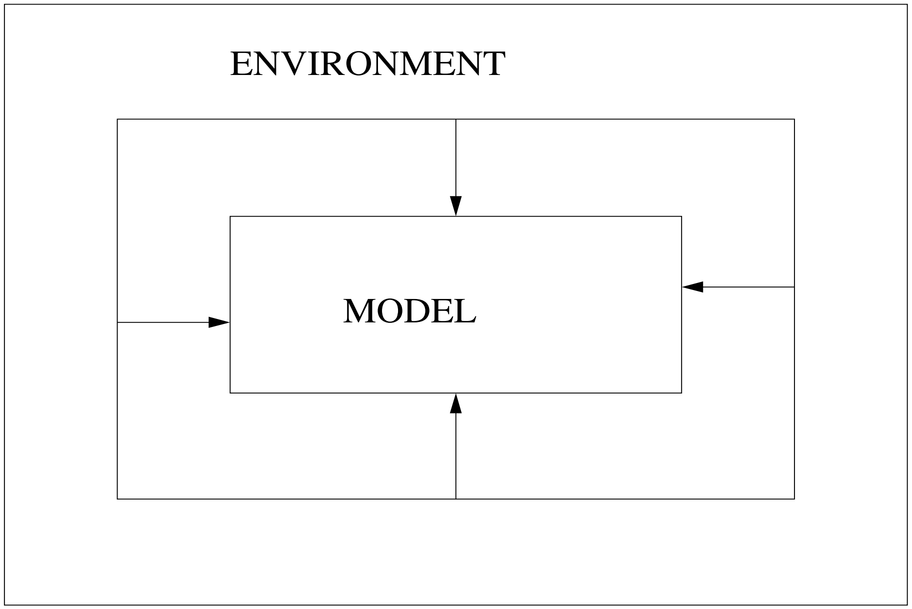
Building Models

- Project identification
- Assumptions
- Flow diagrams
- Sources of equations
- Graphical Methods
- Solving equations

Project identification

Set realistic goals!

Model what?



Exponential Growth Model

Assume rate of growth is proportional to quantity present

Continuous time, deterministic

$$\frac{dp}{dt} = ap \; p(t) = p_0 e^{at}$$

Discrete time, deterministic

$$p_{t+1} = bp_t \; p(t) = p_0 b^t$$

Continuous time, stochastic

$$\text{Prob}\{p(t + \delta t) = p(t) + 1\} = cp(t)\delta t$$
$$\text{Prob}\{p(t + \delta t) = p(t) - 1\} = dp(t)\delta t$$
$$E[p(t)] = p_0 e^{(c-d)t}$$

Deterministic and stochastic models

General birth-death

$$\frac{dp(t)}{dt} = b(p) - d(p)\,,$$

Event	**Effect on population**, p	**Probability of Event**
Birth	$p(t+\delta t) = p(t) + 1$	$b(p)\delta t$
Death	$p(t+\delta t) = p(t) - 1$	$d(p)\delta t$
No change	$p(t+\delta t) = p(t)$	$1 - b(p)\delta t - d(p)\delta t$

Exponential Growth Model

Spatial model

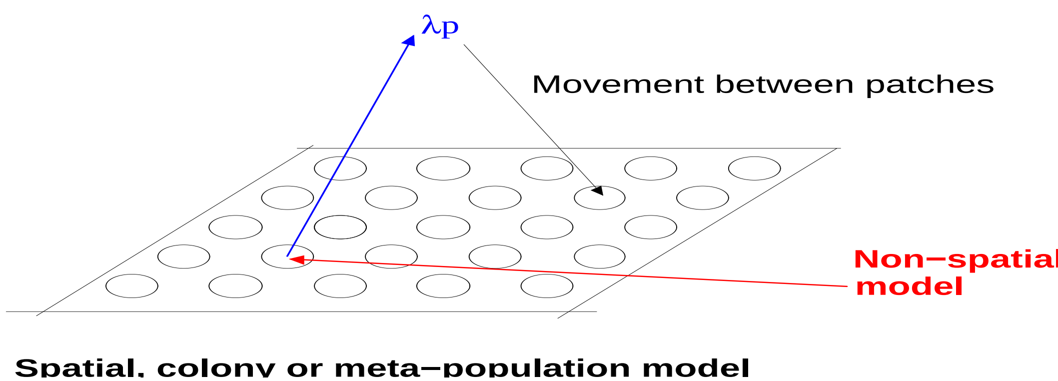


Figure 1: A schematic description of a spatial model

Assumptions

Framework

State explicitly

Relevance of simplifications

Limit applications

Flow Diagrams

Pictorial representation of model structure

Use standard symbols, or set a logical puzzle!

- $\square$ state or level variables
- $\bigcirc$ source or sink
- $\longrightarrow$ channel of material flow
- $\dashrightarrow$ channel of information flow
- $\bowtie$ control on rate of flow

Energy Model for Cattle Growth

Assume

the only relevant information about an animal is its liveweight, W

Assume

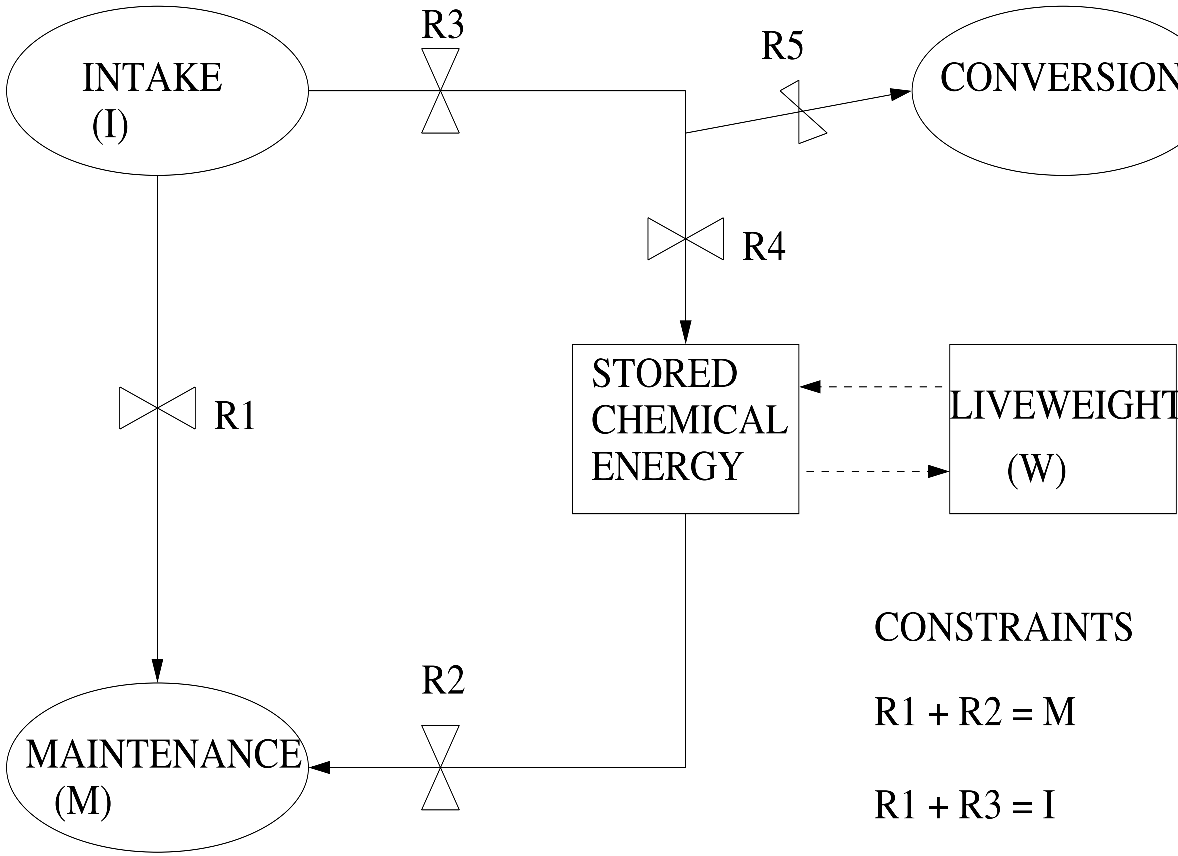
the only source of energy is food intake, I

Assume

energy is lost
either as a result of maintenance (M)
or due to converting food to body tissue

W is a state variable, I is a source, M is a sink

Flow Diagram for Energy Model



CONSTRAINTS

R1 + R2 = M

R1 + R3 = I

R3 = R4 + R5

Sources of Equations

Literature

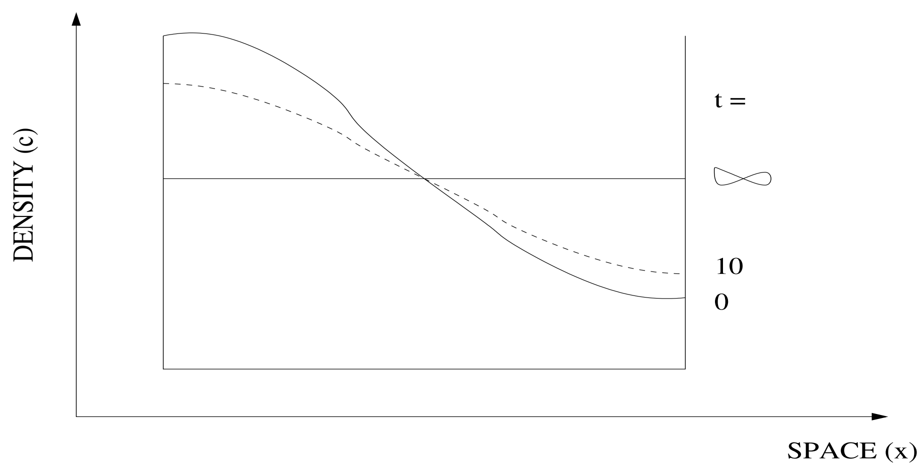
Analogy with existing models

Data analysis

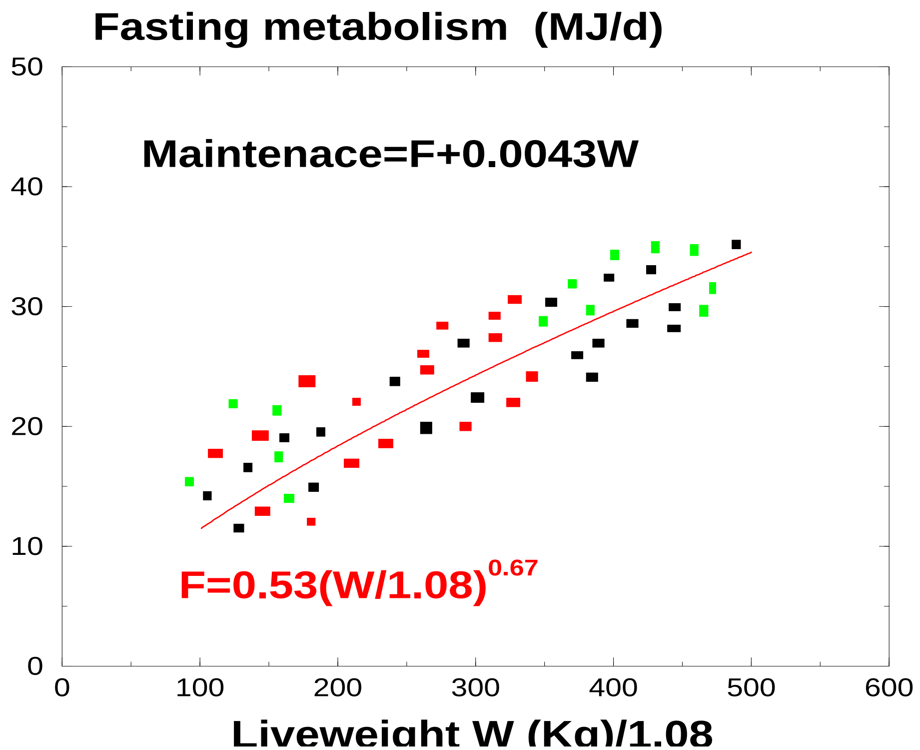
Diffusion

Used to describe the random spread of objects in space from areas of high density to areas of low density

With large numbers of objects, the partial differential equation $\frac{\partial c}{\partial t} = D\frac{\partial^2 c}{\partial x^2}$ describes the change well.



Estimating relationships from data



Meta-analyses
Energy Content of Weight Gain

46 published equations for the linear regression of

log Protein on log Liveweight
AND
log Fat on log Liveweight

need to summarize these by "average" equations

OR

must decide which equation is most appropriate

Graphical Methods for Differential Equations

Based on the approximation

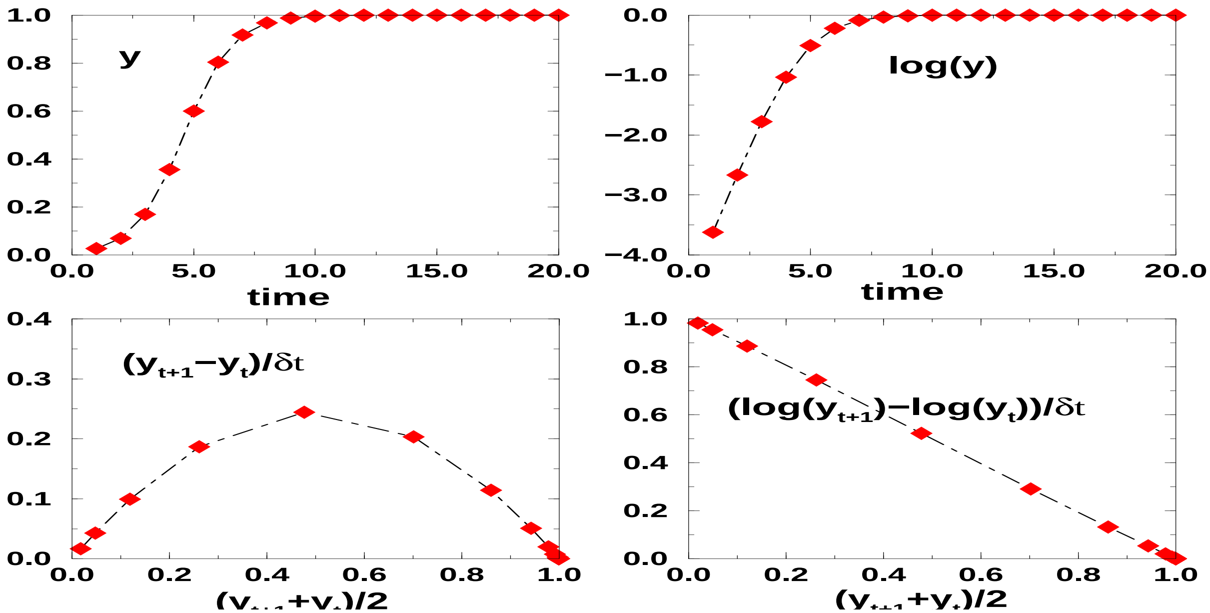
$$\frac{dy}{dt} = \frac{\{y(t+\delta t) - y(t)\}}{\delta t}$$

Observations y_i at times t_i, calculate

$$\frac{(y_{i+1} - y_i)}{(t_{i+1} - t_i)}$$

and plot this against y or t

Logistic growth a population data



For data from some growth curves, find

$$\{\log(y_{i+1} - \log(y_i\}/(t_{i+1} - t_i) \approx A - By_i$$

Why should this be?

$$\begin{aligned} \tfrac{d(\log y)}{dt} &= \tfrac{1}{y}\tfrac{dy}{dt} = A - By \\ \tfrac{dy}{dt} &= y(A - By) \end{aligned}$$

This is the logistic growth curve

Solving Equations

Mathematival solutions exist for linear, deterministic equations

$$\frac{dy}{dt} = ay;\ y = y(0)e^{at}$$

$$\begin{aligned}\frac{dy}{dt} &= ay + bz\\ \frac{dz}{dt} &= cy + dz\end{aligned}$$

Mathematical solution of non-linear equations is much more difficult

Numerical Solution of Model Equations

Computer intensive

Approximate

Case specific

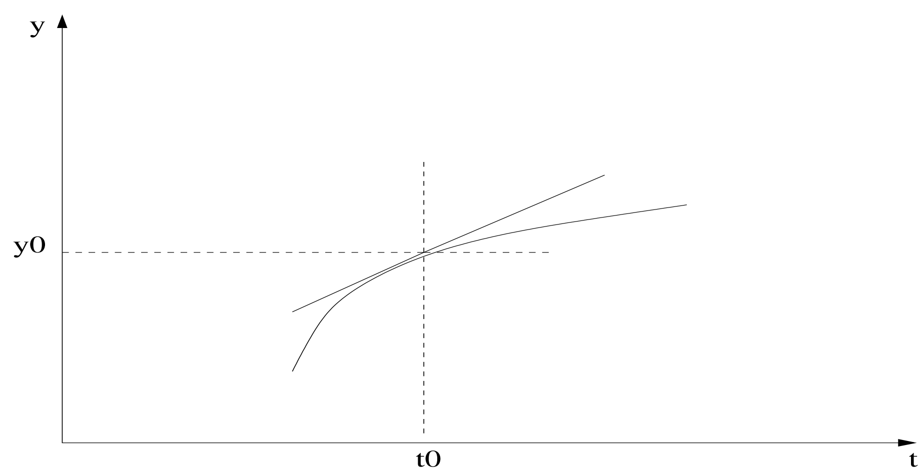
Model structure less critical

Euler's Method

Assume $\frac{dy}{dt}$ is constant over short time intervals

$$y(t_{i+1}) \approx y_i + (t_{i+1} - t_i)\frac{dy}{dt}|_{t_i}$$

Often, $t_{i+1} - t_i$ is held constant, called the step size



Fourth Order Runge-Kutta (RK4)

Uses an average of $\frac{dy}{dt}$ over the interval from t_i to t_{i+1}

Achieves much greater accuracy than assuming $\frac{dy}{dt}$ is constant

However, if vastly different time scales use stiff ODE solver, e.g., Rosenbrock

Numerical Approximation of cos(t)

$\cos(t)$ an exact solution to second order ODE

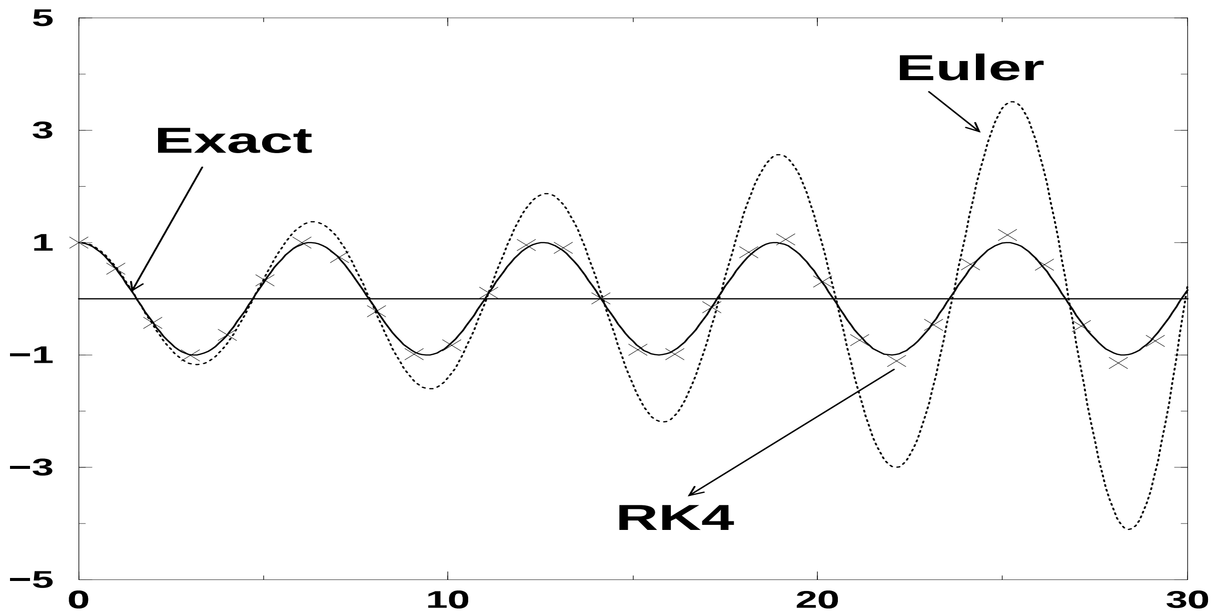
$$\frac{d^2y}{dt^2} = -y$$

Re-write as system of first order ODE's

$$\frac{dy}{dt} = z$$
$$\frac{dz}{dt} = -y$$

Solution will be unit circle in y-z phase space

Numerical estimation of the cosine function



Stochastic simulation (approximate)

General birth-death

Event	Effect on population, p	Probability of Event
Birth	$p(t+\delta t) = p(t) + 1$	$b(p)\delta t$
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Update time: $t \rightarrow t + \delta t$ and draw $y \sim U(0,1)$

Birth if $y < b(p)\delta t$

else Death if $y < b(p)\delta t + d(p)\delta t$

else No event