Special Relativity Sheet 3

1. If Λ^{μ}_{ν} is the matrix that generates the Lorentz transformation from S to $\bar{S}$ (in standard configuration) and $\bar{\Lambda}^{\mu}_{\nu}$ is the corresponding matrix that generates the transformation from $\bar{S}$ to S,

- show that for any 4-vector $\mathbf{A}$, $A^{\gamma} = \bar{\Lambda}^{\gamma}_{\mu}\Lambda^{\mu}_{\nu}A^{\nu}$;
- show also that $A^{\gamma} = \delta^{\gamma}_{\beta}A^{\beta}$, where δ^{γ}_{β} is the Kronecker delta-symbol;
- hence prove that $\bar{\Lambda}^{\gamma}_{\mu}\Lambda^{\mu}_{\beta} = \delta^{\gamma}_{\beta}$;
- prove in the same way that $\Lambda^{\gamma}_{\mu}\bar{\Lambda}^{\mu}_{\beta} = \delta^{\gamma}_{\beta}$;
- it follows from the previous two results that the matrices Λ and $\bar{\Lambda}$ are mutual inverses. Show this explicitly by direct matrix multiplication.

2. Given $\mathbf{A} = (9, -2, 3, 5)$ and $\mathbf{B} = (1, 1, 0, -2)$ in S, find

a) the components of $\mathbf{A}$ and $\mathbf{B}$ in $\overline{S}$, which moves at speed $0.8c$ relative to S in the positive x-direction;

b) the magnitude of $\mathbf{A}$ in both S and $\overline{S}$;

c) the value of $\mathbf{A} \cdot \mathbf{B}$ in both S and $\overline{S}$.

3. Show that

a) if $\mathbf{X}(\neq \mathbf{0})$ is timelike, there exists an inertial frame in which it has zero spatial components;

b) if $\mathbf{X}(\neq \mathbf{0})$ is spacelike, there exists an inertial frame in which it has a zero time component.

4. Two 4-vectors $\mathbf{A}$ and $\mathbf{B}$ are orthogonal if $\mathbf{A} \cdot \mathbf{B} = 0$. Show that

a) if $\mathbf{T}$ is a timelike 4-vector and $\mathbf{T} \cdot \mathbf{V} = 0$ then $\mathbf{V}$ is spacelike;

b) the sum of two timelike vectors which are isochronous (ie. both pointing into the future, or both into the past) is also timelike and isochronous with them;

c) every spacelike 4-vector may be expressed as the difference of two isochronous timelike 4-vectors.

(Hint: use the result of question 3 to find the easiest inertial frames to work in.)

Special relativity solutions 3

1.

- Writing $\bar{A}^\alpha$ as the components of A in $\bar{S}$,

$$\bar{A}^\alpha = \Lambda^\alpha_{\ \beta} A^\beta \tag{1}$$
$$A^\gamma = \bar{\Lambda}^\gamma_{\ \alpha} \bar{A}^\alpha \tag{2}$$
$$A^\gamma = \bar{\Lambda}^\gamma_{\ \alpha} \Lambda^\alpha_{\ \beta} A^\beta \tag{3}$$

- Since

$$\delta^\alpha_\beta = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta, \end{cases} \tag{4}$$

this is obvious.

- Inserting $A^\gamma = \delta^\gamma_\beta A^\beta$ in the first result, we have

$$(\bar{\Lambda}^\gamma_{\ \alpha} \Lambda^\alpha_{\ \beta} - \delta^\gamma_\beta) A^\beta = 0. \tag{5}$$

Since this must be true for any A^β,

$$\bar{\Lambda}^\gamma_{\ \alpha} \Lambda^\alpha_{\ \beta} = \delta^\gamma_\beta. \tag{6}$$

- In the same way

$$\bar{A}^\alpha = \Lambda^\alpha_{\ \beta} A^\beta = \Lambda^\alpha_{\ \beta} \bar{\Lambda}^\beta_{\ \gamma} \bar{A}^\gamma \tag{7}$$

hence, relabelling the dummy indices and using $\bar{A}^\alpha = \delta^\alpha_\gamma \bar{A}^\gamma$, the result follows directly, as above.

- $$\Lambda = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

By substituting v by $-v$

$$\bar{\Lambda} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Then, trivially

$$\Lambda\,\bar{\Lambda} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10}$$

2. $v = 0.8c$, so $\gamma = 5/3$ and this implies that

$$\Lambda = \begin{pmatrix} \frac{5}{3} & \frac{-4}{3} & 0 & 0 \\ \frac{-4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It follows that

$$\begin{pmatrix} \overline{A}^0 \\ \overline{A}^1 \\ \overline{A}^2 \\ \overline{A}^3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & \frac{-4}{3} & 0 & 0 \\ \frac{-4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9 \\ -2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 17\frac{2}{3} \\ -15\frac{1}{3} \\ 3 \\ 5 \end{pmatrix}$$

and

$$\begin{pmatrix} \overline{B}^0 \\ \overline{B}^1 \\ \overline{B}^2 \\ \overline{B}^3 \end{pmatrix} = \begin{pmatrix} \frac{5}{3} & \frac{-4}{3} & 0 & 0 \\ \frac{-4}{3} & \frac{5}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 0 \\ -2 \end{pmatrix}$$

b) in S, $\mathbf{A}^2 = -9^2+(-2)^2+3^2+5^2 = -43$ and in $\overline{S}$, $\mathbf{A}^2 = -(17\frac{2}{3})^2+(15\frac{1}{3})^2+3^2+5^2 = -43$.

c) in S, $\mathbf{A}\cdot\mathbf{B} = -9\cdot 1 + (-2)\cdot 1 + 3\cdot 0 + 5\cdot(-2) = -21$ and in $\overline{S}$, $\mathbf{A}\cdot\mathbf{B} = -(17\frac{2}{3})\cdot\frac{1}{3} + (-15\frac{1}{3})\cdot\frac{1}{3} + 3\cdot 0 + 5\cdot(-2) = -21$.

3. Choose the spatial axes so that $\mathbf{X} = (ct_0, x_0, 0, 0)$, where $t_0, x_0 \neq 0$, in S. Since $\mathbf{X}$ is timelike, $\mathbf{X}^2 < 0$. Thus $-c^2t_0^2 + x_0^2 < 0$, so $x_0^2 < c^2t_0^2$. Let $v = x_0/t_0$ ($|v| < c$), then transform to frame $\overline{S}$ which moves along the x-axis of S with speed v. Now we have

$$\begin{aligned} t_0' &= \gamma(v)\left(t_0 - \tfrac{x_0 v}{c^2}\right) && \neq 0 \ \ (\text{since}|v| < c) \\ x_0' &= \gamma(v)(x_0 - vt_0) && = 0. \end{aligned}$$

That is, in $\overline{S}$, $\mathbf{X}$ has zero spatial components.

b) if $\mathbf{X}$ is spacelike, $\mathbf{X}^2 > 0$, and so $x_0^2 > c^2t_0^2$. Now let $v = \frac{c^2t_0}{x_0}$. Clearly

$$\begin{aligned} t_0' &= \gamma(v)\left(t_0 - \tfrac{x_0}{c^2}\tfrac{c^2t_0}{x_0}\right) && = 0 \\ x_0' &= \gamma(v)\left(x_0 - \tfrac{c^2t_0}{x_0}t_0\right) && \neq 0. \end{aligned}$$

4. a) If $\mathbf{T}$ is timelike, there is a frame such that $\mathbf{T} = (ct_0, \mathbf{0})$. Let $\mathbf{V} = (ct_1, x_1, y_1, z_1)$, so $\mathbf{T}\cdot\mathbf{V} = -c^2t_0t_1 = 0$. This implies that $t_1 = 0$, and so $\mathbf{V}$ is spacelike.

b) Let $\mathbf{T}_1 = (ct_1, \mathbf{0})$ and $\mathbf{T}_2 = (ct_2, x_2, 0, 0)$. Assuming t_1 and t_2 have the same sign, and using $\mathbf{T}_1 + \mathbf{T}_2 = (ct_1 + ct_2, x_2, 0, 0)$,

$$
\begin{aligned}
(\mathbf{T}_1 + \mathbf{T}_2)^2 &= x_2^2 - c^2(t_1 + t_2)^2 \\
&= [x_2^2 - c^2 t_2^2] - c^2 t_1^2 - 2c^2 t_1 t_2.
\end{aligned}
$$

In the above expression, the quantity in the square brackets is negative because $\mathbf{T}_2$ is timelike, the middle term is clearly negative and the last term is also negative as t_1 and t_2 have the same sign. Therefore $(\mathbf{T}_1 + \mathbf{T}_2)^2 < 0$ and so $\mathbf{T}_1 + \mathbf{T}_2$ is timelike.

The time component of $\mathbf{T}_1 + \mathbf{T}_2$ is $c(t_1 + t_2)$ which clearly has the same sign as the corresponding components of $\mathbf{T}_1$ (ct_1) and $\mathbf{T}_2$ (ct_2), and so is isochronous with them.

c)If $\mathbf{V}$ is spacelike, there is a reference frame such that $\mathbf{V} = (0, x_0, 0, 0)$. This may be written $\mathbf{V} = (t_0, x_0, 0, 0) - (t_0, 0, 0, 0)$. These two vectors are isochronous and the second is timelike. If we choose $|t_0| > |x_0|$, then the first is timelike also.