UNIVERSITY OF BRISTOL

Examination for the Degrees of B.Sc. and M.Sci. (Level M)

ANALYTIC NUMBER THEORY
MATH M0007
(Paper Code MATH-M0007)

Summer 2014 ;, 2 hours and 30 minutes

This paper contains five *questions*
A candidate's **FOUR** *best answers will be used for assessment.*
Calculators are not *permitted in this examination.*

Notational convention: Summations are assumed to be restricted to positive integers.

THIS PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. Recall that the arithmetical functions μ, φ, and σ are defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\varphi(n) = \sum_{\substack{a=1 \\ \gcd(a,n)=1}}^{n} 1, \quad \text{and}$$

$$\sigma(n) = \sum_{d|n} d.$$

(a) **(3 marks)**
Prove that for $n \in \mathbb{N}$,

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(b) **(4 marks)**
Use part (a) and the definition of φ to prove that for $x \geq 1$,

$$\sum_{n \leq x} \varphi(n) = \sum_{d \leq x} \mu(d) \sum_{m \leq x/d} \sum_{b \leq m} 1.$$

(c) **(6 marks)**
Use part (b) to prove that for $x \geq 2$,

$$\sum_{n \leq x} \varphi(n) = \frac{3x^2}{\pi^2} + O(x \log x).$$

(d) **(4 marks)**
Recall that for p prime and $n \in \mathbb{N}$ the notation $p^\alpha \| n$ means that $p^\alpha | n$ and $p^{\alpha+1} \nmid n$.
Prove that for $n \in \mathbb{N}$,

$$\sigma(n) = \prod_{p^\alpha \| n} \frac{p^{\alpha+1} - 1}{p - 1}.$$

(e) **(8 marks)**
Find constants $c_1, c_2 > 0$ such that, for all $n \in \mathbb{N}$,

$$c_1 \leq \frac{\varphi(n)\sigma(n)}{n^2} \leq c_2.$$

Continued...

2. Recall that for any finite Abelian group G (assumed in what follows to be written multiplicatively), the dual group $\widehat{G}$ is the group of all homomorphisms from G to the unit circle in $\mathbb{C}$ (viewed as a group under multiplication). The group operation in $\widehat{G}$ is point-wise multiplication, i.e. if $\chi_1, \chi_2 \in \widehat{G}$ then $\chi_1\chi_2$ is the element $\widehat{G}$ defined by $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for all $g \in G$.

   (a) **(8 marks)**
   Prove that if G is a finite cyclic group then $\widehat{G} \cong G$.

   (b) **(8 marks)**
   Use part (a) to prove that $\widehat{G} \cong G$ for any finite Abelian group. You may use the Fundamental Theorem of Abelian Groups and the fact that $\widehat{H_1 \times H_2} \cong \widehat{H_1} \times \widehat{H_2}$ for finite Abelian groups H_1 and H_2.

   (c) **(4 marks)**
   Prove that if G is a finite Abelian group and $g \in G$ is not the identity element, then there exists a $\psi \in \widehat{G}$ with $\psi(g) \neq 1$.

   (d) **(5 marks)**
   Assuming that G is a finite Abelian group, prove that if $g \in G$ then

$$\sum_{\chi \in \widehat{G}} \chi(g) = \begin{cases} |G| & \text{if } g = 1_G, \\ 0 & \text{otherwise,} \end{cases}$$

   where 1_G denotes the identity element in G.

Continued...

3. (a) **(3 marks)**
Suppose q is a positive integer. Define what it means for a function $\chi : \mathbb{Z} \to \mathbb{C}$ to be a Dirichlet character modulo q.

(b) **(4 marks)**
Prove that if χ is a Dirichlet character modulo q which is not the principal character then

$$\sum_{n=1}^{q} \chi(n) = 0.$$

(c) **(5 marks)**
Recall that if χ is a Dirichlet character modulo q then we define $\widehat{\chi} : \mathbb{Z} \to \mathbb{C}$ by

$$\widehat{\chi}(m) = q^{-1/2} \sum_{n=1}^{q} \chi(n) e\left(\frac{-mn}{q}\right),$$

where $e(x) = \exp(2\pi i x)$. Prove that if χ is a Dirichlet character modulo q then

$$\chi(n) = q^{-1/2} \sum_{m=1}^{q} \widehat{\chi}(m) e\left(\frac{mn}{q}\right)$$

for all $n \in \mathbb{Z}$.

(d) **(6 marks)**
Suppose that χ is a Dirichlet character modulo q and that $m \in \mathbb{Z}$ satisfies $\gcd(m, q) = 1$. Prove that

$$\widehat{\chi}(m) = \overline{\chi}(m)\widehat{\chi}(1). \tag{1}$$

(e) **(7 marks)**
Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be defined by

$$f(x) = \lim_{N\to\infty} \sum_{n=1}^{N} \frac{e(nx)}{n}.$$

Prove that if χ is a primitive Dirichlet character modulo q then

$$\mathrm{L}(1, \chi) = q^{-1/2}\widehat{\chi}(1) \sum_{m=1}^{q} \overline{\chi}(m) f(m/q).$$

You may use the fact that, since χ is primitive, equation (1) holds for all integers m.

Continued...

4. (a) **(6 marks)**
Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \frac{s}{s-1} - s\int_1^\infty \frac{\{x\}}{x^{s+1}}\,dx,$$

where $\{x\}$ denotes the fractional part of x.

(b) **(7 marks)**
Prove that the equation in (a) gives an analytic continuation of $(s-1)\zeta(s)$ to the region $\{\mathrm{Re}(s) > 0\}$.

(c) **(4 marks)**
Let $M : [1, \infty) \to \mathbb{R}$ be defined by

$$M(x) = \sum_{n \le x} \mu(n).$$

Prove that for $\mathrm{Re}(s) > 1$,

$$\zeta^{-1}(s) = s\int_1^\infty \frac{M(x)}{x^{s+1}}\,dx.$$

(d) **(4 marks)**
State the Riemann Hypothesis for the Riemann zeta function.

(e) **(4 marks)**
Use part (c) to prove that if, for every $\epsilon > 0$,

$$M(x) \ll x^{1/2+\epsilon},$$

then the Riemann Hypothesis is true.

Continued...

5. A Dirichlet character ψ is called a complex character if there exists an $n \in \mathbb{Z}$ such that $\psi(n) \notin \{0 \pm 1\}$. The goal of this problem is to prove that $L(1,\psi) \neq 0$ if ψ is a complex Dirichlet character.

   (a) **(4 marks)**
   Prove that for any non-principal character χ we have that

   $$L(1,\overline{\chi}) = \overline{L(1,\chi)}.$$

   You may assume that $L(s,\chi)$ is continuous at $s = 1$ when χ is non-principal.

   (b) **(4 marks)**
   Prove that if there is a complex character ψ modulo q such that $L(1,\psi) = 0$, then the function $f : \mathbb{C} \to \mathbb{C}$ defined by

   $$f(s) = \prod_{\chi \bmod q} L(s,\chi)$$

   satisfies $f(1) = 0$. The product here is over all Dirichlet characters modulo q.

   (c) **(12 marks)**
   Prove that for f defined as above,

   $$\log f(s) = \sum_{p} \sum_{\substack{m=1 \\ p^m \equiv 1 \bmod q}}^{\infty} \frac{\varphi(q)}{mp^{ms}}$$

   for $\mathrm{Re}(s) > 1$. The outer sum here is a sum over all prime numbers p. Justify all issues of convergence which arise. (Hint: Start with the Euler product formulas for the L functions.)

   (d) **(5 marks)**
   Use parts (b) and (c) to conclude that $L(1,\psi) \neq 0$ for any complex character ψ.

End of examination.