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Lyapunov Spectra of Periodic Orbits for a Many-Particle System

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The Lyapunov spectrum corresponding to a periodic orbit for a two-dimensional many-particle system with hard core interactions is discussed. Noting that the matrix to describe the tangent space dynamics has the block cyclic structure, the calculation of the Lyapunov spectrum is attributed to the eigenvalue problem of 16×16 reduced matrices regardless of the number of particles. We show that there is the thermodynamic limit of the Lyapunov spectrum in this periodic orbit. The Lyapunov spectrum has a step structure, which is explained by using symmetries of the reduced matrices.

KEY WORDS: Periodic orbits; hard ball systems; Lyapunov spectrum; step structure; thermodynamic limit.

1. INTRODUCTION

Just as low dimensional chaos is revealed by a single positive Lyapunov exponent which indicates exponential sensitivity to initial conditions, systems with phase space of very high dimension can be characterized by their Lyapunov spectra which give information about many possible instabilities in the system. As a concrete example, we consider the system consisting of N disks with hard core interactions and periodic boundary conditions. This is a surprisingly good model of a fluid,[(1)] and yet is sufficiently simple that ergodic properties may be established under fairly general conditions.[(2)]

The following is a brief review of the study of Lyapunov spectra in such systems; for more details and references in numerical work, see ref. 3

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and in analytical work, see ref. 4. It has been known for some time that the Lyapunov exponents of Hamiltonian systems come in plus/minus pairs, that is, the spectrum is symmetric about zero.[5] Non-equilibrium extensions were shown to exhibit symmetry about a point other than zero,[6,7] leading to the discovery that these extensions also contain hidden Hamiltonian structure.[8,9] Also known for more than twenty years is the algorithm for numerical computation of Lyapunov exponents due to Benettin and others.[10,11] Later, a constraint method was introduced.[12] More recently, the effects of the hard collisions have been properly taken into account.[3,13] The existence of a thermodynamic limit in Lyapunov spectra, that is, that the spectrum retains its shape as the number of particles increases, has been put forward using random matrix approximations,[14] numerical evidence,[15] and mathematical arguments,[16] but recent numerical work has suggested in contrast, a logarithmic singularity of the largest Lyapunov exponent with the number of particles.[17] Lyapunov spectra for diatomic molecules (represented by dumb-bells) show an explicit separation of the rotational and translational degrees of freedom if the departure from sphericity is small enough.[18] Finally, and of particular interest to us, careful simulations of sufficiently large systems have revealed a step structure in the Lyapunov spectrum for the *smallest* positive Lyapunov exponents.[3,13,18] The above references give an incomplete description in terms of "Posch Lyapunov modes," phase space perturbations corresponding to these small Lyapunov exponents which are approximately sinusoidal in position space. Eckmann and Gat have suggested an explanation of the Lyapunov modes of a one-dimensional system using a random matrix approximation of the Lyapunov spectrum.[19] Recently the kinetic approach[20] and the master equation approach[21] were also proposed to explain the Lyapunov mode and stepwise structure of the Lyapunov spectrum. In this paper we study step structure in the Lyapunov spectrum of a two-dimensional many-particle system without making any approximations, however we are restricted to periodic orbits.

Periodic orbit theory[22,23] has proven very useful for investigations of the corresponding low dimensional system, the periodic Lorentz gas,[24–27] computing properties such as the diffusion coefficient. These methods cannot generally be applied directly to high dimensional systems due to the difficulty of finding all the periodic orbits, although a notable exception is the Kuramoto-Shivashinsky PDE in a regime where the *effective* number of degrees of freedom is small.[28] However, periodic orbit arguments have been used to justify thermodynamic results such as non-negativity of the entropy production[29] and the Onsager relations[30] without explicitly finding any periodic orbits. Periodic orbits of spatially extended systems in the form of coupled map lattices have been considered previously,[31, 32] leading to block cyclic matrices similar to those observed in this paper.

Here we apply methods similar to those to ref. 32 to reduce the problem of computing the Lyapunov spectrum to that of finding the eigenvalues of a relatively small (16×16) matrix. A step structure is observed, which is related to the symmetries of the system. Our formalism shows that, at least in the case of this periodic orbit, the thermodynamic limit of the Lyapunov spectrum holds exactly.

2. LYAPUNOV EXPONENTS OF PERIODIC ORBITS FOR THE MANY-PARTICLE SYSTEM

The system which we consider in this paper is a two-dimensional Hamiltonian system consisting of N disks, interacting only by hard core collisions. We choose units such that the mass and radius of the particles are one. We write the position and the momentum of the jth particle as $\mathbf{q}_j$ and $\mathbf{p}_j$, respectively, and for a later convenience we represent the phase space vector $\mathbf{\Gamma}$ as a column vector $(\mathbf{q}_1, \mathbf{p}_1, \mathbf{q}_2, \mathbf{p}_2, \ldots, \mathbf{q}_N, \mathbf{p}_N)^T$ with T the transpose operation.

The dynamics of such a many-particle system is simply separated into the free flight part and the collision part, and the tangent vector $\delta\mathbf{\Gamma}_n$ of the phase space just after the nth collision occurs is related to the tangent vector $\delta\mathbf{\Gamma}_0$ at the initial time by $\delta\mathbf{\Gamma}_n = M_n\,\delta\mathbf{\Gamma}_0$ with the $4N \times 4N$ matrix M_n represented as

$$M_n \equiv M_n^{(c)} M_n^{(f)} M_{n-1}^{(c)} M_{n-1}^{(f)} \cdots M_1^{(c)} M_1^{(f)}. \tag{1}$$

Here $M_j^{(f)}$ is the $4N \times 4N$ matrix to specify the jth free flight dynamics, and is given by

$$M_j^{(f)} \equiv \mathrm{Diag}(L_1^{(1)}, L_2^{(1)}, \ldots, L_N^{(1)}) \tag{2}$$

where $\mathrm{Diag}(X_1, X_2, \ldots, X_l)$ means the matrix on whose diagonal are the matrix blocks $X_1, X_2, \ldots, X_l$ for an integer l, and $L_j^{(1)}$ is defined by

$$L_j^{(1)} \equiv \begin{pmatrix} \underline{I} & \tau_j \underline{I} \\ \underline{0} & \underline{I} \end{pmatrix} \tag{3}$$

where $\underline{I}$ and $\underline{0}$ are 2×2 identical and null matrices, respectively, and τ_j is its corresponding free flight time. On the other hand, $M_j^{(c)}$ is the $4N \times 4N$ matrix to specify the jth collision of particles. For the simplest case in which the jth collision involves only the k_jth and the l_jth particles colliding, the matrix $M_j^{(c)}$ is given as the block matrix

$$
\begin{array}{cc}
 & \begin{array}{ccccc} & k_j & & l_j & \\ & \downarrow & & \downarrow & \end{array} \\
M_j^{(c)} \equiv \begin{array}{c} \\ \\ k_j \to \\ \\ l_j \to \\ \\ \\ \end{array} &
\begin{pmatrix}
\tilde{I} & & & & \\
 & \ddots & & & \\
 & & F_j & & G_j & \\
 & & & \ddots & & \\
 & & G_j & & F_j & \\
 & & & & & \ddots & \\
 & & & & & & \tilde{I}
\end{pmatrix}
\end{array} \tag{4}
$$

where $\tilde{I}$ is the 4×4 identity matrix and is put in the part of $\ddots$ of the above representation, and F_j (G_j) is the (k_j, k_j) and (l_j, l_j) (((k_j, l_j) and (l_j, k_j))) block matrix elements, and the 4×4 null matrices are put in the other elements. Here 4×4 matrices F_j and G_j are defined by[13]

$$
F_j \equiv \begin{pmatrix} \underline{I} - L_j^{(2)} & \underline{0} \\ -L_j^{(3)} & \underline{I} - L_j^{(2)} \end{pmatrix} \tag{5}
$$

$$
G_j \equiv \begin{pmatrix} L_j^{(2)} & \underline{0} \\ L_j^{(3)} & L_j^{(2)} \end{pmatrix} \tag{6}
$$

with the 2×2 matrices $L_j^{(2)}$ and $L_j^{(3)}$ defined by

$$
L_j^{(2)} \equiv \mathbf{n}_j \mathbf{n}_j^T \tag{7}
$$

$$
L_j^{(3)} \equiv \mathbf{n}_j^T \,\Delta\mathbf{p}^{(j)} \left(\underline{I} + \frac{\mathbf{n}_j \,\Delta\mathbf{p}^{(j)T}}{\mathbf{n}_j^T \,\Delta\mathbf{p}^{(j)}} \right) \left(\underline{I} - \frac{\Delta\mathbf{p}^{(j)} \mathbf{n}^T}{\Delta\mathbf{p}^{(j)T} \mathbf{n}_j} \right), \tag{8}
$$

(Note that all vectors in this paper are introduced as column vectors, so for example, $\mathbf{n}_j^T \,\Delta\mathbf{p}^{(j)}$ is a scalar and $\mathbf{n}_j \,\Delta\mathbf{p}^{(j)T}$ is a matrix.) where $\mathbf{n}_j$ is the unit vector pointing from the center of the k_jth disk to the center of the l_jth disk at the jth collision, and $\Delta\mathbf{p}^{(j)}$ is the momentum difference $\mathbf{p}_{l_j} - \mathbf{p}_{k_j}$ just before the jth collision.

Now we consider the case that the movement of the system is periodic in time, so that the condition $\mathbf{\Gamma}_{n_p} = \mathbf{\Gamma}_0$ is satisfied for a integer n_p. In this case the Lyapunov exponents λ_j, $j = 1, 2, \ldots, 4N$ of the periodic orbit are defined by $\lambda_j = (1/t(n_p)) \ln |m_j(n_p)|$ in terms of the absolute value of the (generally complex) eigenvalue $m_j(n_p)$ of the matrix M_{n_p}, where $t(n_p) \equiv \sum_{j=1}^{n_p} \tau_j$ is the period of this orbit.

We put the set of the Lyapunov exponents in descending order, namely $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{4N}$. It is well known that in the time-independent

Hamiltonian system the Lyapunov exponents satisfy the pairing rule,[5] namely the condition $\lambda_1 + \lambda_{4N} = \lambda_2 + \lambda_{4N-1} = \cdots = \lambda_{2N} + \lambda_{2N+1} = 0$. Noting this fact, thereafter we consider only the first half λ_1, λ_2,..., λ_{2N} of the full Lyapunov exponents, and refer to their set as the Lyapunov spectrum in this paper.

3. PERIODIC ORBIT MODEL AND THE REDUCED MATRIX

By using the method given in the previous section, we calculate the Lyapunov spectrum for the periodic orbit illustrated in Fig. 1. In this periodic orbit each particle moves in a square orbit with the same absolute value of momentum and the same direction of rotation and with a constant free flight time τ (So the period of the orbit is $t(4) = 4\tau$.). We put $2N_1$ ($2N_2$) as the number of the particles in each horizontal line (each vertical line) so that $N = 4N_1N_2$. We impose periodic boundary conditions, thus requiring the number of particles in each direction to be even.

In this periodic orbit model, there are only two types of collisions illustrated in Fig. 2, in which the trajectories of two colliding particles are drawn with their moving directions shown by the arrows. Now we consider the dynamics of particles for a free flight plus one of these two collisions. Such a dynamics is described by the matrix multiplications

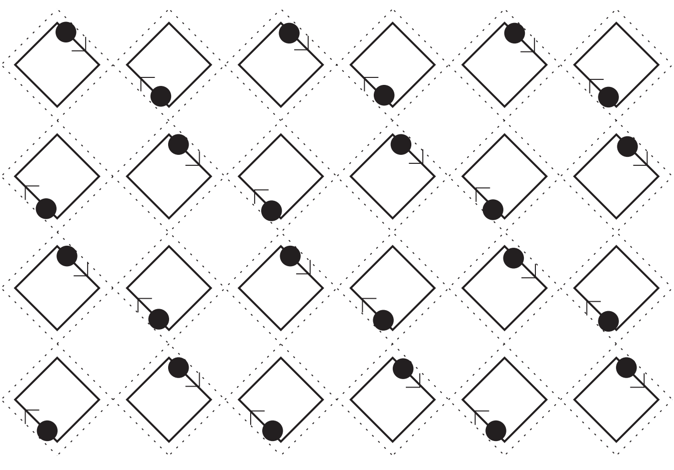

Fig. 1. Periodic orbit of the many-particle system. The circular dots show the positions of particles at the initial time, and the solid lines give the subsequent path in the direction of the arrows.

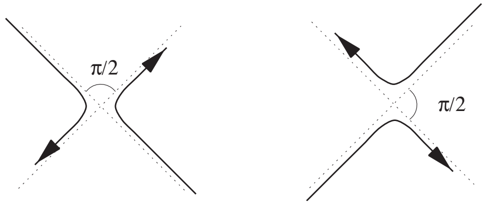

Fig. 2. Two types of particle collisions. The solid lines give the path of the particles, which move in the direction shown by the arrows.

$\tilde{F}_1 \equiv F_1 \,\mathrm{Diag}(L_1^{(1)}, L_1^{(1)})$ and $\tilde{G}_1 \equiv G_1 \,\mathrm{Diag}(L_1^{(1)}, L_1^{(1)})$ ($\tilde{F}_2 \equiv F_2 \,\mathrm{Diag}(L_2^{(1)}, L_2^{(1)})$ and $\tilde{G}_2 \equiv G_2 \,\mathrm{Diag}(L_2^{(1)}, L_2^{(1)})$) corresponding to the left (right) orbit in Fig. 2. By using Eqs. (5)–(8) these matrices are simply given by

$$\tilde{F}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \tau \\ -v & -v & -v\tau & -v\tau \\ v & v & v\tau & 1+v\tau \end{pmatrix} \tag{9}$$

$$\tilde{G}_1 = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 \\ v & v & 1+v\tau & v\tau \\ -v & -v & -v\tau & -v\tau \end{pmatrix} \tag{10}$$

$$\tilde{F}_2 = \begin{pmatrix} 1 & 0 & \tau & 0 \\ 0 & 0 & 0 & 0 \\ v & -v & 1+v\tau & -v\tau \\ v & -v & v\tau & -v\tau \end{pmatrix} \tag{11}$$

$$\tilde{G}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \tau \\ -v & v & -v\tau & v\tau \\ -v & v & -v\tau & 1+v\tau \end{pmatrix}, \tag{12}$$

where one of the components of the collision vector is zero, namely $\mathbf{n} = (1, 0)^T$ or $(0, 1)^T$, and $|v|/\sqrt{2}$ is the speed of the particles.

The $4N \times 4N$ matrices $M_j^{(c)} M_j^{(f)}$, $j = 1, 2, 3, 4$ are represented as

$$M_1^{(c)} M_1^{(f)} = \mathrm{Diag}(P_1, P_2, P_1, P_2, \ldots, P_1, P_2) \tag{13}$$

$$M_2^{(c)} M_2^{(f)} = \begin{pmatrix} Q_0 & Q_1 & & & & & Q_2 \\ Q_1 & Q_0 & Q_2 & & & & \\ & Q_2 & Q_0 & Q_1 & & & \\ & & Q_1 & Q_0 & Q_2 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & Q_2 & Q_0 & Q_1 \\ Q_2 & & & & & Q_1 & Q_0 \end{pmatrix} \tag{14}$$

$$M_3^{(c)} M_3^{(f)} = \mathrm{Diag}(P_2, P_1, P_2, P_1, \ldots, P_2, P_1) \tag{15}$$

$$M_4^{(c)} M_4^{(f)} = \begin{pmatrix} Q_0 & Q_2 & & & & & Q_1 \\ Q_2 & Q_0 & Q_1 & & & & \\ & Q_1 & Q_0 & Q_2 & & & \\ & & Q_2 & Q_0 & Q_1 & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & Q_1 & Q_0 & Q_2 \\ Q_1 & & & & & Q_2 & Q_0 \end{pmatrix} \tag{16}$$

where the $8N_1 \times 8N_1$ matrices P_j and Q_j are defined by

$$P_1 = \begin{pmatrix} \tilde{F}_1 & \tilde{G}_1 & & & & & & & \\ \tilde{G}_1 & \tilde{F}_0 & & & & & & & \\ & & \tilde{F}_1 & \tilde{G}_1 & & & & & \\ & & \tilde{G}_1 & \tilde{F}_1 & & & & & \\ & & & & \tilde{F}_1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \tilde{F}_1 & & \\ & & & & & & & \tilde{F}_1 & \tilde{G}_1 \\ & & & & & & & \tilde{G}_1 & \tilde{F}_1 \end{pmatrix} \tag{17}$$

$$P_2 = \begin{pmatrix} \tilde{F}_1 & & & & & & & & \tilde{G}_1 \\ & \tilde{F}_0 & \tilde{G}_1 & & & & & & \\ & \tilde{G}_1 & \tilde{F}_1 & & & & & & \\ & & & \tilde{F}_1 & \tilde{G}_1 & & & & \\ & & & \tilde{G}_1 & \tilde{F}_1 & & & & \\ & & & & & \ddots & & & \\ & & & & & & \tilde{F}_1 & \tilde{G}_1 & \\ & & & & & & \tilde{G}_1 & \tilde{F}_1 & \\ \tilde{G}_1 & & & & & & & & \tilde{F}_1 \end{pmatrix} \tag{18}$$

$$Q_0 = \mathrm{Diag}(\tilde{F}_2, \tilde{F}_2, \tilde{F}_2, \tilde{F}_2, \ldots, \tilde{F}_2, \tilde{F}_2) \tag{19}$$

$$Q_1 = \mathrm{Diag}(\tilde{G}_2, 0, \tilde{G}_2, 0, \ldots, \tilde{G}_2, 0). \tag{20}$$

$$Q_2 = \mathrm{Diag}(0, \tilde{G}_2, 0, \tilde{G}_2, \ldots, 0, \tilde{G}_2) \tag{21}$$

In order to get these expression for the matrices the first horizontal row of particles is numbered $1, 2, \ldots, 2N_1$ from left to right, the second row numbered $2N_1+1, 2N_1+2, \ldots, 4N_1$ and so on until the last row numbered $(2N_2-1)\, 2N_1+1, (2N_2-1)\, 2N_1+2, \ldots, 4N_2N_1$.

The matrix M_{n_p}, whose eigenvalues lead to the Lyapunov spectrum, is given by $M_{n_p} = M_4^{(c)} M_4^{(f)} M_3^{(c)} M_3^{(f)} M_2^{(c)} M_2^{(f)} M_1^{(c)} M_1^{(f)}$.

It is very important to note that this periodic model is invariant with respect to translations horizontally or vertically by the distance corresponding to two particles. This feature is reflected as block-cyclic structures in the matrix M_{n_p}. By using this fact, as shown in Appendix A, we can show that the eigenvalues of the matrix M_{n_p} are equal to the eigenvalues of the 16×16 matrices $\mathscr{M}(2\pi n_1/N_1, 2\pi n_2/N_2)$, $n_1 = 1, 2, \ldots, N_1$, $n_2 = 1, 2, \ldots, N_2$ with the matrix $\mathscr{M}(k, l)$ defined by

$$\mathscr{M}(k, l) \equiv \begin{pmatrix} S_1(-k, -l) & S_2(-k, -l) & T_1(k, l)\, e^{-il} & T_2(k, l)\, e^{-i(k+l)} \\ S_2(k, l) & S_1(k, l) & T_2(-k, -l)\, e^{ik} & T_1(-k, -l) \\ T_1(-k, -l)\, e^{il} & T_2(-k, -l)\, e^{il} & S_1(k, l) & S_2(k, l)\, e^{-ik} \\ T_2(k, l) & T_1(k, l) & S_2(-k, -l)\, e^{ik} & S_1(-k, -l) \end{pmatrix}. \tag{22}$$

Here, $S_1(k, l)$, $S_2(k, l)$, $T_1(k, l)$ and $T_2(k, l)$ are defined by

$$S_1(k, l) \equiv (\tilde{F}_2\tilde{F}_1)^2 + (\tilde{G}_2\tilde{G}_1)^2 + (\tilde{F}_2\tilde{G}_1)^2\, e^{ik} + (\tilde{G}_2\tilde{F}_1)^2\, e^{il} \tag{23}$$

$$S_2(k, l) \equiv \tilde{F}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{G}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{F}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 e^{ik} + \tilde{G}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{il} \tag{24}$$

$$T_1(k,l) \equiv \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{G}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{F}_1 + \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{G}_1 e^{ik} + \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{F}_1 e^{il} \tag{25}$$

$$T_2(k,l) \equiv \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 e^{ik} + \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{il}. \tag{26}$$

In the next section we investigate the Lyapunov spectrum of this periodic orbit model by using the eigenvalues of the matrix $\mathscr{M}(k,l)$.

4. LYAPUNOV SPECTRUM AND ITS STEP STRUCTURE

The reduced matrix (22) is useful, not only to reduce the calculation time to get the full Lyapunov spectrum, but also to allow us to consider some properties of the Lyapunov spectrum itself. One of important results obtained by such a consideration using the reduced matrix is the existence of the thermodynamic limit in the Lyapunov spectrum. It should be noted that the N_1N_2 matrices $\mathscr{M}(k,l)$, with $k = 2\pi n_1/N_1$, $l = 2\pi n_2/N_2$, $n_1 = 1, 2, \ldots, N_1$ and $n_2 = 1, 2, \ldots, N_2$ have the same form of the matrix given by Eq. (22), so in the limit $N_1 \to +\infty$ and $N_2 \to +\infty$ the Lyapunov spectrum is given through the eigenvalues of the matrices $\mathscr{M}(k,l)$, $k \in [0, 2\pi)$ and $l \in [0, 2\pi)$. This also implies that the maximum Lyapunov exponent takes a finite value in the thermodynamic limit in this periodic orbit model.

Now we calculate the Lyapunov spectrum in our model. Figure 3 is the Lyapunov spectrum in the case of $v = 1.8$, $\tau = 2.3$, $N_1 = 11$ and $N_2 = 9$ (So the total number of particles is $N = 4N_1N_2 = 396$). One of the remarkable features of this Lyapunov spectrum is its step structure. It is important to note that some steps of the Lyapunov spectrum are explained by using symmetries of the reduced matrix (22). Actually we can show (at least

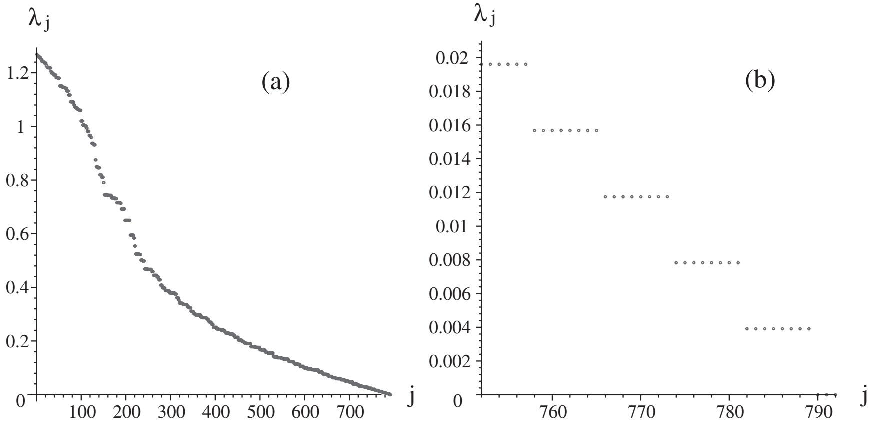

Fig. 3. Lyapunov spectrum in the case of $(N_1, N_2) = (11, 9)$. (a) Full scale. (b) Large j part.

numerically) that in the case of Fig. 3 the Lyapunov exponents calculated using the matrix $\mathscr{M}(k, l)$ are invariant under the transformations $k \to 2\pi - k$ and $l \to 2\pi - l$, leading to steps in the Lyapunov spectrum. Performing both of these transformations at once has the effect of taking the complex conjugate of $\mathscr{M}$ so the result is obvious, however it is not immediately obvious that the spectrum should be invariant if k and l are transformed separately.

We can investigate the relation between a symmetry of the system and the step structure in the Lyapunov spectrum another way. For this purpose we consider the Lyapunov spectra in the square case and the rectangular case with the same number of particles. Here, the square system has the symmetry for the exchange of the vertical and the horizontal directions, which the rectangular system does not have. Figure 4 is the Lyapunov spectra in the case of $(N_1, N_2) = (12, 12)$ (the upper two graphs) and in the case $(N_1, N_2) = (24, 6)$ (the lower two graphs) with $v = 1.8$ and $\tau = 2.3$. These graphs show that there is not a remarkable difference in the global shapes of the Lyapunov spectrum in these two cases, but the square system has (even twice) longer steps in the Lyapunov spectrum than in the rectangular system, shown in Figs. 4(b) and (d). These longer steps come from an additional symmetry in the square system. Actually we can check numerically that the Lyapunov exponents obtained from the reduced matrix (22) are invariant under $k \leftrightarrow l$, which leads to more degeneracy in the case $N_1 = N_2 = 12$ than when $(N_1, N_2) = (24, 6)$. For $N_1 = N_2$ all that is required is that n_1 and n_2 are interchanged. For $(N_1, N_2) = (24, 6)$ degeneracy by this mechanism only occurs if n_1 is divisible by 4 which is much rarer.

It should be noted that some steps of the Lyapunov spectra are too close to be distinguished in Fig. 4. For example, if we can investigate more precisely, we can see 5 different steps in the long flat part of the Lyapunov exponents λ_j in $j \in [1018, 1097]$ (in $j \in [1086, 1125]$) in the Lyapunov spectrum in the case of $(N_1, N_2) = (12, 12)$ (in the case of $(N_1, N_2) = (24, 6)$). Similarly, most of the Lyapunov exponents λ_j, $j \in [1106, 1152]$ ($j \in [1130, 1152]$)) in the case of $(N_1, N_2) = (12, 12)$ (in the case of $(N_1, N_2) = (24, 6)$) are not zero, just too small for the scale of the plot.

5. COMPARISON WITH ANOTHER ORBIT

In general, the block cyclic technique in the previous sections can be applied to calculate Lyapunov spectra for space and time periodic orbits. Figure 5(a) is another example of such periodic orbits. In this periodic orbit each particle follows a zigzag trajectory with the same absolute value of momentum and with a constant free flight time, and have only two collision types as shown in Fig. 5(b). We impose periodic boundary conditions

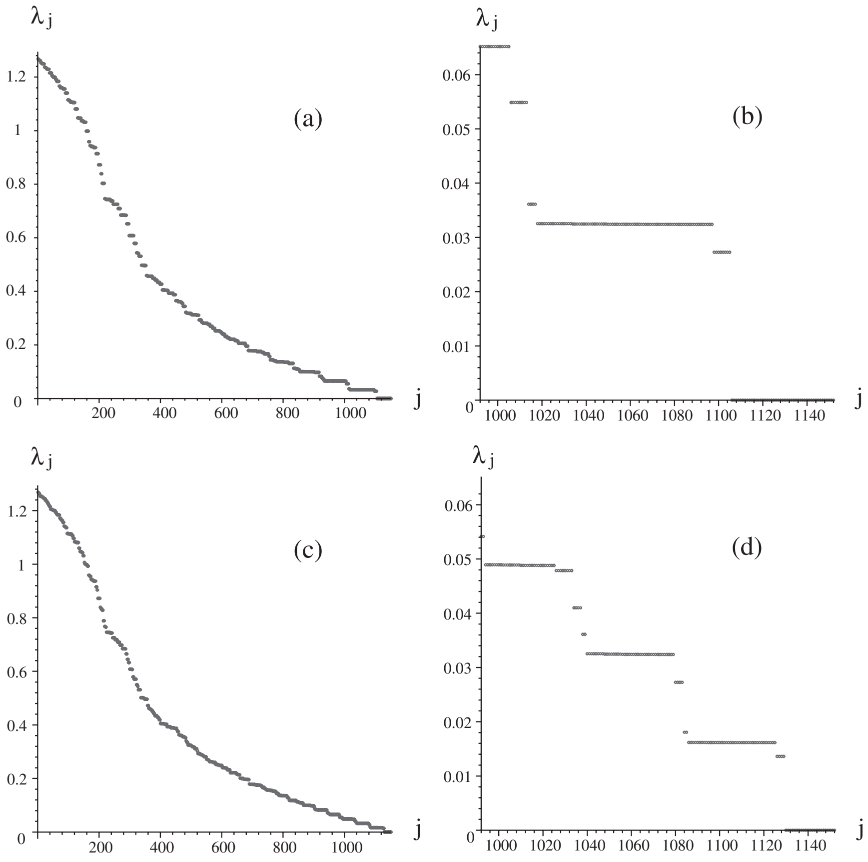

Fig. 4. Lyapunov spectra in the cases of a square system $(N_1, N_2) = (12, 12)$ ((a) Full scale. (b) Large j part.) and a rectangular system $(N_1, N_2) = (24, 6)$ ((c) Full scale. (d) Large j part.).

and put $2\bar{N}_1$ and $\bar{N}_2$ as the number of the zigzag lines and the number of the particles in each zigzag line, respectively.

Figure 5(c) is the Lyapunov spectrum for the periodic orbit shown in Fig. 5(a) in the case of $(\bar{N}_1, \bar{N}_2) = (20, 3)$ with its enlarged graph (d) in the region of small positive Lyapunov exponents. Here we used the same speed of the particle and the same free flight time as in the Lyapunov spectra given in Figs. 3 and 4. (The explicit calculation of this Lyapunov spectrum is similar to that in Section 3 and Appendix A, so it is omitted.) We can see clearly a stepwise structure of the Lyapunov spectrum in this periodic orbit, like the previous periodic orbit model. The Lyapunov spectrum looks broadly similar, lending hope to the idea that the final Lyapunov spectrum

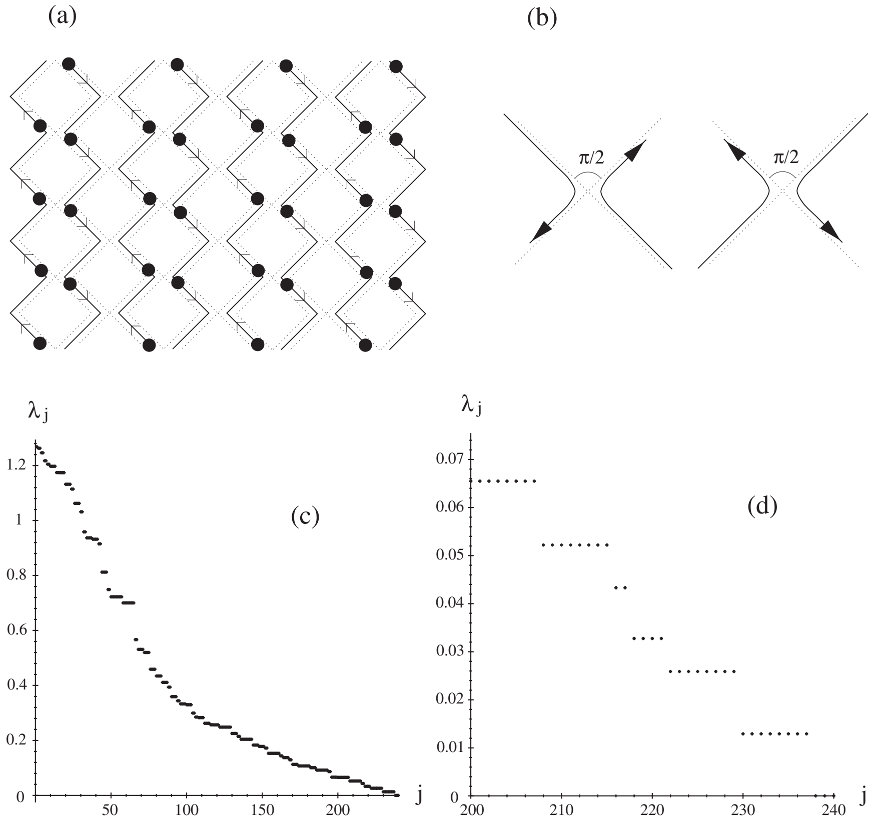

Fig. 5. Another periodic orbit (a) with the collision types (b) and its Lyapunov spectrum ((c) Full scale. (d) Large j part.) in the case of $(\bar{N}_1, \bar{N}_2) = (20, 3)$.

caused by a general orbit can be obtained by a combination of periodic orbits. A larger sample size is required to make a more definitive statement.

It is important to note that the step structure itself of the periodic orbit in Fig. 5(a) is different from in the previous periodic orbit in Fig. 1. This difference is due to the different individual symmetries involved and it is expected that we can distinguish such individual symmetries by comparing a variety of periodic orbits.

6. CONCLUSION AND REMARKS

In this paper we have investigated the Lyapunov spectrum of a periodic orbit of a two-dimensional system, which consists of many disks with hard-core interactions. The system has a rectangular shape (or a square shape in a special case), and we used periodic boundary conditions. By the block cyclic structure of the matrix, which describes the tangent space dynamics, the calculation of the Lyapunov exponents is simplified to the eigenvalue problem of reduced 16×16 matrices however many particles. The reduced matrix is used to consider the relation between such a step structure and symmetries of the system, and to show the existence of the thermodynamic limit in the Lyapunov spectrum. In particular we showed that the difference of the aspect ratio of the system appears in the stepwise structure of the Lyapunov spectrum rather than in the global shape of the Lyapunov spectrum. We also gave another periodic orbit model, in which we can calculate the Lyapunov spectrum by using the block cyclic technique, and which also shows a stepwise structure of the Lyapunov spectrum.

Our techniques apply to more general space and time periodic orbits, and in particular can readily be extended to three spatial dimensions and soft potentials, although the matrices are more likely to be numerical rather than analytic in the latter case. The size of the final reduced matrix depends on the periodic orbit considered; more complicated orbits will lead to larger matrices. In such a general space and time periodic orbit, we expect to reduce the Lyapunov spectrum to a finite matrix independent of the size of the system, and by making the latter tend to infinity, deduce the thermodynamic limit.

The Lyapunov spectrum of the periodic orbit discussed in this paper shows a step structure, but it should be noted that this step structure is different from the step structure obtained by the numerical work.[(3)] It is expected that the final Lyapunov spectrum caused by a general orbit can be obtained by a combination of periodic orbits, but more systematic investigations of the Lyapunov spectra of periodic orbits may be required to explain the numerical results.

It should be emphasized that there are many periodic orbits in which we can calculate their Lyapunov spectra in many-hard-disk systems by using the block cyclic technique shown in this paper. On the other hand we are still far from the position where we can calculate the general Lyapunov spectrum of the many-hard-disk system by using the periodic orbit expansion technique. One of the problems is that in many-particle system we could not know how to systematically find all of the periodic orbits whose periods are less than a given length. In addition, the periodic orbits of many-particle systems are typically distributed continuously, not isolated from each other. These problems make the application of the periodic orbit expansion technique to the many-particle systems difficult, and not yet solved.

APPENDIX A. BLOCK CYCLIC STRUCTURE AND THE REDUCED MATRIX

In this appendix we show that the eigenvalues of the matrix M_{n_p} are equal to the eigenvalues of the matrices $\mathscr{M}(2\pi n_1/N_1, 2\pi n_2/N_2)$, $n_1 = 1, 2, \ldots, N_1$, $n_2 = 1, 2, \ldots, N_2$, given by Eq. (22).

First we calculate the matrix M_{n_p}. We multiply the matrices $M_j^{(c)} M_j^{(f)}$, $j = 1, 2, 3, 4$ given by Eqs. (13)–(16), and obtain

$$M_{n_p} = \begin{pmatrix} K_0 & K_1 & & & & K_2 \\ K_2 & K_0 & K_1 & & & \\ & K_2 & K_0 & K_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & K_2 & K_0 & K_1 \\ K_1 & & & & K_2 & K_0 \end{pmatrix} \tag{27}$$

where K_j, $j = 0, 1, 2$ are defined by

$$K_0 \equiv \begin{pmatrix} L_{02}L_{01} + L_{21}L_{11} + L_{11}L_{21} & L_{02}L_{12} + L_{21}L_{02} \\ L_{01}L_{11} + L_{22}L_{01} & L_{01}L_{02} + L_{22}L_{12} + L_{12}L_{22} \end{pmatrix} \tag{28}$$

$$K_1 \equiv \begin{pmatrix} L_{21}^2 & \tilde{0} \\ L_{01}L_{21} + L_{12}L_{01} & L_{12}^2 \end{pmatrix} \tag{29}$$

$$K_2 \equiv \begin{pmatrix} L_{11}^2 & L_{11}L_{02} + L_{02}L_{22} \\ \tilde{0} & L_{22}^2 \end{pmatrix} \tag{30}$$

with the matrices $L_{jk} \equiv Q_j P_k$ and $8N_1 \times 8N_1$ null matrix $\tilde{0}$. Equation (27) implies that the matrix M_{n_p} has the block cyclic structure, so it is block-diagonalized by the orthogonal matrix $U(16N_1, N_2)$ introduced through

$$U(j,k) \equiv \frac{1}{\sqrt{k}} \begin{pmatrix} I_j e^{2\pi i 1 \times 1/k} & I_j e^{2\pi i 1 \times 2/k} & \cdots & I_j e^{2\pi i 1 \times k/k} \\ I_j e^{2\pi i 2 \times 1/k} & I_j e^{2\pi i 2 \times 2/k} & \cdots & I_j e^{2\pi i 2 \times k/k} \\ \vdots & \vdots & & \vdots \\ I_j e^{2\pi i k \times 1/k} & I_j e^{2\pi i k \times 2/k} & \cdots & I_j e^{2\pi i k \times k/k} \end{pmatrix} \tag{31}$$

with the $j \times j$ identical matrix I_j so that we obtain the matrix $U(16N_1, N_2)^\dagger M_{n_p} U(16N_1, N_2) = \mathrm{Diag}(A(2\pi 1/N_2), A(2\pi 2/N_2), \ldots, A(2\pi N_2/N_2))$ with $\dagger$ meaning to take its Hermitian conjugate. Here the matrix $A(l)$ is defined by

$$A(l) \equiv K_0 + K_1 e^{il} + K_2 e^{-il}. \tag{32}$$

The eigenvalues of the matrix M_{n_p} are equal to the eigenvalues of the matrices $A(2\pi n_2/N_2)$, $n_2 = 1, 2, \ldots, N_2$.

The matrix $A(l)$ is represented as

$$A(l) = \begin{pmatrix} B^{(1)}(l) & B^{(2)}(l) \\ B^{(3)}(l) & B^{(4)}(l) \end{pmatrix} \tag{33}$$

where $B^{(j)}(l)$ is defined by

$$B^{(j)}(l) \equiv \begin{pmatrix} B_0^{(j)}(l) & B_1^{(j)}(l) & & & & B_2^{(j)}(l) \\ B_2^{(j)}(l) & B_0^{(j)}(l) & B_1^{(j)}(l) & & & \\ & B_2^{(j)}(l) & B_0^{(j)}(l) & B_1^{(j)}(l) & & \\ & & \ddots & \ddots & \ddots & \\ & & & B_2^{(j)}(l) & B_0^{(j)}(l) & B_1^{(j)}(l) \\ B_1^{(j)}(l) & & & & B_2^{(j)}(l) & B_0^{(j)}(l) \end{pmatrix}. \tag{34}$$

Here the matrices $B_k^{(j)}(l)$, $j = 1, 2, 3, 4$ and $k = 0, 1, 2$ are defined by

$$B_0^{(1)}(l) \equiv \begin{pmatrix} (\tilde{F}_2\tilde{F}_1)^2 + (\tilde{G}_2\tilde{G}_1)^2 + (\tilde{G}_2\tilde{F}_1)^2\, e^{-il} & \tilde{F}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{G}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{-il} \\ \tilde{F}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{G}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{il} & (\tilde{F}_2\tilde{F}_1)^2 + (\tilde{G}_2\tilde{G}_1)^2 + (\tilde{G}_2\tilde{F}_1)^2\, e^{il} \end{pmatrix} \tag{35}$$

$$B_1^{(1)}(l) \equiv \begin{pmatrix} \hat{0} & \hat{0} \\ \tilde{F}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 & (\tilde{F}_2\tilde{G}_1)^2 \end{pmatrix} \tag{36}$$

$$B_2^{(1)}(l) \equiv \begin{pmatrix} (\tilde{F}_2\tilde{G}_1)^2 & \tilde{F}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 \\ \hat{0} & \hat{0} \end{pmatrix} \tag{37}$$

$$B_0^{(2)}(l) \equiv \begin{pmatrix} \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{F}_1 + (\tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{F}_1 + \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{G}_1)\, e^{-il} & \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 e^{-il} \\ \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 & \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{G}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{F}_1 + \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{F}_1 e^{-il} \end{pmatrix} \tag{38}$$

$$B_1^{(2)}(l) \equiv \begin{pmatrix} \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{G}_1 e^{-il} & \hat{0} \\ \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{-il} & \hat{0} \end{pmatrix} \tag{39}$$

$$B_2^{(2)}(l) \equiv \begin{pmatrix} \hat{0} & \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 + (\tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1)\, e^{-il} \\ \hat{0} & \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{G}_1 \end{pmatrix} \tag{40}$$

$$B_0^{(3)}(l) \equiv \begin{pmatrix} \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{F}_1 + (\tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{G}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{F}_1)\, e^{il} & \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 + (\tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1)\, e^{il} \\ \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{il} & \tilde{F}_2\tilde{G}_1\tilde{G}_2\tilde{G}_1 + \tilde{G}_2\tilde{F}_1\tilde{F}_2\tilde{F}_1 + \tilde{F}_2\tilde{F}_1\tilde{G}_2\tilde{F}_1 e^{il} \end{pmatrix} \tag{41}$$

$$B_1^{(3)}(l) \equiv \begin{pmatrix} \hat{0} & \hat{0} \\ \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 & \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{G}_1 \end{pmatrix} \tag{42}$$

$$B_2^{(3)}(l) \equiv \begin{pmatrix} \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{G}_1 e^{il} & \tilde{G}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 e^{il} \\ \hat{0} & \hat{0} \end{pmatrix} \tag{43}$$

$$B_0^{(4)}(l) \equiv \begin{pmatrix} (\tilde{F}_2\tilde{F}_1)^2 + (\tilde{G}_2\tilde{G}_1)^2 + (\tilde{G}_2\tilde{F}_1)^2\, e^{il} & \tilde{F}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 \\ \tilde{F}_2\tilde{G}_1\tilde{F}_2\tilde{F}_1 & (\tilde{F}_2\tilde{F}_1)^2 + (\tilde{G}_2\tilde{G}_1)^2 + (\tilde{G}_2\tilde{F}_1)^2\, e^{-il} \end{pmatrix} \tag{44}$$

$$B_1^{(4)}(l) \equiv \begin{pmatrix} (\tilde{F}_2\tilde{G}_1)^2 & \hat{0} \\ \tilde{F}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{G}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{-il} & \hat{0} \end{pmatrix} \tag{45}$$

$$B_2^{(4)}(l) \equiv \begin{pmatrix} \hat{0} & \tilde{F}_2\tilde{F}_1\tilde{F}_2\tilde{G}_1 + \tilde{G}_2\tilde{G}_1\tilde{G}_2\tilde{F}_1 + \tilde{G}_2\tilde{F}_1\tilde{G}_2\tilde{G}_1 e^{il} \\ \hat{0} & (\tilde{F}_2\tilde{G}_1)^2 \end{pmatrix} \tag{46}$$

with the 4×4 null matrix $\hat{0}$. It follows from Eqs. (31), (33) and (34) that

$$\mathrm{Diag}(U(8, N_1), U(8, N_1))^\dagger A(l)\, \mathrm{Diag}(U(8, N_1), U(8, N_1)) = \begin{pmatrix} C^{(1)}(l) & C^{(2)}(l) \\ C^{(3)}(l) & C^{(4)}(l) \end{pmatrix} \tag{47}$$

where $C^{(j)}(l)$, $j = 1, 2, 3, 4$ are defined by $C^{(j)}(l) \equiv \mathrm{Diag}(D^{(j)}(2\pi 1/N_1, l), D^{(j)}(2\pi 2/N_1, l), \ldots, D^{(j)}(2\pi N_1/N_1, l))$ with $D^{(j)}(k, l) \equiv B_0^{(j)} + B_1^{(j)} e^{ik} + B_2^{(j)} e^{-ik}$. The matrix $\mathscr{M}(k, l)$ is introduced as

$$\mathscr{M}(k, l) = \begin{pmatrix} D^{(1)}(k, l) & D^{(2)}(k, l) \\ D^{(3)}(k, l) & D^{(4)}(k, l) \end{pmatrix}, \tag{48}$$

which is equal to Eq. (22). Therefore the eigenvalues of the matrix $A(l)$ are equal to the eigenvalues of the matrices $\mathscr{M}(2\pi n_1/N_1, l)$, $n_1 = 1, 2,..., N_1$. It implies that the eigenvalues of the matrix M_{n_p} are equal to the eigenvalues of the matrices $\mathscr{M}(2\pi n_1/N_1, 2\pi n_2/N_2)$, $n_1 = 1, 2,..., N_1$, $n_2 = 1, 2,..., N_2$.

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