Special Relativity Sheet 5

1. Prove that the 4-velocity and 4-momentum are 4-vectors. Explain why

$$\left(\frac{dt^0}{dt}, \frac{dX^1}{dt}, \frac{dX^2}{dt}, \frac{dX^3}{dt}\right) \tag{1}$$

is not a 4-vector.

2. The world-line of a particle in an inertial frame S is $x(t) = at + b\sin(\omega t)$; $y(t) = b\cos(\omega t)$; $z(t) = 0$. Compute the particle's 4-velocity and 4-acceleration.

3. A and B are two particles of equal rest mass. In an inertal frame S, A is located at the origin and is stationary, and B impacts it with velocity $(u, 0, 0)$ along the x-axis. Find another inertial frame S_{com}, in standard configuration with S, so that in S_{com} the velocities of A and B are $(-v, 0, 0)$ and $(v, 0, 0)$, respectively.

Show that

$$2\gamma^2(v) = \gamma(u) + 1. \tag{2}$$

By applying conservation of 4-momentum in S_{com}, deduce that in this frame the two particles move with equal speeds but in opposite directions after the impact. By transforming the post-collision velocities back to S, show that

$$\tan(\theta_A)\tan(\theta_B) = \frac{-2}{\gamma(u)+1}, \tag{3}$$

where θ_A and θ_B are the angles of the velocities to the x-axis of A and B after the impact.

Show that in non-relativistic Newtonian Mechanics

$$\tan(\theta_A)\tan(\theta_B) = -1. \tag{4}$$

4. Show that Maxwell's equations imply the continuity equation (conservation of charge) using both the 3 and 4 dimensional formulations. Which is easier?

5. An observer S observes waves moving with velocity $v < c$ and satisfying

$$\frac{1}{v^2}\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0$$

in her inertial frame. Construct a relativistically invariant equation for these waves, using the 4-velocity of S. Assume that ϕ is a 4-scalar. Find but do not solve, the equation relating ω and k for these waves in a general reference frame S', and show that $\omega = kv$ in S.

Relativity solutions 5

1.

$$\begin{aligned} U^{\alpha'} &= \frac{dX^{\alpha'}}{d\tau} = \frac{\partial X^{\alpha'}}{\partial X^{\beta}}\frac{dX^{\beta}}{d\tau} && (5)\\ \text{therefore} \quad U^{\alpha'} &= \Lambda^{\alpha'}_{\ \beta}\, U^{\beta} && (6)\end{aligned}$$

That is, U^{α} transforms like the component of a 4-vector.

Since $P^{\alpha} = mU^{\alpha}$, where m is a constant, P^{α} transforms like U^{α} and so $\underline{P}$ is also a 4-vector.

$$\frac{dX^{\alpha'}}{dt'} = \frac{\partial X^{\alpha'}}{\partial X^{\beta}}\frac{dX^{\beta}}{dt'} = \frac{\partial X^{\alpha'}}{\partial X^{\beta}}\frac{dX^{\beta}}{dt}\frac{dt}{dt'} = \left(\Lambda^{\alpha'}_{\ \beta}\frac{dX^{\beta}}{dt}\right)\frac{dt}{dt'}. \tag{7}$$

But $dt/dt' \neq 1$ and so dX^{α}/dt does not transform like a 4-vector.

2.

$$\begin{aligned} x(t) &= at + b\sin(\omega t) && (8)\\ y(t) &= b\cos(\omega t) && (9)\\ z(t) &= 0 && (10)\end{aligned}$$

$$\begin{aligned} \mathbf{u} &= (a + b\omega\cos(\omega t), -b\omega\sin(\omega t), 0) && (11)\\ \mathbf{a} &= (-b\omega^2\sin(\omega t), -b\omega^2\cos(\omega t), 0) && (12)\\ u &= \sqrt{(a^2 + b^2\omega^2 + 2ab\omega\cos(\omega t))} && (13)\end{aligned}$$

As usual, $\gamma(u) = 1/\sqrt{1 - u^2/c^2}$, so we have the 4-velocity

$$\underline{U} = \gamma(u)(c, \mathbf{u}), \tag{14}$$

and the 4-acceleration

$$\begin{aligned} \mathbf{a} &= \gamma(u)\left(c\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\mathbf{u} + \gamma(u)\mathbf{a}\right) && (15)\\ &= \gamma(u)\left(c\frac{u}{c^2}\gamma^3(u)\frac{du}{dt}, \frac{u}{c^2}\gamma^3(u)\frac{du}{dt} + \gamma(u)\mathbf{a}\right), && (16)\end{aligned}$$

where

$$\frac{du}{dt} = \frac{-ab\omega^2\sin(\omega t)}{u}. \tag{17}$$

3.

In S_{com} we have $v'_A = \frac{0-v_{com}}{1} = -v$, so $v_{com} = v$.
Then $v'_B = \frac{u-v}{1-uv/c^2} = v$ and this implies

$$u = \frac{2v}{1+v^2/c^2}. \tag{18}$$

Thus

$$\gamma(u) = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1+v^2/c^2}{1-v^2/c^2} \tag{19}$$

and so

$$1 + \gamma(u) = 2\gamma^2(v). \tag{20}$$

Still in S_{com}, we equate the pre- and post- 4-momenta:

$$(2\gamma(v), \mathbf{0}) = (\gamma(v''_A) + \gamma(v''_B), \mathbf{v}''_A\gamma(v''_A) + \mathbf{v}''_B\gamma(v''_B)) \tag{21}$$

The solution of this is $v''_A = v''_B = v$ and $\mathbf{v}''_A = -\mathbf{v}''_B$.

Now we transform back into S, and after the collision the velocities of A and B must be

$$\mathbf{v}_A = \left(\frac{v\cos\theta' + v}{1 + \frac{v^2}{c^2}\cos\theta'}, \frac{v\sin\theta'}{\gamma(v)\left(1 + \frac{v^2}{c^2}\cos\theta'\right)}, 0 \right) \tag{22}$$

$$\mathbf{v}_B = \left(\frac{-v\cos\theta' + v}{1 - \frac{v^2}{c^2}\cos\theta'}, \frac{-v\sin\theta'}{\gamma(v)\left(1 - \frac{v^2}{c^2}\cos\theta'\right)}, 0 \right) \tag{23}$$

Hence

$$\tan\theta_A = \frac{\sin\theta'}{(1+\cos\theta')\gamma(v)} \tag{24}$$

$$\tan\theta_B = \frac{-\sin\theta'}{(1-\cos\theta')\gamma(v)}. \tag{25}$$

Thus

$$\tan\theta_A \;\tan\theta_B = \frac{-\sin^2\theta'}{\gamma^2(v)(1-\cos^2\theta')} \tag{26}$$

$$= \frac{-1}{\gamma^2(v)} \tag{27}$$

$$= \frac{-2}{1+\gamma(u)}. \tag{28}$$

In the Newtonian limit $c \to \infty$, $\gamma \to 1$ and so $\tan\theta_A \;\tan\theta_B \to -1$.

4. For the three dimensional formulation we need two Maxwell equations:

$$\nabla \cdot \mathbf{e} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \mathbf{b} = \mu_0(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{e}}{\partial t})$$

Taking the time derivative of the first and the divergence of the second, we find

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = \epsilon_0 \frac{\partial}{\partial t}\nabla \cdot \mathbf{e} + \frac{1}{\mu_0}\nabla \cdot (\nabla \times \mathbf{b}) - \epsilon_0 \nabla \cdot \frac{\partial \mathbf{e}}{\partial t}$$

The right hand side is zero, because the first and third terms cancel (the time derivative and divergence commute), and the second is zero (the divergence of a curl is zero). Thus we have the continuity equation.

For the four dimensional formulation, we have

$$E^{\mu\nu}_{\ \ ,\mu} = J^\nu/(c\epsilon_0)$$

Taking a 4-divergence, we find

$$J^{\nu}_{,\nu} = c\epsilon_0 E^{\mu\nu}{}_{,\mu\nu} = 0$$

since $E^{\mu\nu}$ is antisymmetric and the derivatives are symmetric. This is surely simpler than the three dimensional version.

5. We know that

$$\phi^{,\mu}{}_{\mu} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2\phi$$

but the coefficent of the time derivatives needs to be modified. The 4-velocity of S in reference frame S is $U^\mu = (c, 0, 0, 0)$, so that the combination

$$\phi_{,\mu\nu}U^\mu U^\nu = \frac{\partial^2\phi}{\partial t^2}$$

extracts the time derivatives. Thus we can write

$$\phi^{,\mu}{}_{\mu} + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\phi_{,\mu\nu}U^\mu U^\nu = 0$$

which is a tensor equation, valid in S and therefore valid in all inertial reference frames.

If we substitute $\phi(t, x, y, z) = Ce^{ik_\mu x^\mu}$ where C is a constant and $k^\mu = (\omega/c, \mathbf{k})$, the resulting equation is

$$k^\mu k_\mu + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)(k_\mu U^\mu)^2 = 0$$

which in three dimensional form reads (in a general reference frame S')

$$\left(\frac{\omega^2}{c^2} - k^2\right) + \left(\frac{1}{v^2} - \frac{1}{c^2}\right)\gamma(u)^2(\omega - \mathbf{k}\cdot\mathbf{u})^2 = 0$$

where $\mathbf{u}$ is the velocity of S in frame S'. If $S = S'$ this reduces to $\omega = \pm kv$. We take the positive solution as ω and k are usually defined to be magnitudes, ie positive.