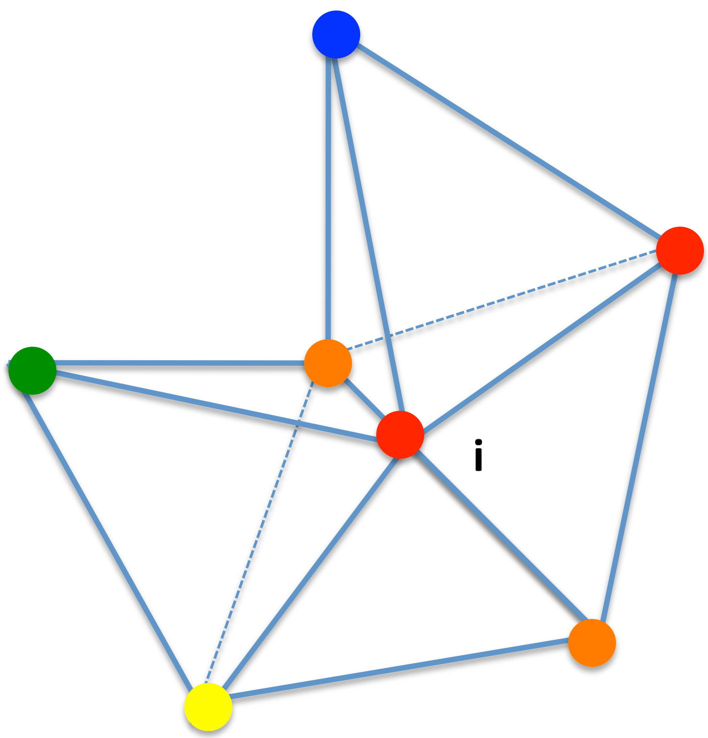
Connection with the Apollonian network



NGF and Apollonian networks

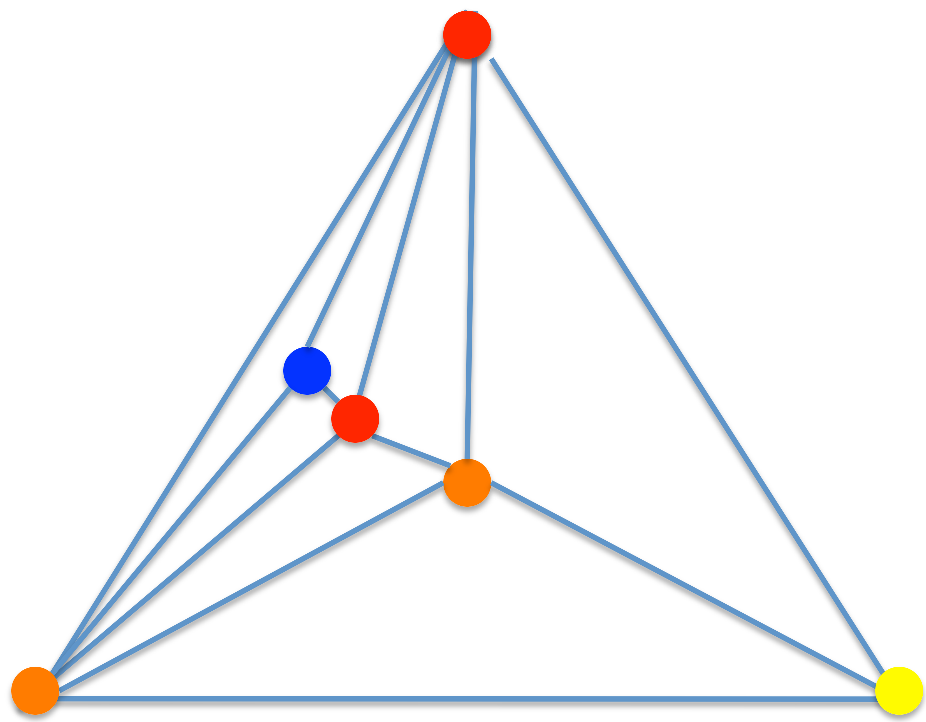
Network Geometry with Flavor(NGF)

s=-1 d=3

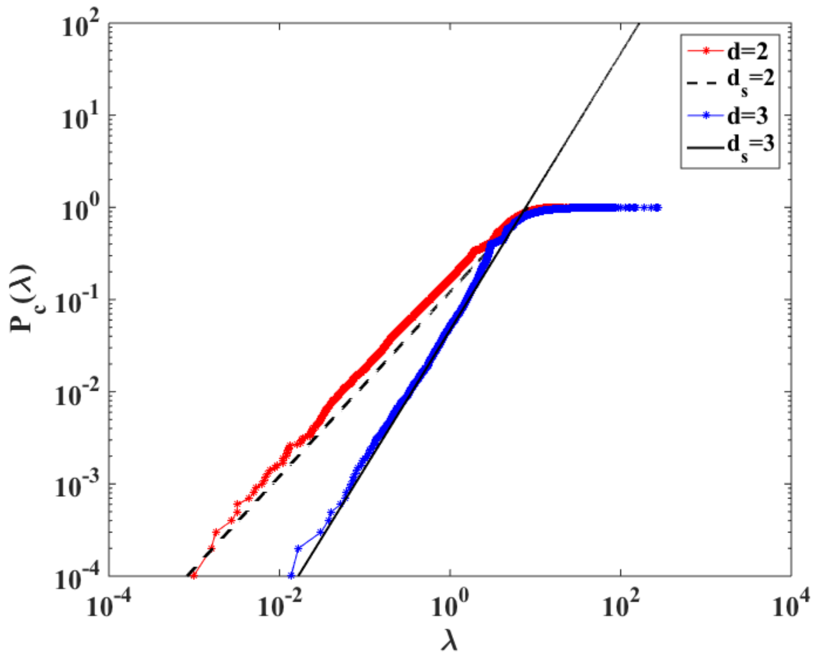


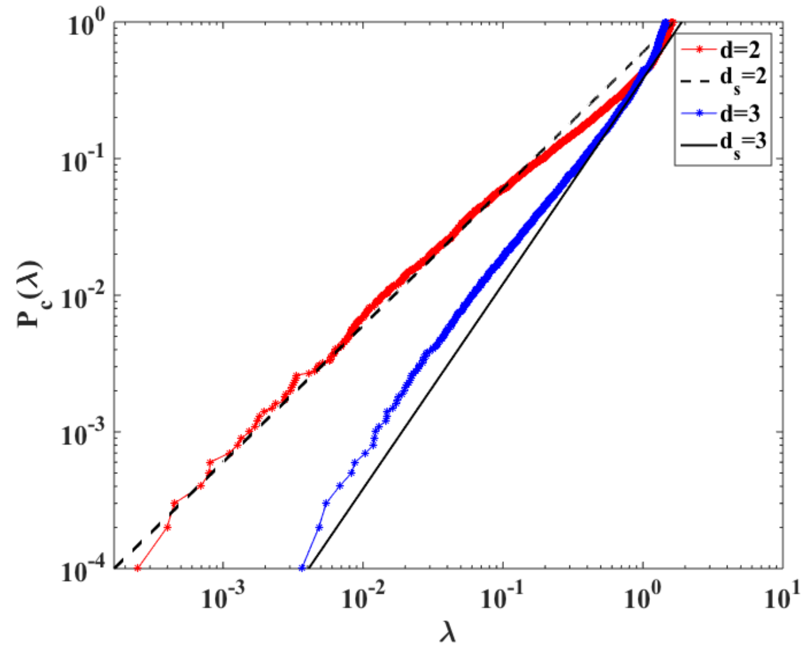
Apollonian random network

Planar projection of the NGF



Spectral dimensions of NGF with s=-1





$$L_{ij} = k_i \delta_{ij} - a_{ij}$$
$$\rho(\lambda) \approx \lambda^{-(d_s/2-1)}$$
$$P_c(\lambda) \approx \lambda^{-d_s/2}$$

$$L_{ij} = \delta_{ij} - a_{ij}/k_i$$
$$\rho(\lambda) \approx \lambda^{-(d_s/2-1)}$$
$$P_c(\lambda) \approx \lambda^{-d_s/2}$$

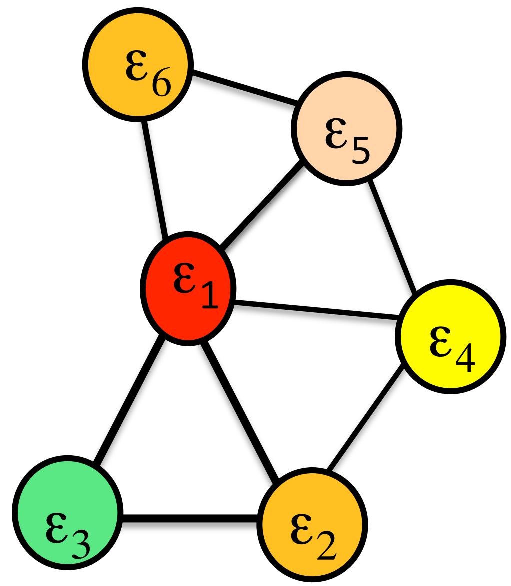
Energies of the nodes

Not all the nodes are the same!

Let assign to each node i

an energy ε *from a*

$g(\varepsilon)$ *distribution*



Energy of the δ-faces

Every δ-face α is associated to an energy which is the sum of the energy of the nodes belonging to α

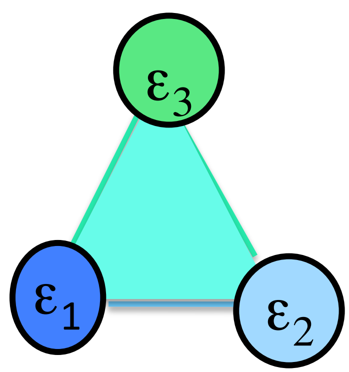
$$\varepsilon_\alpha = \sum_{i \subset \alpha} \varepsilon_i$$

For example, in d=3

the energy of a link

is $\varepsilon_1+\varepsilon_2$

the energy of a face



is $\varepsilon_1+\varepsilon_2+\varepsilon_3$

Network Geometry with Flavor

Starting from a single d-dimensional simplex

(1) GROWTH :

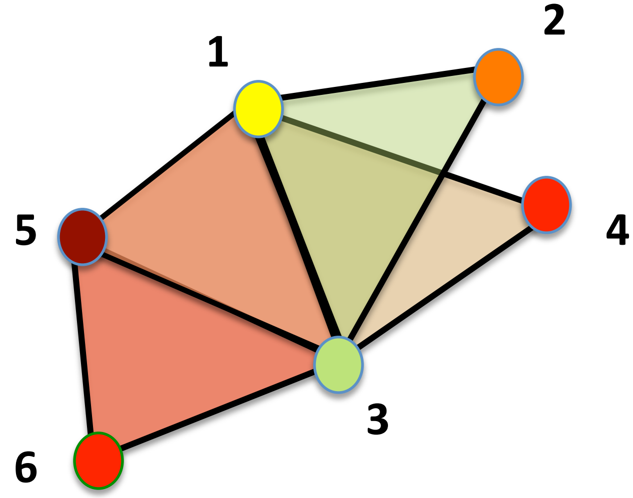
At every timestep we add a new node d simplex

(formed by one new node and an existing (d-1)-face).

The new node has energy ε drawn from the distribution $g(\varepsilon)$

(2) ATTACHMENT:

The probability that a new node will be connected to a face α depends on the flavor s=-1,0,1 and is given by



$$\Pi_{\mu}^{[s]} = \frac{e^{-\beta\varepsilon_{\mu}}(1+sn_{\mu})}{\sum_{\mu'} e^{-\beta\varepsilon_{\mu'}}(1+sn_{\mu'})}$$

Bianconi & Rahmede (2016)

NGF
with flavor s=-1
Manifolds

Manifolds in d=3

In NGF with s=-1 and d=3
also called
Complex Quantum Network Manifolds
the average of the generalized degree follow
the ***Fermi-Dirac, Boltzmann and Bose-Einstein***
distribution
respectively for
triangular faces, links and nodes

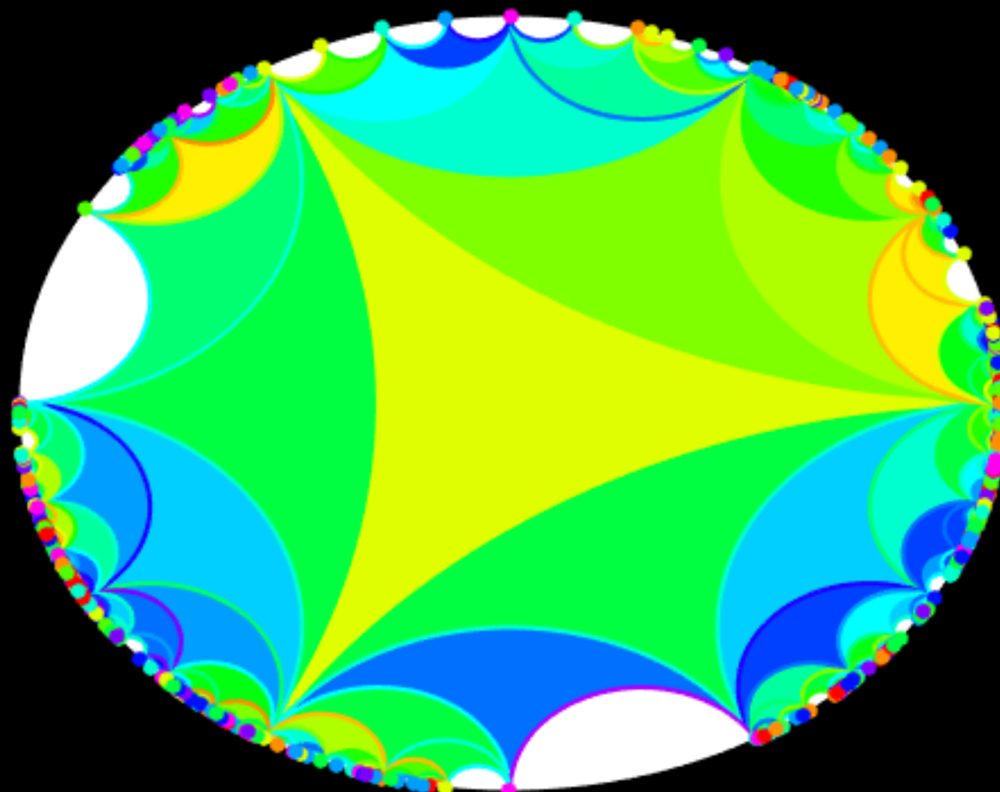
The average of the generalized degree of the NGF over δ-faces of energy ε

$$\langle [k_{d,\delta} - 1] \| \varepsilon \rangle$$

follows a regular pattern

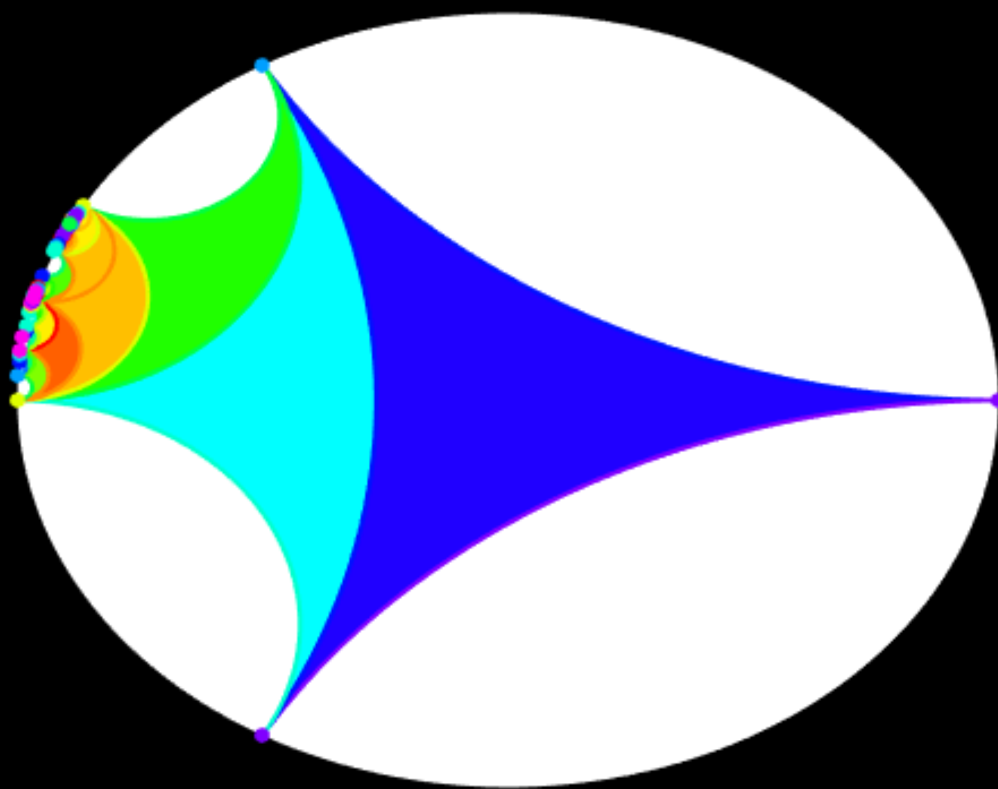
flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Fermi-Dirac	Boltzmann	Bose-Einstein
$\delta = d - 2$	Boltzmann	Bose-Einstein	Bose-Einstein
$\delta \leq d - 3$	Bose-Einstein	Bose-Einstein	Bose-Einstein

Emergent geometry at high temperature



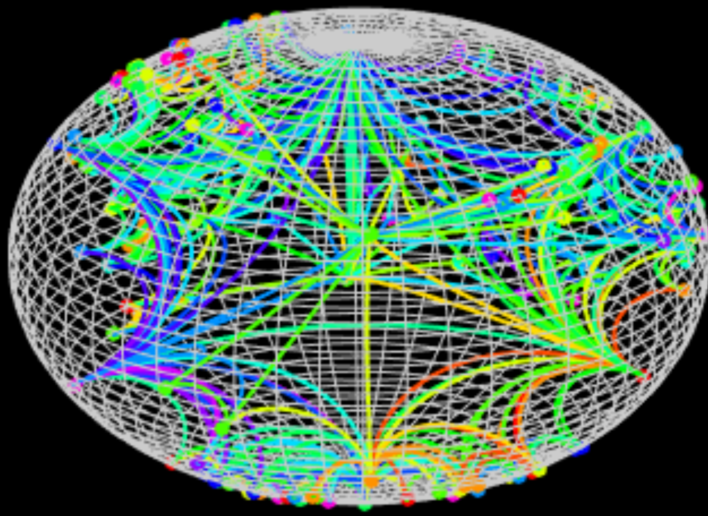
d=2
β=0.01

Emergent geometry at low temperature



d=2
β=5

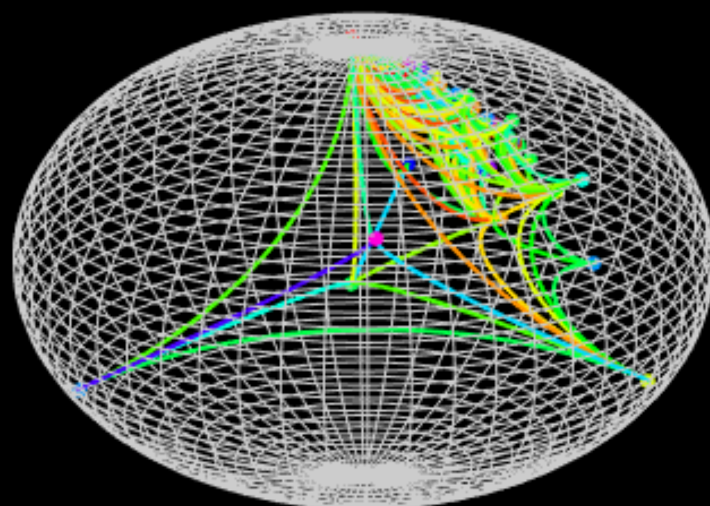
Emergent geometry at high temperature



d=3

β=0.01

Emergent geometry at low temperature



d=3
β=5

Conclusions

Emergent complex network geometries

- **Show small world behavior, finite clustering coefficient, high modularity, finite spectral dimension** *which are properties of many real network datasets*
- ***display a distribution of the local curvature that can be exponential or scale-free***
- ***for m=2 they generate random manifolds***

Network Geometry with Flavor

- ***are scale-free for d>1 also when they do not include an explicit preferential attachment. The dimension d=3 is the lowest dimension for obtaining scale-free manifold.***
- ***They have generalized degrees displaying Fermi-Dirac, Boltzmann or Bose-Einstein distribution depending on the dimensionality of the δ-face and on the flavour s.***

Collaborators and References

Emergent network geometry

G. Bianconi C. Rahmede Network geometry with flavor PRE 93, 032315 (2016)

G. Bianconi and C. Rahmede Scientific Reports 5, 13979 (2015)

G.Bianconi and C. Rahmede Scientific Reports 7, 41974 (2017)

Z. Wu, G. Menichetti, C. Rahmede and G. Bianconi Scientific Reports 5, 10073 (2015).

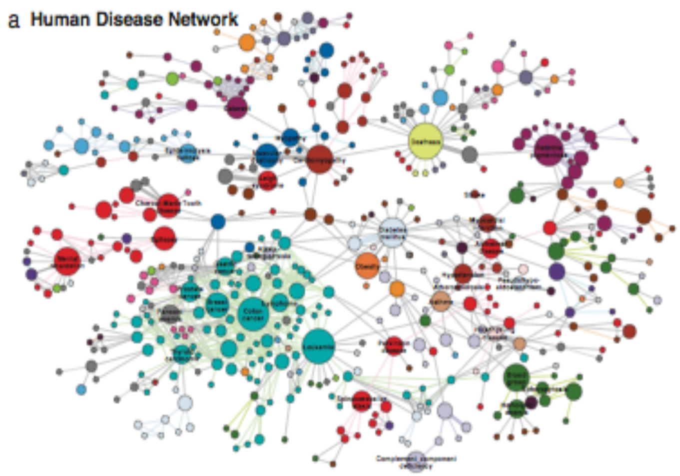
O. T. Courtney and G. Bianconi PRE 95, 062301(2017)

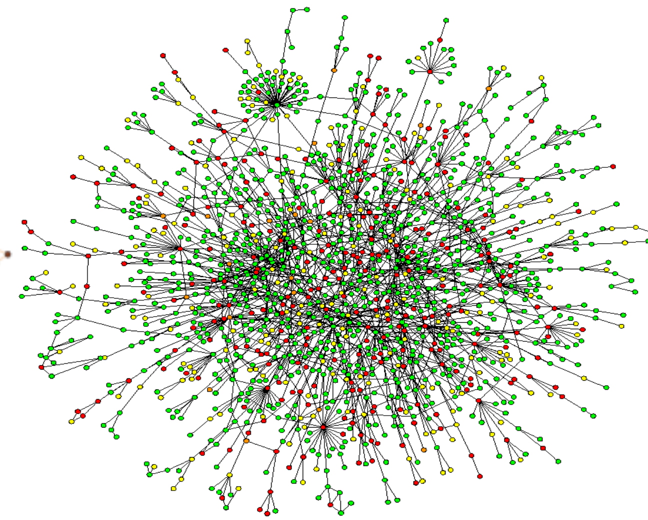
Ensembles of simplicial complexes

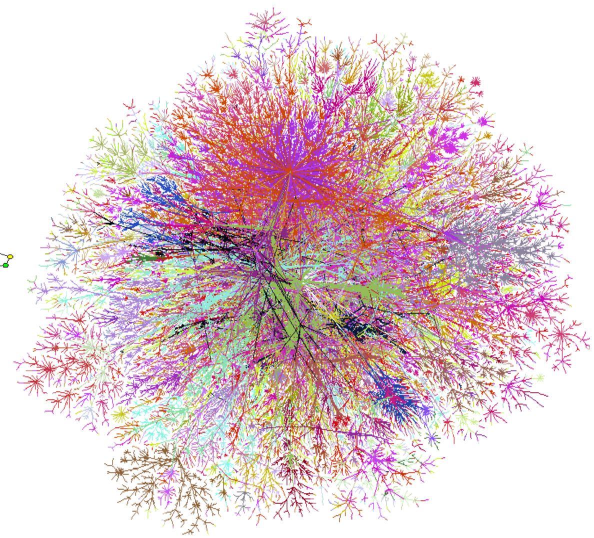
O. T. Courtney and G. Bianconi PRE *93*, 062311 (2016)

A primer on Network Theory

Complex Networks







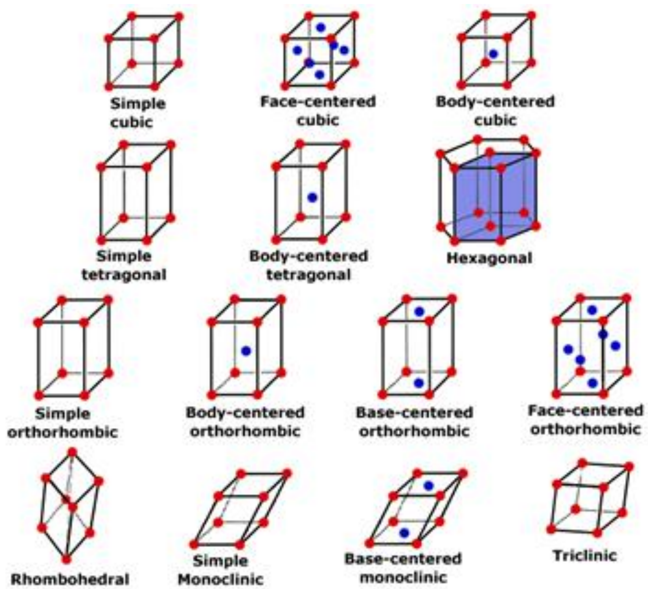
describe

the interactions between the elements of large complex

Biological, Social and Technological systems.

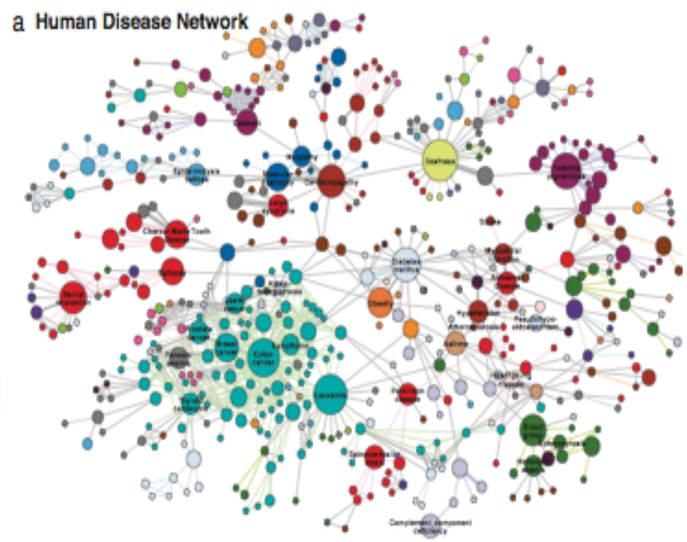
Complexity: between randomness and order

LATTICES



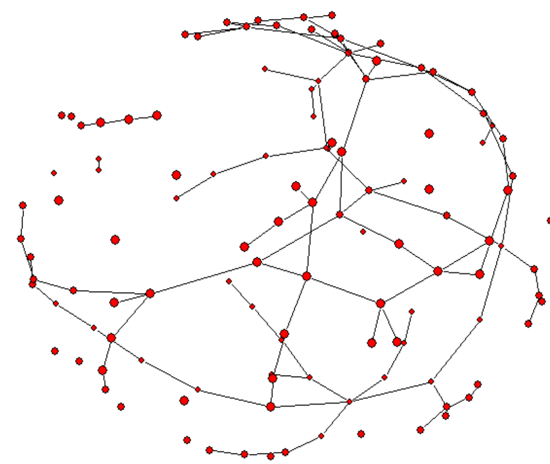
Regular networks
Symmetric

COMPLEX NETWORKS



Scale free networks
Small world
With communities
ENCODING INFORMATION IN THEIR STRUCTURE

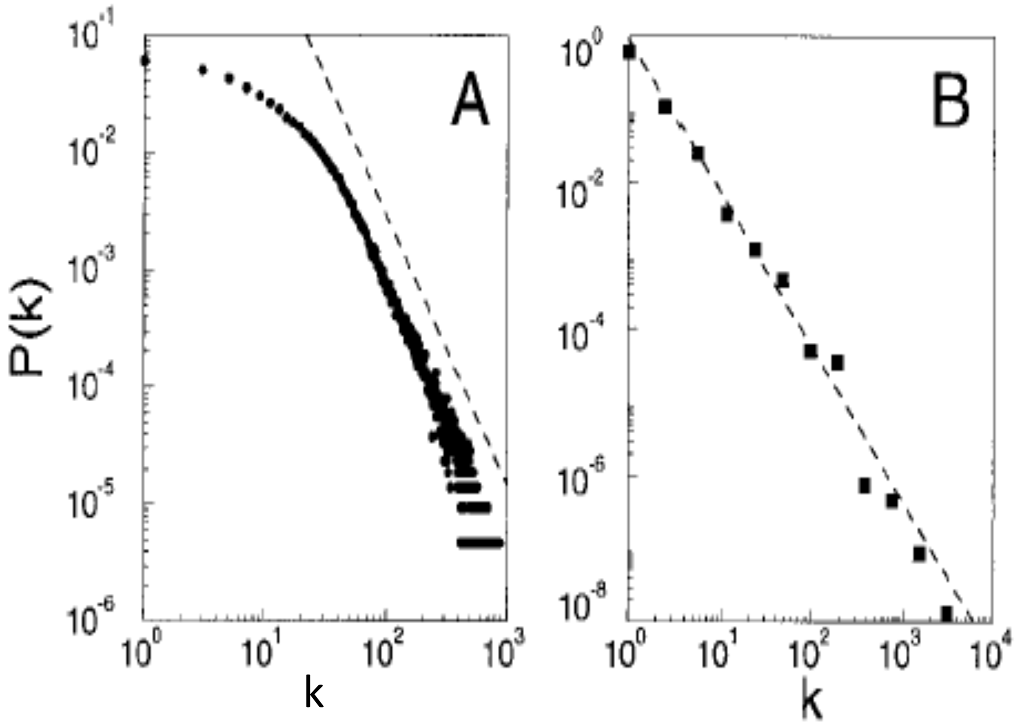
RANDOM GRAPHS



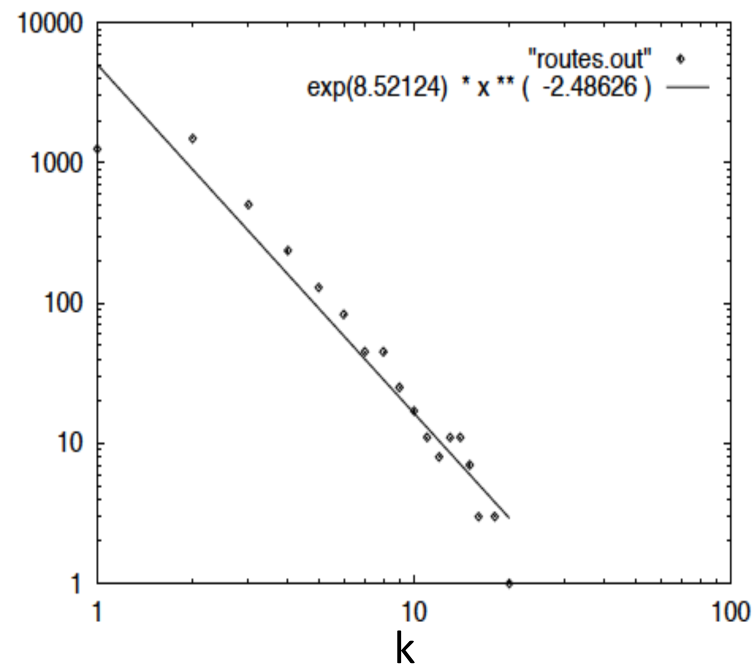
Totally random
Binomial degree distribution

Scale-free networks

Actor networks **WWW** **Internet**



Barabasi-Albert 1999



Faloutsos, Faloutsos and Faloutsos 1999

$$P(k) \propto k^{-\gamma} \qquad \gamma \in (2,3]$$

$$\langle k \rangle \quad finite$$

$$\langle k^2 \rangle \to \infty$$

Scale-free networks

Technological networks
Internet
World-Wide Web

Biological networks
Metabolic networks,
Protein-interaction networks,
Transcription networks

Transportation networks
Airport networks

Social networks
Collaboration networks
Citation networks
Facebook

Economical networks
Networks of shareholders
The World Trade Web

BA scale-free model

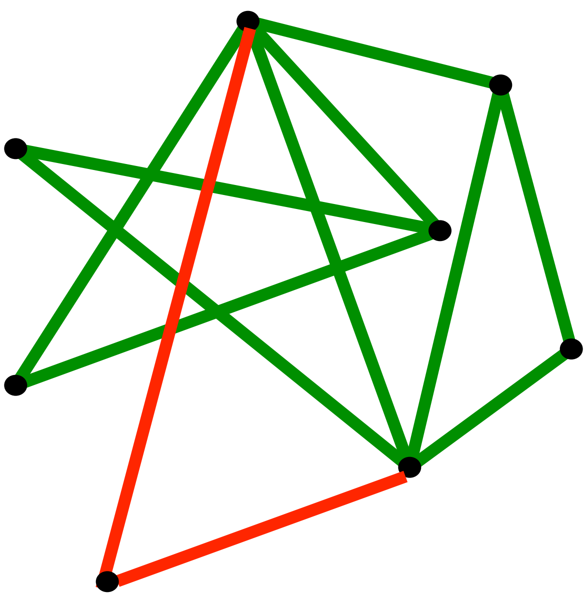
(1) GROWTH :

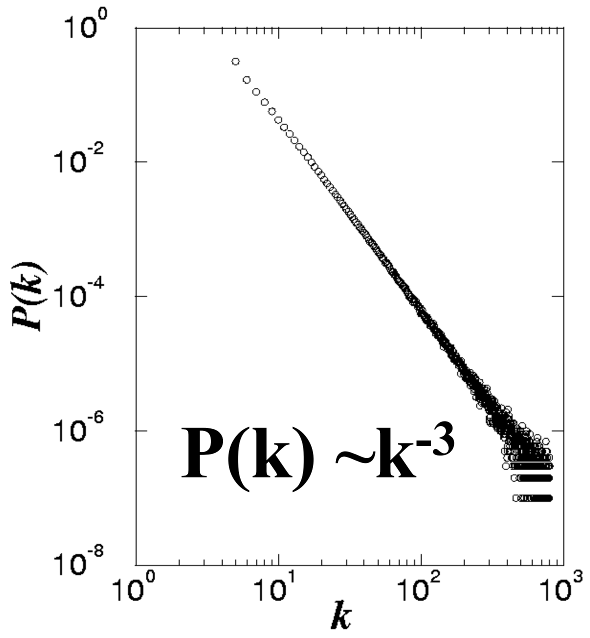
At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

(2) PREFERENTIAL ATTACHMENT :

The probability $\Pi(k_i)$ that a new node will be connected to node *i* depends on the connectivity k_i of that node

$$\Pi(k_i) = \frac{k_i}{\Sigma_j k_j}$$





Barabási et al. Science (1999)

Growth by uniform attachment of links

(1) GROWTH :

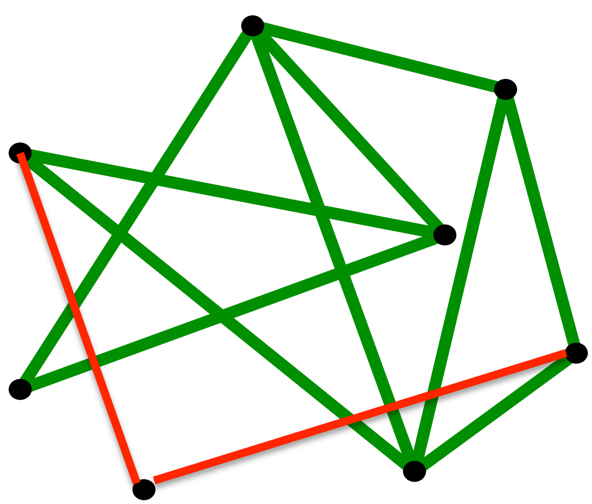
At every timestep we add a new node with *m* edges (connected to the nodes already present in the system).

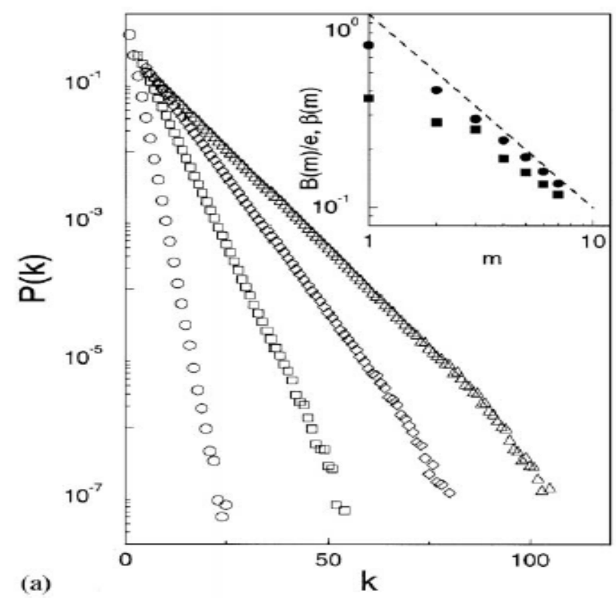
(2) UNIFORM ATTACHMENT :

The probability Π_i that a new node will be connected to node *i is uniform*

$$\Pi_i = \frac{1}{N}$$

Exponential



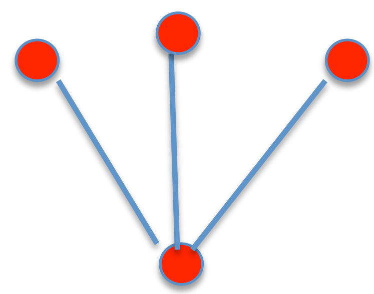


Barabási & Albert, Physica A (1999)

Growing networks
and quantum statistics

The unitary cell of growing networks

Consider the following "unitary cell" of a network



To grow a network need to attach the unitary cell to the exiting network. We have two options

The unitary cell of growing networks

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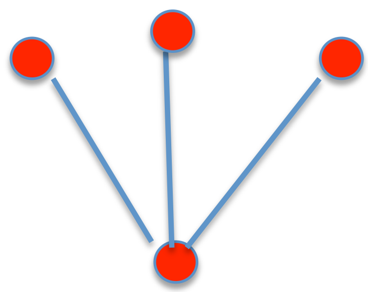

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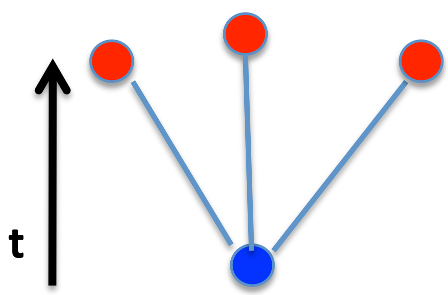

An old node
is attached
to m new nodes

The unitary cell of growing networks

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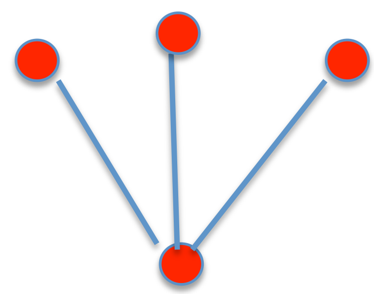

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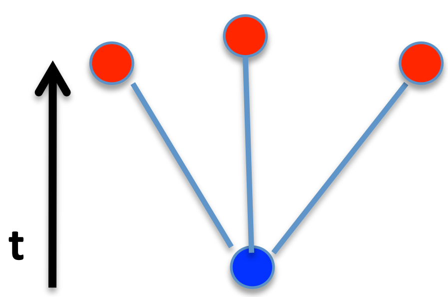
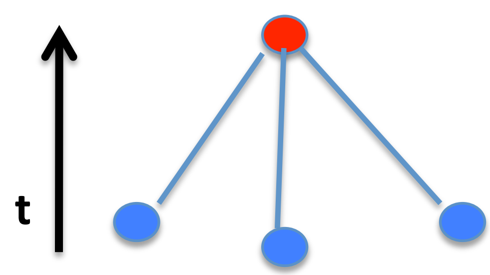


An old node is attached to m new nodes



A new node is attached to m old nodes

Intrinsic properties of the nodes

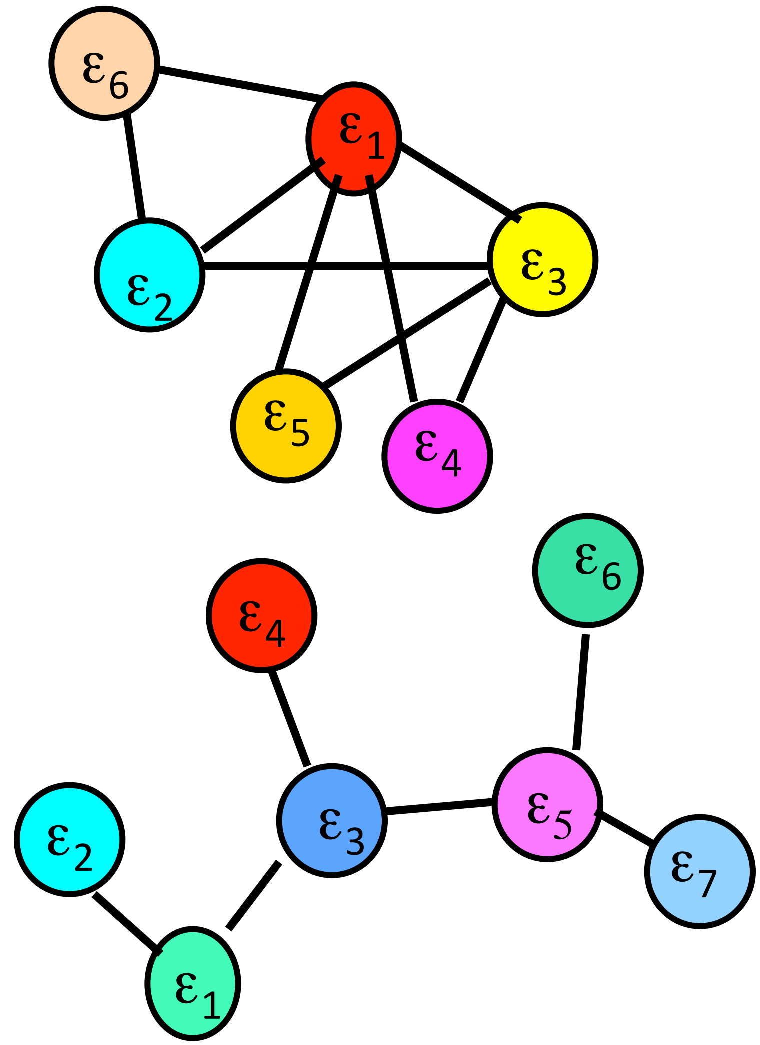
Not all the nodes are the same!

Let assign to each node

an energy ε *from a*

$g(\varepsilon)$ *distribution*

that describes an intrinsic quality of the node



Fitness

The fitness of a node i is given by

$$\eta_i = e^{-\beta \varepsilon_i}$$

where $\beta=1/T$ is the inverse temperature

If $\beta=0$ all the nodes have same fitness

If $\beta>>1$ small differences in energy have large impact on the fitness of the faces

The Complex Growing Cayley tree model

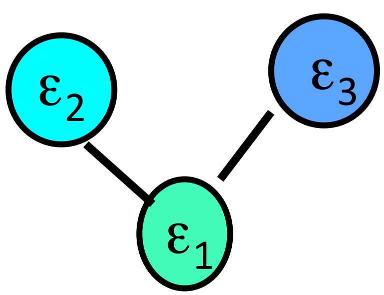
Growth:

–At each time attach a old node with ρ_i=1 to *m* links are added to the network and then we set ρ_i=0.

–To each node *i* we assign a energy ε_i from a $g(\varepsilon)$ distribution

Attachment towards high energy nodes:

–The node *i* to which we attach the new “unitary cell” is chosen with probability



$$\Pi_i = \frac{e^{-\beta\varepsilon_i}\rho_i}{\sum_j e^{-\beta\varepsilon_j}\rho_j}$$

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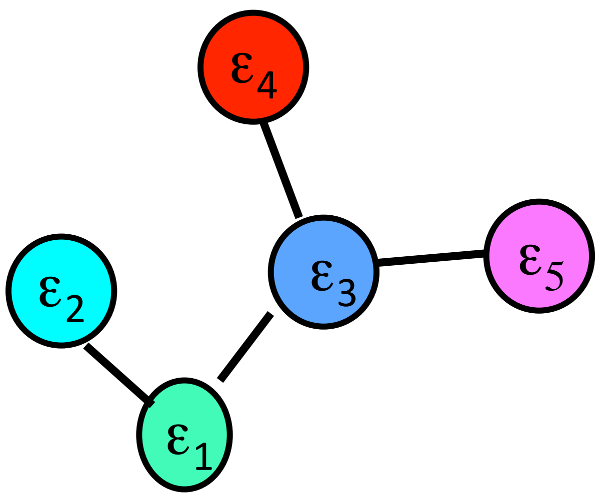

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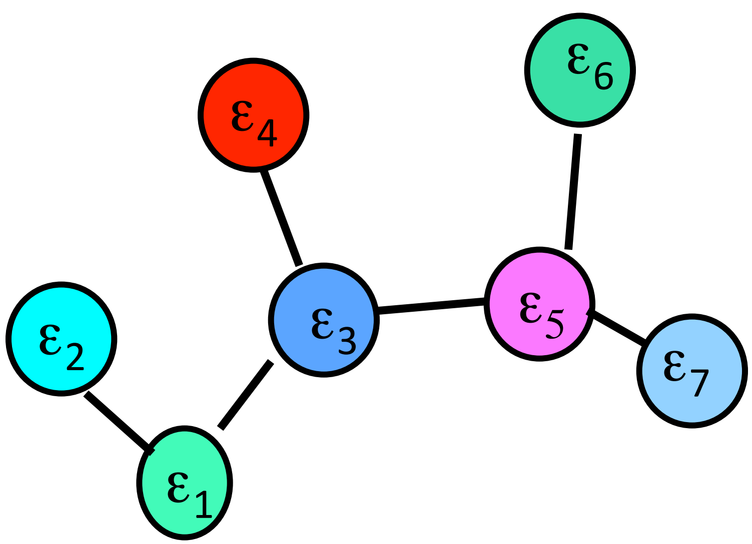
Attachment towards high energy nodes:

–The node *i* to which we attach the new "unitary cell" is chosen with probability



$$\Pi_i = \frac{e^{-\beta\varepsilon_i}\rho_i}{\sum_j e^{-\beta\varepsilon_j}\rho_j}$$

MF Equations
for the growing scale-free network and the complex growing Cayley tree network

- Scale-free

Bianconi-Barabasi model

$$\frac{d\bar{k}_i(t)}{dt} = m \frac{e^{-\beta\varepsilon_i}\bar{k}_i}{\sum_j e^{-\beta\varepsilon_j}\bar{k}_j}$$

- Complex Growing

  Cayley tree

$$\frac{d\bar{\rho}_i(t)}{dt} = -\frac{e^{-\beta\varepsilon_i}\bar{\rho}_i}{\sum_j e^{-\beta\varepsilon_j}\bar{\rho}_j}$$

Classical and quantum statistics

Classical particles

Boltzmann statitics-(s=0)

occupation number n=0,1,2,...

Quantum particles

-Fermi particles-(s=-1)

occupation number n=0,1

-Bose particles-(s=1)

occupation number n=0,1,2...

Average occupation number

Classical and quantum statistics

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} + s}$$

Solution of the Bianconi-Barabasi model

The average degree of node increases in time as a power-law with an exponent depending on its energy, β and a self-consistent constant μ_B

$$\bar{k}_i(t) = m\left(\frac{t}{t_i}\right)^{f_B(\varepsilon)} \qquad f_B(\varepsilon) = e^{-\beta(\varepsilon - \mu_B)}$$

The self consistent constant μ_B is determined by the same equation fixing the chemical potential in a Bose gas!

$$1 = \int d\varepsilon\; g(\varepsilon)\frac{1}{e^{\beta(\varepsilon - \mu_B)} - 1}$$

Solution of the complex growing Cayley model

The average ρ of node (determining the probability that a node is at the interface) decreases in time as a power-law with an exponent depending on its energy, β and a self-consistent constant μ_F

$$\overline{\rho}_i(t) = m\left(\frac{t}{t_i}\right)^{-f_F(\varepsilon)} \qquad f_F(\varepsilon) = e^{-\beta(\varepsilon-\mu_F)}$$

The self consistent constant μ_F is determined by the same equation fixing the chemical potential in a Fermi gas!

$$\frac{1}{m} = \int d\varepsilon\, g(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu_F)}+1}$$

The Bianconi-Barabasi model

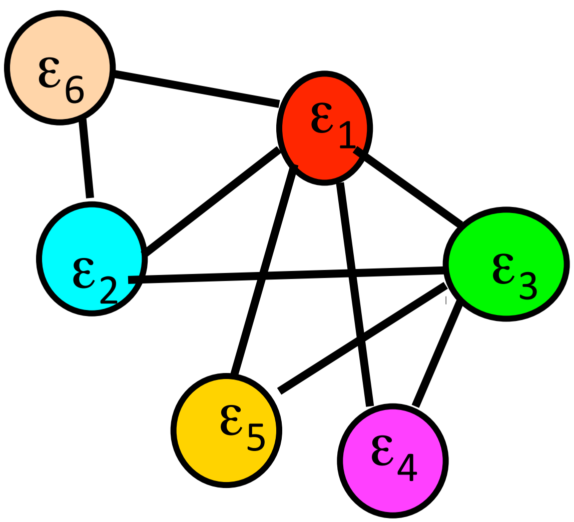
Growth:

–At each time a new node and *m* links are added to the network.

–To each node *i* we assign a energy ε_i from a $g(\varepsilon)$ distribution

Preferential attachment towards high degree low energy nodes:

–Each node connects to the rest of the network by *m* links attached preferentially to well connected, low energy nodes.

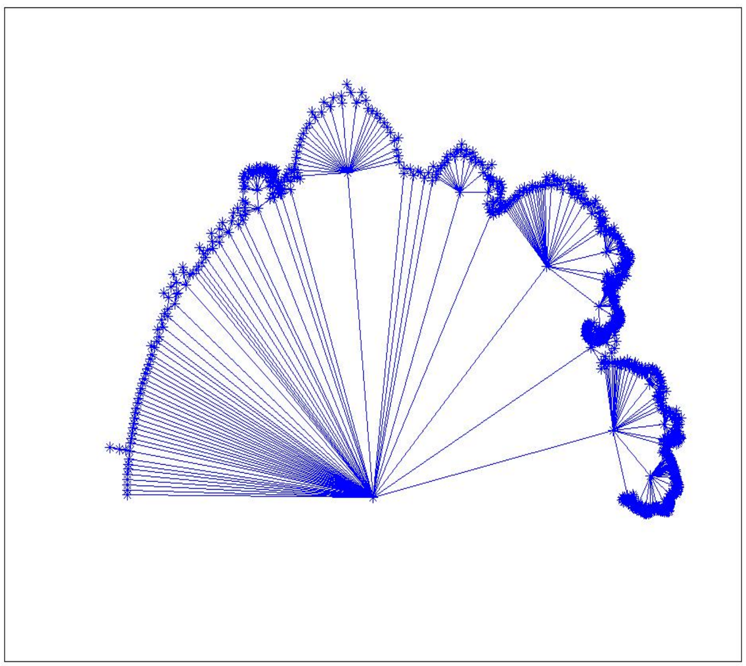


$$\Pi_i = \frac{e^{-\beta\varepsilon_i} k_i}{\sum_j e^{-\beta\varepsilon_j} k_j}$$

Bose-Einstein condensation in complex networks

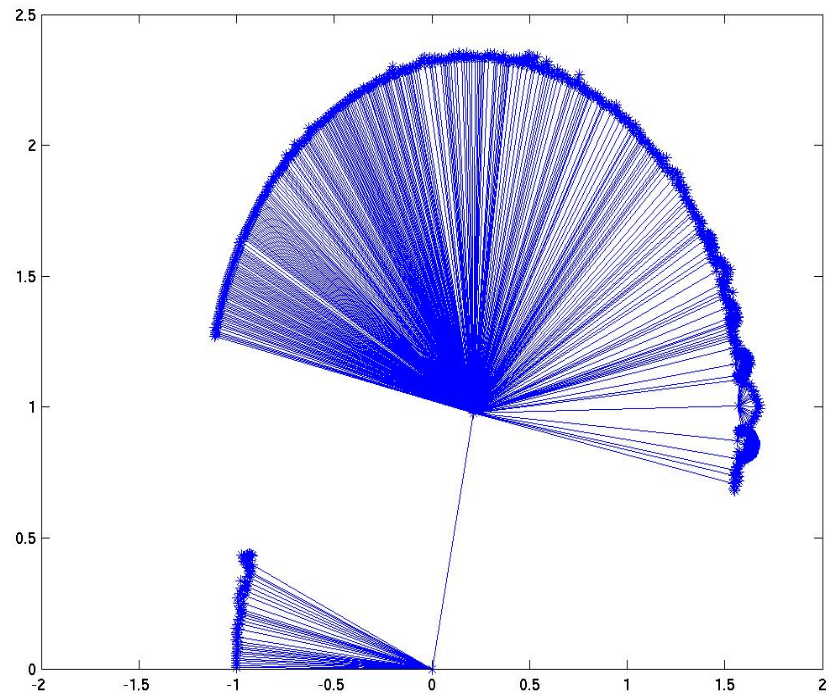
Scale-Free Fit-get-rich Phase

$$\beta < \beta_C$$



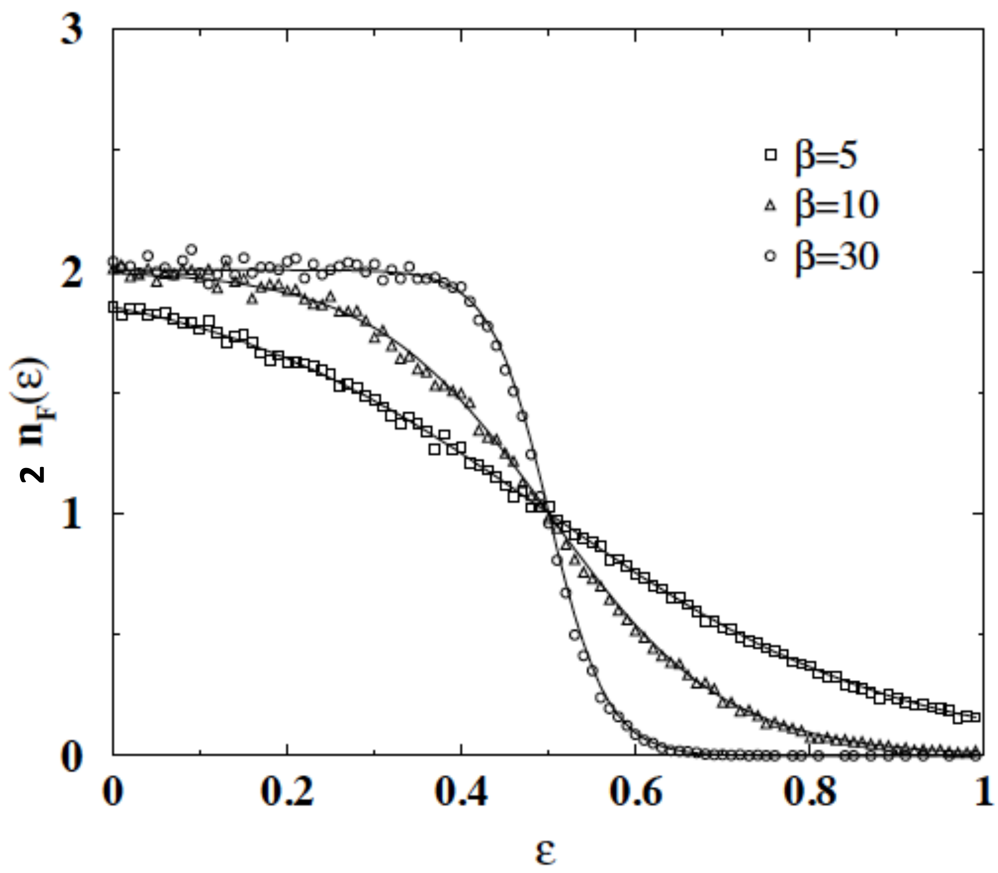
Bose-Einstein condensate Phase

$$\beta > \beta_C$$



G. Bianconi, A.-L. Barabási 2001

Energy distribution of the nodes at the bulk of the growing Cayley tree network



G. Bianconi (2002)