Special Relativity Sheet Two

1. Two clocks, A and B, move with constant velocities. They meet momentarily, when both are set to zero. Later, upon registering a time T_A, clock A emits a light pulse which is received by clock B when B registers a time T_B. Show on a space-time diagram, in the inertial frame of clock B, the three events of the clocks meeting, and the emission and reception of the light pulse, as well as the world lines of the two clocks and the light pulse. Show that the relative velocity of the two clocks is

$$c\,\frac{T_B^2 - T_A^2}{T_B^2 + T_A^2}.$$

2. Show that the Fitzgerald (length) contraction formula, derived in the lectures by defining length to be the difference in the space separation of the ends of a rod at the same time, may also be derived by a single observer measuring the time interval it takes the rod to pass her.

3. The rapidity of a particle moving with speed u is defined to be $\phi = \tanh^{-1}(u/c)$. Prove that *colinear* rapidities are additive - ie. if A has rapidity ϕ relative to B, and B has rapidity ψ relative to C, then A has rapidity $\psi + \phi$ relative to C.

4. In a frame S, consider a rectilinearly moving particle having velocity u, rapidity ϕ, and proper acceleration α, and let τ be the proper time measured by a clock carried with the particle. Prove that $c\frac{d\phi}{d\tau} = \alpha$. Hence show that $\frac{d}{dt}(\gamma(u)\,u) = \alpha$.

5. Debris from a supernova explosion in a distant galaxy is observed to have redshifts ranging from $z = 1$ to $z = 2$. Recall that $1 + z = f'/f$ where f' and f are the emitted and observed frequencies respectively.
(a)Assuming that both redshifts correspond to velocities along our line of sight, show that the debris is moving away from us at speeds between $3c/5$ and $4c/5$.
(b)What is the relative velocity of the two extremes?
(c)What is the velocity of the galaxy, assuming that in its frame of reference, the debris moves at the same speed in all directions? Hint: use the result of question 3.

Sheet Two: Solutions

1.

Because of time dilation, B measures the time at which A sends off the signal to be γT_A. The distnace travelled by A relative to B in this time is $v\gamma\tau_A$, hence the time for the light to travel from $A \to B$ is $v\gamma T_A/c$. Thus

$$T_B = \gamma T_A + \frac{v\gamma T_A}{c}$$

Therefore

$$T_B = \frac{\left(1 + \frac{v}{c}\right) T_A}{\sqrt{1 - \frac{v^2}{c^2}}},$$

and this leads to

$$v = c\,\frac{T_B^2 - T_A^2}{T_B^2 + T_A^2}.$$

2.

Consider two events: when end B passes the observer O, and when end A passes O. In S',

$$\begin{aligned} \Delta x' &= -L' \quad (L' \text{ is length of rod in } S') \\ \Delta t' &= \frac{L'}{V}. \end{aligned}$$

In S,

$$\begin{aligned}
\Delta x &= 0 \quad \text{(observer is stationary in } S\text{)} \\
\Delta t &= \gamma\left(\Delta t' + \frac{v}{c^2}\Delta x'\right) \\
&= \gamma\left(\frac{L'}{v} - \frac{v}{c^2}L'\right) \\
&= \gamma\frac{L'}{v}\frac{1}{\gamma^2}
\end{aligned}$$

Therefore, the time for the rod to pass O is $\frac{L'}{\gamma v}$, where v is its velocity. Hence the length measured in S is $L = L'/\gamma$.

3. Let $u_1 = c\tanh\phi$ and $u_2 = c\tanh\psi$.
In the rest-frame of B

Therefore, using the velocity addition formula, velocity of A relative to C is

$$\begin{aligned}
\frac{u_1 + u_2}{1 + \frac{u_1 u_2}{c^2}} &= c\left[\frac{\tanh\phi + \tanh\psi}{1 + \tanh\phi\ \tanh\psi}\right] \\
&= c\tanh(\phi + \psi),
\end{aligned}$$

using the addition formula for $\tanh(x + y)$.

4. $u = c\tanh\phi$, at time $t = t_0$, if $u = u_0$ and $\phi = \phi_0$, then with respect to the irf at time t_0, $u' = c\tanh(\phi' - \phi_0)$ (since rapidities are additive). Since τ is the time in the irf, by definition $\alpha = \frac{du'}{d\tau}\big|_{t_0}$ is the proper acceleration.

Thus

$$\begin{aligned}
\alpha &= c\frac{d}{d\tau}\tanh(\phi' - \phi_0)\big|_{t=t_0} \\
&= c\,\mathrm{sech}^2(\phi' - \phi_0)\frac{d\phi'}{d\tau}\big|_{t=t_0}
\end{aligned}$$

and at $t = t_0$, $\phi = \phi_0$ and $\frac{d\phi'}{d\tau} = \frac{d\phi}{d\tau}$, thus $\alpha = c\frac{d\phi}{d\tau}$.

Now,

$$
\begin{aligned}
u\gamma(u) &= c\tanh\phi\left(\sqrt{1-\tanh^2\phi}\right)^{-1} \\
&= c\sinh\phi \\
\therefore \frac{d}{dt}(u\gamma(u)) &= c\cosh\phi\frac{d\phi}{d\tau}\frac{d\tau}{dt},
\end{aligned}
$$

but

$$
\begin{aligned}
t &= \gamma(u)\left(\tau + \frac{ux'}{c^2}\right) \\
\Rightarrow \frac{dt}{d\tau} &= \gamma(u) = \cosh\phi \\
\therefore \frac{d}{dt}(u\gamma(u)) &= c\frac{d\phi}{d\tau} = \alpha.
\end{aligned}
$$

5. (a) We have

$$1 + z = \frac{f'}{f} = \sqrt{\frac{c+v}{c-v}}$$

Squaring and multiplying up, we find

$$\frac{v}{c} = \frac{(1+z)^2+1}{(1+z)^2-1}$$

into which we substitute $z = 1$ and $z = 2$ to obtain $v = 3c/5$ and $v = 4c/5$ respectively.

(b) The relative velocity is given by

$$\frac{u-v}{1-uv/c^2} = \frac{4/5-3/5}{1-12/25}c = \frac{5c}{13}$$

(c) Since the rapidity (see question 3) is additive in the velocity addition formula, the rapidity of the galaxy at which both pieces of debris appear to recede at the same velocity is given by the arithmetic mean of the two rapidities. Thus we have for the galaxy

$$\phi = \frac{1}{2}(\mathrm{artanh}\frac{3}{5} + \mathrm{artanh}\frac{4}{5})$$

Using the algebraic formula for the inverse hyperbolic tangent

$$\mathrm{artanh} x = \frac{1}{2}\ln\frac{1+x}{1-x}$$

we find

$$\phi = \frac{1}{2}(\ln 2 + \ln 3) = \ln\sqrt{6}$$

and finally

$$v = c\tanh\phi = \frac{e^\phi - e^{-\phi}}{e^\phi + e^{-\phi}}c = 5c/7$$