Special Relativity Sheet 4

1. If $A^\mu_{\ \nu}$ and $B^\mu_{\ \nu}$ are both components of tensors, show that

- $A_{\mu\nu}$
- $A^\mu_{\ \nu} + B^\mu_{\ \nu}$
- $A^\mu_{\ \nu} B^\rho_{\ \sigma}$

are components of tensors too.

2. Given the numbers

$$\begin{gathered}
A^0 = 5, \quad A^1 = 0, \quad A^2 = -1, \quad A^3 = -6, \\
B_0 = 0, \quad B_1 = -2, \quad B_2 = 4, \quad B_3 = 0, \\
\begin{array}{cccc}
C_{00} = 1, & C_{01} = 2, & C_{02} = 2, & C_{03} = 3, \\
C_{10} = 5, & C_{11} = -2, & C_{12} = -2 & C_{13} = 0, \\
C_{20} = 4, & C_{21} = 5, & C_{22} = 2, & C_{23} = -2, \\
C_{30} = -1, & C_{31} = -1, & C_{32} = -3 & C_{33} = 0
\end{array}
\end{gathered}$$

find

a) $A^\alpha B_\alpha$;
b) $A^\alpha C_{\alpha\beta}$ for all β;
c) $A^\gamma C_{\gamma\sigma}$ for all σ;
d) $A^\gamma C_{\mu\gamma}$ for all μ;
e) $A^\alpha B_\beta$ for all α and β.

3. If $a_{\alpha\beta}$ are constant and symmetric show that

$$\left(a_{\alpha\beta} A^\alpha A^\beta\right)_{,\gamma} = 2 a_{\alpha\beta} A^\alpha A^\beta_{\ ,\gamma}. \tag{1}$$

What happens if $a_{\alpha\beta}$ is antisymmetric instead?

4. If $F_{\alpha\beta}$ are the components of a tensor, so that in an inertial frame S

$$F_{\alpha\beta} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & cb_3 & -cb_2 \\ -e_2 & -cb_3 & 0 & cb_1 \\ -e_3 & cb_2 & -cb_1 & 0 \end{pmatrix}. \tag{2}$$

- Calculate the components $\bar{F}_{\mu\nu}$ in the frame $\bar{S}$ which is in standard configuration with S.
- Calculate $F^{\alpha\beta}$.
- Calculate $F^{\mu\nu} F_{\nu\mu}$ and hence prove that $|\mathbf{e}|^2 - c^2 |\mathbf{b}|^2$is a scalar.

Special Relativity Solutions 4

1.

- $A_{\mu\nu} = g_{\mu\rho} A^{\rho}{}_{\nu}$ and since both $g_{\mu\nu}$ and $A^{\mu}{}_{\nu}$ are tensors

$$A_{\mu'\nu'} = \Lambda^{\alpha}_{\mu'} \Lambda^{\beta}_{\rho'} g_{\alpha\beta} \Lambda^{\rho'}{}_{\gamma} \Lambda^{\delta}_{\nu'} A^{\gamma}_{\delta} = \Lambda^{\alpha}_{\mu'} \left(\Lambda^{\beta}_{\rho'} \Lambda^{\rho'}{}_{\gamma} \right) \Lambda^{\delta}_{\nu'} g_{\alpha\beta} A^{\gamma}_{\delta}. \tag{3}$$

  But $\Lambda^{\beta}_{\rho'} \Lambda^{\rho'}{}_{\gamma} = \delta^{\beta}_{\gamma}$, therefore

$$A_{\mu'\nu'} = \Lambda^{\alpha}_{\mu'} \Lambda^{\delta}_{\nu'} g_{\alpha\beta} A^{\beta}_{\delta} = \Lambda^{\alpha}_{\mu'} \Lambda^{\delta}_{\nu'} A_{\alpha\delta}, \tag{4}$$

  i.e. $A_{\mu\nu}$ transforms like a $\binom{0}{2}$-tensor, a tensor with no upper and two lower indices.

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$$A^{\mu'}_{\nu'} + B^{\mu'}_{\nu'} = \Lambda^{\mu'}{}_{\alpha} \Lambda^{\beta}_{\nu'} A^{\alpha}_{\beta} + \Lambda^{\mu'}{}_{\alpha} \Lambda^{\beta}_{\nu'} B^{\alpha}_{\beta} = \Lambda^{\mu'}{}_{\alpha} \Lambda^{\beta}_{\nu'} \left(A^{\alpha}_{\beta} + B^{\alpha}_{\beta} \right). \tag{5}$$

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$$A^{\mu'}_{\nu'} B^{\rho'}_{\sigma'} = \Lambda^{\mu'}{}_{\alpha} \Lambda^{\beta}_{\nu'} \Lambda^{\rho'}{}_{\gamma} \Lambda^{\delta}_{\sigma'} A^{\alpha}_{\beta} B^{\gamma}_{\delta} \tag{6}$$

  transforms like a $\binom{2}{2}$-tensor, a tensor with two upper and two lower indices.

2. a) $A^{\alpha} B_{\alpha} = 5 \cdot 0 + 0 \cdot (-2) + (-1) \cdot 4 + (-6) \cdot 0 = -4$

b)

$$\begin{array}{lll} \beta = 0 & A^{\alpha} C_{\alpha 0} = & 5 \cdot 1 + 0 + (-1) \cdot 4 + (-6) \cdot (-1) = 7 \\ \beta = 1 & A^{\alpha} C_{\alpha 1} = & 5 \cdot 2 + 0 + (-1) \cdot 5 + (-6) \cdot (-1) = 11 \\ \beta = 2 & A^{\alpha} C_{\alpha 2} = & 5 \cdot 2 + 0 + (-1) \cdot 2 + (-6) \cdot (-3) = 26 \\ \beta = 3 & A^{\alpha} C_{\alpha 3} = & 5 \cdot 3 + 0 + (-1) \cdot (-2) + (-6) \cdot (0) = 17 \end{array}$$

c) same as (b) with $\sigma = \beta$.

d)

$$\begin{array}{lll} \mu = 0 & A^{\gamma} C_{0\gamma} = & 5 \cdot 1 + 0 + (-1) \cdot 2 + (-6) \cdot 3 = -15 \\ \mu = 1 & A^{\gamma} C_{1\gamma} = & 5 \cdot 5 + 0 + (-1) \cdot (-2) + (-6) \cdot 0 = 27 \\ \mu = 2 & A^{\gamma} C_{2\gamma} = & 5 \cdot 4 + 0 + (-1) \cdot 2 + (-6) \cdot (-2) = 30 \\ \mu = 3 & A^{\gamma} C_{3\gamma} = & 5 \cdot (-1) + 0 + (-1) \cdot (-3) + (-6) \cdot 0 = -2 \end{array}$$

e)

$$A^{\alpha} B_{\beta} = \begin{pmatrix} 0 & -10 & 20 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & 12 & -24 & 0 \end{pmatrix}$$

where $\alpha = 0, 1, 2, 3$ labels the rows and $\beta = 0, 1, 2, 3$ labels the columns in order.

3. We have

$$\left(a_{\alpha\beta} A^{\alpha} A^{\beta} \right)_{,\gamma} = a_{\alpha\beta,\gamma} A^{\alpha} A^{\beta} + a_{\alpha\beta} A^{\alpha}{}_{,\gamma} A^{\beta} + a_{\alpha\beta} A^{\alpha} A^{\beta}{}_{,\gamma}. \tag{7}$$

Since $a_{\alpha\beta}$ are constant $a_{\alpha\beta},_{\gamma} = 0$. And since $a_{\alpha\beta} = a_{\beta\alpha}$,

$$a_{\alpha\beta}A^{\alpha},_{\gamma} A^{\beta} = a_{\beta\alpha}A^{\alpha},_{\gamma} A^{\beta} = a_{\alpha\beta}A^{\alpha}A^{\beta},_{\gamma} . \tag{8}$$

In the last step we have used the fact that α and β are dummy indices. Finally,

$$\left(a_{\alpha\beta}A^{\alpha}A^{\beta}\right),_{\gamma} = 2a_{\alpha\beta}A^{\alpha}A^{\beta}_{,\gamma}. \tag{9}$$

If $a_{\alpha\beta}$ is antisymmetric, the result is zero:

$$a_{\alpha\beta}A^{\alpha}A^{\beta} = -a_{\beta\alpha}A^{\alpha}A^{\beta} = -a_{\alpha\beta}A^{\beta}A^{\alpha} = -a_{\alpha\beta}A^{\alpha}A^{\beta}$$

The first equality uses the antisymmetry, the second exchanges the dummy indices, and the third uses the commutativity of products.

4.

- $$F_{\alpha'\beta'} = \Lambda^{\mu}_{\ \alpha'}F_{\mu\nu}\Lambda^{\nu}_{\ \beta'} \tag{10}$$

$$F_{\alpha'\beta'} = \begin{pmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & cb_3 & -cb_2 \\ -e_2 & -cb_3 & 0 & cb_1 \\ -e_3 & cb_2 & -cb_1 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \frac{v}{c}\gamma & 0 & 0 \\ \frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{11}$$

$$= \begin{pmatrix} 0 & -e_1 & -\gamma e_2 + \gamma v b_3 & -\gamma e_3 - \gamma v b_2 \\ e_1 & 0 & \gamma\frac{v}{c}e_2 + \gamma c b_3 & -\gamma\frac{v}{c}e_3 - \gamma c b_2 \\ \gamma e_2 - \gamma v b_3 & -e_2\gamma\frac{v}{c} - \gamma c b_3 & 0 & cb_1 \\ \gamma e_3 + \gamma v b_2 & e_3\gamma\frac{v}{c} + \gamma c b_2 & -cb_1 & 0 \end{pmatrix}. \tag{12}$$

- $$F^{\mu\nu} = g^{\mu\alpha}F_{\alpha\beta}g^{\beta\nu} \tag{13}$$

$$F^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & cb_3 & -cb_2 \\ -e_2 & -cb_3 & 0 & cb_1 \\ -e_3 & cb_2 & -cb_1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

$$= \begin{pmatrix} 0 & -e_1 & -e_2 & -e_3 \\ e_1 & 0 & cb_3 & -cb_2 \\ e_2 & -cb_3 & 0 & cb_1 \\ e_3 & cb_2 & -cb_1 & 0 \end{pmatrix}. \tag{15}$$

- $F^{\mu\nu}F_{\nu\mu}$ = Trace $(F^{\alpha\nu}F_{\nu\beta})$= sum of diagonal elements. By multiplying $F^{\alpha\nu}F_{\nu\beta}$ the diagonal elements are

$$\begin{pmatrix} e_1^2 + e_2^2 + e_3^2 & & & \\ & e_1^2 - c^2(b_2^2 + b_3^2) & & \\ & & e_2^2 - c^2(b_1^2 + b_3^2) & \\ & & & e_3^2 - c^2(b_1^2 + b_2^2) \end{pmatrix}. \tag{16}$$

It follows that

$$F^{\mu\nu}F_{\nu\mu} = 2\left[e_1^2 + e_2^2 + e_3^2 - c^2(b_1^2 + b_2^2 + b_3^2)\right] = 2\left(|\mathbf{e}|^2 - c^2\,|\mathbf{b}|^2\right) \tag{17}$$

is a scalar.